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Extremal structure of Lipschitz free spaces

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Abstract

We analyse the relationship between different extremal notions in Lipschitz free spaces. We completely characterise strongly exposed points in the unit ball of a free space. We prove that every preserved extreme point of the unit ball is also a denting point. We show in some particular cases that every extreme point is a molecule, and that a molecule is extreme whenever the two points, say x and y, which define it satisfy that the metric segment $[x, y]$ only contains x and y. As an application, we get some new consequences about norm-attainment in spaces of vector-valued Lipschitz functions.
This is based on a joint work with A. Procházka, C. Petitjean and A. Rueda Zoca.

Introduction

We are interested in studying the following families of distinguished points in the unit ball B_X of a Banach space X.

Definition

A point x in the unit ball B_X of a Banach space X is said to be:

- an **extreme point** of B_X if $x = \lambda y + (1-\lambda)z$, $y, z \in B_X$, $\lambda \in (0,1)$, then $x = y = z$.
- a **preserved extreme point** of B_X if x is an extreme point of $B_{X^{**}}$,
- a **denting point** of B_X if the slices of B_X containing x are a neighbourhood basis of x in $(B_X, \|\ \|)$.
- an **exposed point** of B_X if there is $f \in X^*$ such that
$$f(x) > f(y) \text{ for every } y \in B_X \setminus \{x\}.$$
- a **weak-strongly exposed point** of B_X if there is $f \in X^*$ such that for every sequence $(x_n)_n$ in B_X we have $x_n \xrightarrow{w} x$ whenever $f(x_n) \to f(x)$, equivalently, the slices of B_X provided by f are a neighbourhood basis of x in (B_X, w).
- a **strongly exposed point** of B_X if there is $f \in X^*$ such that for every sequence $(x_n)_n$ in B_X we have $x_n \to x$ whenever $f(x_n) \to f(x)$, equivalently, the slices of B_X provided by f are a neighbourhood basis of x in $(B_X, \|\ \|)$.

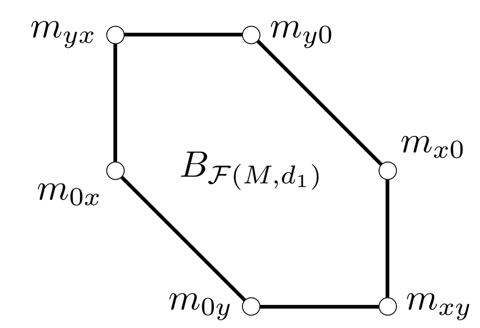
It is not difficult to check that the above concepts are related in the following way:

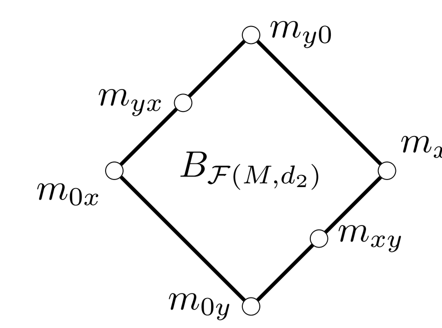
$$\begin{array}{ccccc} \text{strongly exposed} & \Longrightarrow & \text{weak-strongly exposed} & \Longrightarrow & \text{exposed} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{denting} & \Longrightarrow & \text{preserved extreme} & \Longrightarrow & \text{extreme} \end{array}$$

