Maps with the Radon-Nikodým property

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Outline

1. Introduction
2. Properties of dentable maps
3. Relation with $\mathcal{DC}$ maps

The Radon-Nikodým property

Let X be a Banach space, $C \subset X$ be convex and closed.
Let $\mathbb{H}$ be the set of all the open half-spaces of a Banach space X.

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Definition

C has the **Radon-Nikodým property** if for every bounded subset A of C and every $\varepsilon > 0$, there is $H \in \mathbb{H}$ such that $A \cap H \neq \emptyset$ and $\mathrm{diam}(A \cap H) < \varepsilon$.

Dentable maps

Let M be a metric space.

Definition

A map $f\colon C \to M$ is said to be **dentable** if for every nonempty bounded set $A \subset C$ and $\varepsilon > 0$, there is $H \in \mathbb{H}$ such that $A \cap H \neq \emptyset$ and $\operatorname{diam}(f(A \cap H)) < \varepsilon$.

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- C has the RNP if and only if the identity $\mathbb{I}\colon C \to C$ is dentable.
- The RNP was extended to linear operators by Reĭnov (1975) and Linde (1976). In 1977 Reĭnov characterised RN-operators as those bounded operators $T\colon X \to Y$ satisfying that for every nonempty bounded set $A \subset X$ and every $\varepsilon > 0$ there exists $x \in A$ such that $x \notin \overline{\mathrm{conv}}(A \setminus T^{-1}(B_Y(T(x), \varepsilon)))$.

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- If $M = \mathbb{R}$, then every bounded above lower semicontinuous convex function defined on C is dentable.

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Dentable maps and dentable sets

Let $C \subset X$ be a closed convex set. Then the following are equivalent:

(i) the set C has the RNP;

(ii) for every metric space (M, d), every continuous map $f\colon C \to M$ is dentable;

(iii) every Lipschitz function $f\colon C \to \mathbb{R}$ is dentable.

The proof is based on a result by García-Castaño, Oncina, Orihuela and Troyanski (2004).

The space of dentable maps

We denote by

$\mathcal{D}_U(C, M)$ the set of dentable maps from C to M which are uniformly continuous on bounded subsets of C.

If M is a vector space, then $\mathcal{D}_U(C, M)$ is a vector space. Assume moreover that C is bounded. Then:

(a) if M is a complete metric space, then $\mathcal{D}_U(C, M)$ is complete for the metric of uniform convergence on C;
(b) if M is a Banach space, then $\mathcal{D}_U(C, M)$ is a Banach space;
(c) if M is a Banach algebra (resp. lattice), then $\mathcal{D}_U(C, M)$ is a Banach algebra (resp. lattice).

The space of dentable maps

Given a bounded subset A of C, $x^* \in X^*$ and $t > 0$, we denote

$$S(A, x^*, t) = \{x \in A : x^*(x) > \sup\{x^*, A\} - t\},$$

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Definition

We say that x^* is f-**strongly slicing** on $A \subset C$ if $\lim_{t\to 0^+} \mathrm{diam}(f(S(A, x^*, t))) = 0$.

Let $f \in \mathcal{D}_U(C, M)$ and $A \subset C$ be a bounded subset. The set of f-strongly slicing functionals on A is a $\mathcal{G}_\delta$ dense in X^*.

Equidentability

Given C bounded, $f_1, \ldots, f_n \in \mathcal{D}_U(C, M)$ and $\varepsilon > 0$, there is $H \in \mathbb{H}$ such that $C \cap H \neq \emptyset$ and $\max\{\mathrm{diam}(f_1(C \cap H)), \ldots, \mathrm{diam}(f_n(C \cap H))\} < \varepsilon$.

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The following corollary was observed by Bourgain.

Given C bounded and $x_1^*, \ldots, x_n^* \in X^*$ and $\varepsilon > 0$, there is $H \in \mathbb{H}$ such that $C \cap H \neq \emptyset$ and $\max\{\mathrm{diam}(x_1^*(C \cap H)), \ldots, \mathrm{diam}(x_n^*(C \cap H))\} < \varepsilon$.

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Assume that $f_1, \dots, f_n \in \mathcal{D}_U(C, M)$ and $f \colon C \to M$ is a continuous map such that $f(x) \in \{f_1(x), \dots, f_n(x)\}$ for every $x \in C$. Then $f \in \mathcal{D}_U(C, M)$.

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Assume that $f \colon C \to \mathbb{R}$ is uniformly continuous on bounded sets. Then f is dentable if and only if $|f|$ is dentable.

The above result fails when the modulus is replaced by the norm for dentable maps.

Dentable sets with respect to a metric

We have also considered the dentability of the identity map $I\colon (C, \|\cdot\|) \to (C, d)$ where d is a metric which is uniformly continuous with respect to the norm.

