f SéNeCa (+)
Agencia de Ciencia y Tecnología
Región de Murcia

GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD

A characterization of the Daugavet property in spaces of Lipschitz functions

L. García-Lirola, A. Procházka, A. Rueda Zoca

Introduction

Definition

A Banach space X is said to have the **Daugavet property** if

$$\|I+T\| = 1 + \|T\|$$

for every rank-one operator $T\colon X \to X$.

Examples of spaces with the Daugavet property are $C[0,1]$ (Daugavet, 1963), $L_1[0,1]$ (Lozanovskii, 1966), $L_\infty[0,1]$ (Pełczyński, 1965), and preduals of spaces with the Daugavet property.
We will need the following geometric characterization of the Daugavet property.

Theorem (Kadets– Shvidkoy–Sirotkin–Werner, [6])

X has the Daugavet property if and only if for every $x_0 \in S_X$, every $\varepsilon > 0$ and every slice S of B_X there is a slice $S' \subset S$ such that $\|x_0 + x\| > 2 - \varepsilon$ for every $x \in S'$.

Given a complete metric space (M, d) and a distinguished point $0 \in M$, the space

$$\mathrm{Lip}_0(M) := \{f\colon M \to \mathbb{R} : f \text{ is Lipschitz}, f(0) = 0\}$$

is a dual Banach space when equipped with the norm

$$\|f\|_L := \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : x \neq y\right\}.$$

The **Lipschitz-free space** over M is given by

$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(x) : x \in M\} \subset \mathrm{Lip}_0(M)^*,$$

where $\langle \delta(x), f\rangle = f(x)$ for $f \in \mathrm{Lip}_0(M)$. Let us highlight that for every Banach space Y and every Lipschitz function $f\colon M \to Y$ such that $f(0) = 0$ there is a unique bounded linear operator $\hat{f}\colon \mathcal{F}(M) \to Y$ such that $\hat{f} \circ \delta = f$. Moreover, $\|\hat{f}\| = \|f\|_L$. It follows that $\mathcal{F}(M)^*$ is isometric to $\mathrm{Lip}_0(M)$ (see e.g. [4, 7]).

Note that $\mathrm{Lip}_0([0,1])$ is isometric to $L_\infty[0,1]$ and thus it has the Daugavet property.

Does $\mathrm{Lip}_0([0,1]^2)$ have the Daugavet property? (D. Werner, [8])

This problem was solved by Ivakhno, Kadets and Werner in [5]. Before stating their result we need to introduce some notions. A metric space M is said to be:

- a **length space** if for every $x, y \in M$, the distance $d(x,y)$ is equal to the infimum of the length of rectifiable curves joining them. Moreover, if that infimum is always attained then we will say that M is a **geodesic space**.
- **local** if for every $\varepsilon > 0$ and every $f \in \mathrm{Lip}_0(M)$ there exist $u, v \in M$ such that $0 < d(u,v) < \varepsilon$ and $\frac{f(u)-f(v)}{d(u,v)} > \|f\|_L - \varepsilon$.
- **spreadingly local** if for every $\varepsilon > 0$ and every $f \in \mathrm{Lip}_0(M)$ the set

