Extremal structure of Lipschitz-free spaces

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Colloquium KSU

December, 2018

Research partially supported by

UNIVERSIDAD DE MURCIA

GOBIERNO DE ESPAÑA

MINISTERIO DE ECONOMÍA Y COMPETITIVIDAD

Unión Europea

Fondo Europeo de Desarrollo Regional

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True if M:

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Theorem (Weaver)

Every **preserved** *extreme point of* $B_{\mathcal{F}(M)}$ *is of the form* $\frac{\delta_x - \delta_y}{d(x,y)}$.

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If $\frac{\delta_{x_\alpha}-\delta_{y_\alpha}}{d(x_\alpha,y_\alpha)}$ *converges weakly to* $\frac{\delta_x-\delta_y}{d(x,y)}$*, then* $d(x_\alpha, x) \to 0$ *y* $d(y_\alpha, y) \to 0$*.*

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Proof.

If $d(x_\alpha, x) \not\to 0$, then there is $0 < \varepsilon < \min\{d(x,y), \limsup_\alpha d(x_\alpha, x)\}$.

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$$f(t)=\max\{\varepsilon-d(x,t),0\}$$

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On the one hand, $\frac{f(x)-f(y)}{d(x,y)} = \frac{\varepsilon}{d(x,y)} > 0$.

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Assume $\frac{\delta_x-\delta_y}{d(x,y)}$ is not a denting point. Then there is $\varepsilon > 0$ such that $\mathrm{diam}(S) > \varepsilon$ for every slice S containing $\frac{\delta_x-\delta_y}{d(x,y)}$.

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- **Case 1**. For every slice S containing $\frac{\delta_x-\delta_y}{d(x,y)}$ there is $\frac{\delta_{x_S}-\delta_{y_S}}{d(x_S,y_S)} \in S$ such that $\left\|\frac{\delta_{x_S}-\delta_{y_S}}{d(x_S,y_S)} - \frac{\delta_x-\delta_y}{d(x,y)}\right\| > \varepsilon/4$.

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  Since the slices containing $\frac{\delta_x-\delta_y}{d(x,y)}$ are a neighbourhood basis for the weak topology, we have $\frac{\delta_{x_S}-\delta_{y_S}}{d(x_S,y_S)}\xrightarrow{w}\frac{\delta_x-\delta_y}{d(x,y)}$.

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$$B_{\mathcal{F}(M)} = \overline{\text{conv}}(V) = \overline{\text{conv}}(V \cap S) \cup (V \setminus S))$$
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Consider

$$C_r=\left\{\lambda x+(1-\lambda)y : x\in B(\frac{\delta_x-\delta_y}{d(x,y)},\varepsilon/4), y\in\overline{\text{conv}}(V\setminus S),\lambda\in[0,r]\right\}.$$

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Since $\frac{\delta_x-\delta_y}{d(x,y)}$ is a preserved extreme point, we have $\frac{\delta_x-\delta_y}{d(x,y)} \in B_{\mathcal{F}(M)} \setminus \overline{C_r}$.

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Since $\frac{\delta_x-\delta_y}{d(x,y)}$ is a preserved extreme point, we have $\frac{\delta_x-\delta_y}{d(x,y)}\in B_{\mathcal{F}(M)}\setminus\overline{C_r}$. If $r\approx 0$, then $\operatorname{diam}(B_{\mathcal{F}(M)}\setminus\overline{C_r})<\varepsilon$.

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Proof.

- **Case 2** (based on the Superlemma of Asplund-Bourgain-Namioka). There is a slice S containing $\frac{\delta_x-\delta_y}{d(x,y)}$ with $V \cap S \subset B(\frac{\delta_x-\delta_y}{d(x,y)}, \varepsilon/4)$, where $V = \{\frac{\delta_u-\delta_v}{d(u,v)}, u \neq v\}$.We have

$$B_{\mathcal{F}(M)} = \overline{\text{conv}}(V) = \overline{\text{conv}}(V \cap S) \cup (V \setminus S))$$
$$\subset \overline{\text{conv}} \left(B(\frac{\delta_x - \delta_y}{d(x,y)}, \varepsilon/4) \cup \overline{\text{conv}}(V \setminus S) \right)$$

Consider

$$C_r = \left\{ \lambda x + (1-\lambda)y : x \in B(\frac{\delta_x - \delta_y}{d(x,y)}, \varepsilon/4), y \in \overline{\text{conv}}(V \setminus S), \lambda \in [0, r] \right\}.$$

Since $\frac{\delta_x-\delta_y}{d(x,y)}$ is a preserved extreme point, we have $\frac{\delta_x-\delta_y}{d(x,y)} \in B_{\mathcal{F}(M)} \setminus \overline{C_r}$. If $r \approx 0$, then $\mathrm{diam}(B_{\mathcal{F}(M)} \setminus \overline{C_r}) < \varepsilon$. Now, we can take a slice S with $\frac{\delta_x-\delta_y}{d(x,y)} \in S \subset B_{\mathcal{F}(M)} \setminus \overline{C_r}$.

Question

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No! To see that we need the following result.

Theorem (Aliaga-Guirao, 2017)

$\frac{\delta_x - \delta_y}{d(x,y)}$ is a preserved extreme point of $B_{\mathcal{F}(M)}$ if and only if for all $\varepsilon > 0$ there is $\delta > 0$ such that

$$(1-\delta)(d(x,z)+d(z,y)) < d(x,y) \Rightarrow \min\{d(x,z), d(y,z)\} < \varepsilon$$

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Example

Let $M = \{0\} \cup \{x_n\} \subset c_0$, where $x_1 = 2e_n$, $x_n = e_1 + (1+1/n)e_n$ if $n \geq 2$. Then $\frac{\delta_{x_1}-\delta_0}{d(x_1,0)}$ is an extreme point of $B_{\mathcal{F}(M)}$ which is not a preserved extreme point.

Definition (Schachermayer, 1983)

A Banach space has **property** α if there is $\Gamma = \{x_\lambda\} \subset X$ and $\Gamma^* = \{x^*_\lambda\}$ such that

1. $\|x_\lambda\| = \|x^*_\lambda\| = |x^*_\lambda(x_\lambda)| = 1$.
2. There is $\leq \alpha < 1$ such that $|x^*_\lambda(x_\mu)| \leq \alpha$ if $\lambda \neq \mu$.
3. $\overline{\text{conv}}(\Gamma) = B_X$.

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Question

When $\mathcal{F}(M)$ has property α?

A metric space is said to be **concave** if every molecule is a preserved extreme point of $B_{\mathcal{F}(M)}$.

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For instance, every Holder metric space (that is, the metric is of the form d^θ for some $0 < \theta < 1$) is concave.

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Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca, 2018)

Let M be a concave metric space. Then $\mathcal{F}(M)$ has property α if and only if M is uniformly discrete and bounded and there is $\varepsilon > 0$ such that

$$d(x,z) + d(z,y) - d(x,y) \geq \varepsilon$$

Thank you for your attention!