Moreover, none of these implications reverse in general.
Our aim is to study the former notions in the particular case in which the Banach space is the Lipschitz free space $\mathcal{F}(M)$ over a complete metric space (M, d). Recall that the space $\mathrm{Lip}_0(M)$ of Lipschitz functions on M vanishing at a distinguised point $0 \in M$ is a Banach space when it is endowed with the norm given by the best Lipschitz constant. Then
$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(x) : x \in M\} \subset \mathrm{Lip}_0(M)^*,$$
where $\langle \delta(x), f \rangle = f(x)\rangle$ for $f \in \mathrm{Lip}_0(M)$. We refer the reader to [1, 2] for the fundamental properties and applications of these spaces. Let us highlight that for every Banach space Y and every Lipschitz function $f\colon M \to Y$ such that $f(0) = 0$ there is a unique bounded linear operator $\hat{f}\colon \mathcal{F}(M) \to Y$ such that $\hat{f} \circ \delta = f$. Moreover, $\|\hat{f}\| = \|f\|$. It follows from this fact that $\mathcal{F}(M)^*$ is isometric to $\mathrm{Lip}_0(M)$.
The study of the extremal structure of $B_{\mathcal{F}(M)}$ probably was started by Weaver in [2], where it is proved that **every preserved extreme point of $B_{\mathcal{F}(M)}$ is a molecule**, that is, an element of the form
$$m_{xy} = \frac{\delta(x) - \delta(y)}{d(x,y)}, x, y \in M, x \neq y.$$
We denote V the set of molecules in $\mathcal{F}(M)$. Note that
$$\|f\| = \sup\left\{\frac{f(x) - f(y)}{d(x,y)} : x, y \in M, x \neq y\right\} = \sup\{\langle f, m_{xy}\rangle : m_{xy} \in V\}$$
and so V is 1-norming for $\mathrm{Lip}_0(M)$. Equivalently, $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(V)$. That provides a useful way for describing the norm in $\mathcal{F}(M)$.

Extreme points in $B_{\mathcal{F}(M)}$

In order to get some intuition, let us assume first that M just consists of three points, $M = \{0, x, y\}$. Let us define the following metrics on M:
$$d_1(x,0) = d_1(y,0) = d_1(x,y) = 1 \qquad d_2(x,0) = d_2(y,0) = 1, d_2(x,y) = 2$$





Note that in $\mathcal{F}(M, d_1)$ the set of extreme points of the ball coincides with the set of molecules. On the other hand, the molecule m_{xy} is not an extreme point of the ball of $\mathcal{F}(M, d_2)$. The reason is that 0 belongs to the metric segment $[x, y]$ in (M, d_2).
More generally, for a molecule m_{xy} to be an extreme point of $B_{\mathcal{F}(M)}$ it is necessary that the metric segment $[x, y]$ between x and y reduces to $\{x, y\}$. Indeed, if $d(x,z) + d(z,y) = d(x,y)$ for some $z \in M \setminus \{x, y\}$ then
$$m_{xy} = \frac{d(x,z)}{d(x,y)} m_{xz} + \frac{d(z,y)}{d(x,y)} m_{zy}$$
and so m_{xy} is not an extreme point of $B_{\mathcal{F}(M)}$. This fact motivates the following question:

Open problem

Assume that $[x, y] = \{x, y\}$. Is m_{xy} an extreme point of $B_{\mathcal{F}(M)}$?

Aliaga and Guirao have recently proved that the above problem has an affirmative answer if M is compact. We have shown the following:

Let M be a bounded uniformly discrete (i.e. $\inf_{x \neq y} d(x,y) > 0$) metric space. If $[x, y] = \{x, y\}$, then m_{xy} is an extreme point of $B_{\mathcal{F}(M)}$.

The above result allows us to find an example of a bounded uniformly discrete countable metric space M such that $\mathcal{F}(M)$ is not isometric to a dual Banach space.
Another natural question is the following:

Open problem

If μ is an extreme point of $B_{\mathcal{F}(M)}$, is μ necessarily a molecule $\mu = m_{xy}$?

We have shown that this question has an affirmative answer in some particular cases.

Theorem 1 (GL – Procházka – Petitjean – Rueda Zoca, 2017)

Let M be a bounded separable metric space. Assume that there is a subspace of $\mathrm{lip}_0(M)$ (little-Lipschitz functions) which is predual of $\mathcal{F}(M)$ and $\delta(M)$ is weak*-closed. Then given $\mu \in B_{\mathcal{F}(M)}$ the following are equivalent:
(i) μ is an extreme point of $B_{\mathcal{F}(M)}$.
(ii) μ is an exposed point of $B_{\mathcal{F}(M)}$.
(iii) There are $x, y \in M$, $x \neq y$, such that $[x, y] = \{x, y\}$ and $\mu = m_{xy}$.