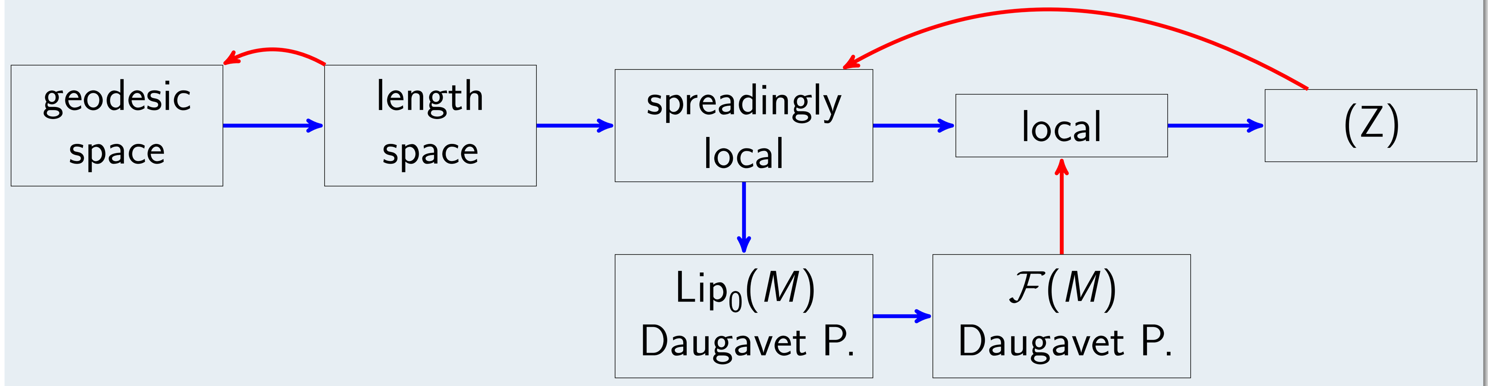
$$\left\{x \in M : \inf_{\delta > 0} \left\|f|_{B(x,\delta)}\right\|_L > \|f\|_L - \varepsilon\right\}$$

is infinite.
- **(Z)** if for every $\varepsilon > 0$ and every $x, y \in M$, $x \neq y$, there is $z \in M \setminus \{x, y\}$ such that

$$d(x,z) + d(z,y) \leq d(x,y) + \varepsilon \min\{d(x,z), d(z,y)\}.$$

Theorem (Ivakhno–Kadets–Werner, [5])

Blue implications hold for complete metric spaces. Red implications hold for compact metric spaces.



Main result

Clearly, $[0,1]^2$ is a geodesic metric space, so the theorem of Ivakhno–Kadets–Werner gives a positive answer to the problem of Werner mentioned above. Moreover, it provides a metric characterization of compact metric spaces M such that $\mathrm{Lip}_0(M)$ has the Daugavet property: they are exactly the spaces having (Z). Our goal here is to provide a metric characterization for complete metric spaces.

Theorem (GL – Procházka – Rueda Zoca, 2018)

Let M be a complete metric space. The following statements are equivalent:
(i) M is local.
(ii) M is a length space.
(iii) $\mathrm{Lip}_0(M)$ has the Daugavet property.
(iv) $\mathcal{F}(M)$ has the Daugavet property.

Remark that a complete metric M is a length space if, and only if, for every $x, y \in M$ and every $\delta > 0$ the balls $B(x, \frac{1+\delta}{2}d(x,y))$ and $B(y, \frac{1+\delta}{2}d(x,y))$ intersect.

Sketch of the proof. (i)⇒(ii). If M is not a length space, then there exists $x, y \in M$ and $\delta > 0$ such that the balls $B\left(x, \frac{1+\delta}{2}d(x,y)\right)$ and $B\left(y, \frac{1+\delta}{2}d(x,y)\right)$ do not intersect. One can check that the Lipschitz function

$$f(t) = \left(\frac{d(x,y)}{2} - \frac{d(x,t)}{1+\delta}\right)^+ - \left(-\frac{d(x,y)}{2} + \frac{d(y,t)}{1+\delta}\right)^+$$

satisfies $\|f\|_L \geq 1$ and

$$\sup\left\{\frac{f(u)-f(v)}{d(u,v)} : d(u,v) < \frac{\delta}{2}d(x,y)\right\} \leq \frac{1}{1+\delta}.$$

(ii)⇒(iii) was proved in [5] and (iii)⇒(iv) is trivial.

(iv)⇒(i). First we prove the following:

Claim: For every $f \in S_{\mathrm{Lip}_0(M)}$, $\varepsilon > 0$ and $x, y \in M$ there are $u, v \in M$ such that $\frac{f(u)-f(v)}{d(u,v)} > 1 - \varepsilon$ and

$$(1-\varepsilon)(d(x,y) + d(u,v)) < \min\{d(x,v) + d(u,y), d(x,u) + d(v,y)\}.$$

Indeed, the geometric characterization of the Daugavet property provides $u, v \in M$ such that $\langle f, \frac{\delta(u)-\delta(v)}{d(u,v)}\rangle > 1 - \varepsilon$ and

$$\left\|\frac{\delta(x)-\delta(y)}{d(x,y)} \pm \frac{\delta(u)-\delta(v)}{d(u,v)}\right\| > 2 - \varepsilon.$$

The claim follows easily.