Let C be a closed convex subset which is dentable with respect to a complete metric d defined on it. Assume moreover that d is uniformly continuous on bounded sets with respect to the norm and induces the norm topology. Then C has the RNP.

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$\mathcal{DC}$ functions

Definition

A function $f\colon C \to \mathbb{R}$ is said to be $\boldsymbol{\mathcal{DC}}$ (or **delta-convex**) if it can be represented as the difference of two convex continuous functions on C, and it is said to be **$\boldsymbol{\mathcal{DC}}$-Lipschitz** if it is the difference of two convex Lipschitz functions.

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Theorem (Cepedello, 1998)

A Banach space X is superreflexive if, and only if, every Lipschitz function $f: X \to \mathbb{R}$ can be approximated uniformly on bounded sets by $\mathcal{DC}$ functions which are Lipschitz on bounded sets.

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Theorem (Raja, 2008)

A Lipschitz function $f\colon C \to \mathbb{R}$ is finitely dentable if, and only if, f is uniform limit of $\mathcal{DC}$-Lipschitz functions.

Finitely dentable functions

Given a dentable map $f\colon C \to M$ defined on a bounded set we may consider the following "derivation"

$$\begin{aligned}
[D]'_\varepsilon &= \{x \in D : \mathrm{diam}(f(D \cap H)) > \varepsilon,\ \forall H \in \mathbb{H}, x \in H\} \\
[C]_\varepsilon^{\alpha+1} &= [[C]_\varepsilon^\alpha]'_\varepsilon \\
[C]_\varepsilon^\alpha &= \bigcap_{\beta<\alpha} [C]_\varepsilon^\beta \text{ if } \alpha \text{ is a limit ordinal .}
\end{aligned}$$

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$$[C]^\alpha_\varepsilon = \bigcap_{\beta<\alpha} [C]^\beta_\varepsilon \text{ if } \alpha \text{ is a limit ordinal .}$$

Definition

We say that
f is **finitely dentable** if for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $[C]^n_\varepsilon = \emptyset$.
f is **countably dentable** if for every $\varepsilon > 0$ there is $\alpha < \omega_1$ s.t. $[C]^\alpha_\varepsilon = \emptyset$.

Dentability and $\mathcal{DC}$-functions

A set D is said to be a **$(\mathcal{C} \setminus \mathcal{C})_\sigma$-set** if $D = \bigcup_{n=1}^{\infty}(A_n \setminus B_n)$, where A_n and B_n are convex closed sets. A function $f\colon C \to \mathbb{R}$ is said to be **$(\mathcal{C} \setminus \mathcal{C})_\sigma$-measurable** if the sets $f^{-1}(-\infty, r)$ and $f^{-1}(r, +\infty)$ are both $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of X for each $r \in \mathbb{R}$.

Let $f\colon C \to \mathbb{R}$ be a uniformly continuous function defined on a bounded closed convex set. Consider the following statements:

(i) f is uniform limit of $\mathcal{DC}$ functions;

(ii) f is uniform limit of $\mathcal{DC}$-Lipschitz functions;

(iii) f is finitely dentable;

(iv) f is countably dentable;

(v) f is $(\mathcal{C} \setminus \mathcal{C})_\sigma$-measurable;

(vi) f is pointwise limit of $\mathcal{DC}$-Lipschitz functions.

Then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \not\Rightarrow (v) \not\Rightarrow (iv) \not\Rightarrow (iii)$.

Relation with $\mathcal{DC}$ maps

Definition (Veselý-Zajíček, 1989)

A continuous map $F\colon C \to Y$ is said to be a $\boldsymbol{\mathcal{DC}}$ **map** if there exists a continuous (necessarily convex) function $f\colon C \to \mathbb{R}$, called **control function for** F, such that $f + y^* \circ F$ is a convex continuous function on A for every $y^* \in S_{Y^*}$

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Let us notice that f is a control function for F if, and only if,

$$\left\| \sum_{i=1}^{n} \lambda_i F(x_i) - F(\sum_{i=1}^{n} \lambda_i x_i) \right\| \leq \sum_{i=1}^{n} \lambda_i f(x_i) - f(\sum_{i=1}^{n} \lambda_i x_i)$$

whenever $x_1, \ldots, x_n \in A$, $\lambda_1, \ldots, \lambda_n \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

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Let $D \subset Y$ be a closed convex set. Then the following are equivalent:

(i) the set D has the RNP;

(ii) for every Banach space X and every convex subset $C \subset X$, every bounded continuous $\mathcal{DC}$ map $F\colon C \to D$ admitting a bounded control function is dentable.

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Thank you for your attention