Now, given $f \in S_{\mathrm{Lip}_0(M)}$, $\varepsilon > 0$ and $x_1, y_1 \in M$ such that $\frac{f(x_1)-f(y_1)}{d(x_1,y_1)} > 1 - \varepsilon$, apply the Claim to the function $h = \frac{f+g}{2}$, where

$$g(t) = \frac{d(x_1,y_1)}{2}\frac{d(t,y_1) - d(t,x_1)}{d(t,y_1) + d(t,x_1)},$$

to get $x_2, y_2 \in M$ such that $\frac{f(x_2)-f(y_2)}{d(x_2,y_2)} > 1 - \varepsilon$ and $d(x_2, y_2) < \frac{\varepsilon}{(1-\varepsilon)^2}d(x_1, y_1)$. Iterating this process we find sequences $(x_n), (y_n)$ with

$$d(x_n, y_n) < \left(\frac{\varepsilon}{(1-\varepsilon)^2}\right)^n d(x_0, y_0)$$

and $\frac{f(x_n)-f(y_n)}{d(x_n,y_n)} > 1 - \varepsilon$.

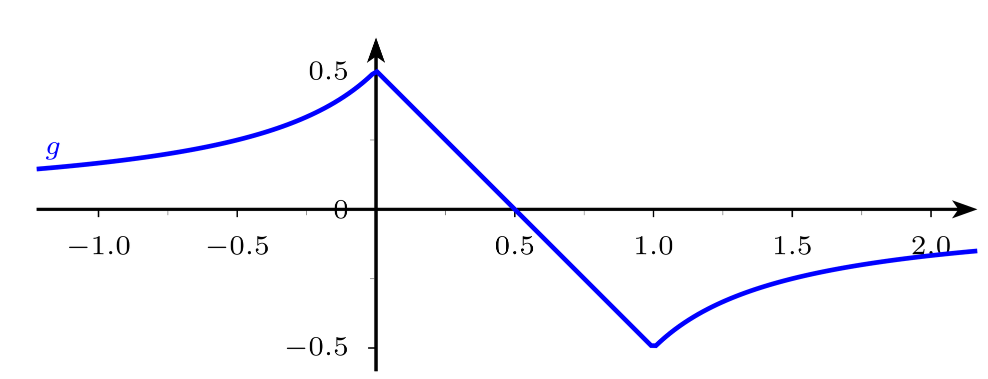


Figure: The function g for $M = \mathbb{R}$, $x_1 = 0$ and $y_1 = 1$

Strongly exposed points in $B_{\mathcal{F}(M)}$

The study of the extremal structure of $B_{\mathcal{F}(M)}$ probably was started by Weaver in [7], where it is proved that **every extreme point** $B_{\mathrm{Lip}_0(M)}$ **that belongs to** $\mathcal{F}(M)$ **is a molecule**, that is, an element of the form

$$m_{xy} = \frac{\delta(x) - \delta(y)}{d(x,y)}, x, y \in M, x \neq y.$$

Weaver also proved that m_{xy} is an extreme point of $B_{\mathrm{Lip}_0(M)^*}$ whenever there is $f \in \mathrm{Lip}_0(M)$ *peaking at* (x,y), that is, $\langle f, m_{xy}\rangle = 1$ and $\sup_{(u,v)\notin U}\langle f, m_{uv}\rangle < 1$ for every open subset U of $M^2 \setminus \Delta$ containing (x,y) and (y,x).
Indeed, peaking functions characterise strongly exposed points in $B_{\mathcal{F}(M)}$. Recall that a point $x \in B_X$ is said to be a **strongly exposed point** if there is $f \in X^*$ such that for every sequence $(x_n)_n$ in B_X we have $x_n \to x$ whenever $f(x_n) \to f(x)$.

Theorem (GL – Procházka – Rueda Zoca, 2018)

Let $x, y \in M$, $x \neq y$. The following statements are equivalent:
(i) The molecule m_{xy} is a strongly exposed point of $B_{\mathcal{F}(M)}$.
(ii) There is $f \in \mathrm{Lip}_0(M)$ peaking at (x,y).
(iii) There is $\varepsilon > 0$ such that

$$d(x,z) + d(z,y) - d(x,y) > \varepsilon \min\{d(x,z), d(z,y)\} \quad \text{for all } z \in M \setminus \{x,y\}.$$

This result extends the characterization of peaking functions in $\mathbb{R}$-trees given in [2]. Note that condition (iii) fails for every pair of distinct points in M if and only if M has (Z). As a consequence, the following dichotomy holds:

Let M be a compact metric space. Then either $\mathrm{Lip}_0(M)$ has the Daugavet property (and so every slice of $B_{\mathcal{F}(M)}$ has diameter 2) or $B_{\mathcal{F}(M)}$ has a strongly exposed point.

Strongly exposed points are related with points of differentiability of the dual norm via the Šmulyan Lemma. This allows us to obtain the following curious consequence:

The norm of $\mathrm{Lip}_0(M)$ is Gâteaux differentiable at f if and only if it is Fréchet differentiable at f.

Let us end by giving the following characterization under compactness assumptions, which improves the ones in [5].

Let M be a compact metric space. The following statements are equivalent:
(i) M is geodesic.
(ii) For every $x \neq y \in M$ there is $z \in M \setminus \{x,y\}$ with $d(x,z) + d(z,y) = d(x,y)$.
(iii) $\mathrm{Lip}_0(M)$ has the Daugavet property.
(iv) No extreme point of $B_{\mathrm{Lip}_0(M)^*}$ belongs to $\mathcal{F}(M)$.
(v) The unit ball of $\mathcal{F}(M)$ does not have any strongly exposed point.
(vi) The norm of $\mathrm{Lip}_0(M)$ does not have any point of Gâteaux differentiability.
(vii) The norm of $\mathrm{Lip}_0(M)$ does not have any point of Fréchet differentiability.

Very recently, Avilés and Martínez-Cervantes [1] have proved that every complete metric space with (Z) is a length space. As a consequence, statements (iii) to (vii) are equivalent for a complete metric space.

References

[1] A. Avilés and G. Martínez-Cervantes. "Complete metric spaces with property (Z) are length metric spaces". arXiv:1808.00715. 2018.
[2] A. Dalet, P. L. Kaufmann, and A. Procházka. "Characterization of metric spaces whose free space is isometric to ℓ_1". *Bull. Belg. Math. Soc. Simon Stevin* 23.3 (2016), pp. 391–400.
[3] L. García-Lirola, A. Procházka, and A. Rueda Zoca. "A characterisation of the Daugavet property in spaces of Lipschitz functions". *J. Math. Anal. Appl.* 464.1 (2018), pp. 473–492.
[4] G. Godefroy. "A survey on Lipschitz-free Banach spaces". *Comment. Math.* 55.2 (2015), pp. 89–118.
[5] Y. Ivakhno, V. Kadets, and D. Werner. "The Daugavet property for spaces of Lipschitz functions". *Math. Scand.* 101.2 (2007), pp. 261–279.
[6] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner. "Banach spaces with the Daugavet property". *Trans. Amer. Math. Soc.* 352.2 (2000), pp. 855–873.
[7] N. Weaver. *Lipschitz algebras*. World Scientific Publishing Co., Inc., River Edge, NJ, 1999, pp. xiv+223.
[8] D. Werner. "Recent progress on the Daugavet property". *Irish Math. Soc. Bull.* 46 (2001), pp. 77–97.

L. García-Lirola was supported by the grants MINECO/FEDER MTM2014-57838-C2-1-P and Fundación Séneca CARM 19368/PI/14. L. García-Lirola is also supported by a postdoctoral grant in the framework of *Programa Regional de Talento Investigador y su Empleabilidad* from Fundación Séneca - Agencia de Ciencia y Tecnología de la Región de Murcia.