This applies for instance in the following cases:

- M compact countable.
- (M, d^α), $0 < \alpha < 1$ compact α-snowflaking of a metric space (M, d).
- M bounded uniformly discrete admitting a compact topology τ such that d is τ-lsc.

Theorem 1 has the following application to the norm-attainment of Lipschitz functions, which extends a result in [1].

Let Y be a Banach space and M be a metric space satisfying the hypotheses of Theorem 1. Then every Lipschitz function $f\colon M \to Y$ which attains its norm as an operator from $\mathcal{F}(M)$ to Y also attains its Lipschitz norm on a pair of points in M.

Strongly exposed points in $B_{\mathcal{F}(M)}$

Weaver proved that m_{xy} is a preserved extreme point of $B_{\mathcal{F}(M)}$ whenever there is $f \in \mathrm{Lip}_0(M)$ *peaking at* (x, y), that is, $\langle f, m_{xy}\rangle = 1$ and $\sup_{(u,v)\notin U} \langle f, m_{uv}\rangle < 1$ for every open subset U of $M^2 \setminus \Delta$ containing (x, y) and (y, x).
Indeed, peaking functions characterise strongly exposed points in $B_{\mathcal{F}(M)}$.

Theorem 2 (GL – Procházka – Rueda Zoca, 2017)

Let $x, y \in M$, $x \neq y$. The following assertions are equivalent:
(i) The molecule m_{xy} is a strongly exposed point of $B_{\mathcal{F}(M)}$.
(ii) There is $f \in \mathrm{Lip}_0(M)$ peaking at (x, y).
(iii) There is $\varepsilon > 0$ such that
$$d(x,z) + d(z,y) - d(x,y) > \varepsilon \min\{d(x,z), d(z,y)\} \quad \text{for all } z \in M \setminus \{x, y\}.$$

This result extends a characterisation of peaking functions in subsets of $\mathbb{R}$-trees due to Dalet, Kaufmann and Procházka [3].
It was shown in [4] that if M is compact then $\mathrm{Lip}_0(M)$ has the Daugavet property if and only if condition (iii) fails for every pair of distinct points in M. As a consequence, the following dichotomy holds:

Let M be a compact metric space. Then either $\mathrm{Lip}_0(M)$ has the Daugavet property (and so every slice of $B_{\mathcal{F}(M)}$ has diameter 2) or $B_{\mathcal{F}(M)}$ has a strongly exposed point.

Preserved extreme points in $B_{\mathcal{F}(M)}$

Aliaga and Guirao have recently proved a characterisation of preserved extreme points in $B_{\mathcal{F}(M)}$ in the spirit of Theorem 2. Namely, they prove in [5] that m_{xy} is a preserved extreme point of $B_{\mathcal{F}(M)}$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that
$$(1-\delta)(d(x,z) + d(z,y)) < d(x,y), z \in M \setminus \{x, y\} \Rightarrow \min\{d(x,z), d(y,z)\} < \varepsilon.$$
We have proved the following result:

Theorem 3 (GL – Procházka – Petitjean – Rueda Zoca, 2017)

Every preserved extreme point of $B_{\mathcal{F}(M)}$ is a denting point, and every weak-strongly exposed point of $B_{\mathcal{F}(M)}$ is a strongly exposed point.

Now, one may wonder if some more implications in the diagram hold in the particular case of $B_{\mathcal{F}(M)}$. However, we have shown:

- There is a compact metric space M with a denting point of $B_{\mathcal{F}(M)}$ which is not strongly exposed.
- There is a bounded uniformly discrete countable metric space M with an extreme point of $B_{\mathcal{F}(M)}$ which is not a preserved extreme point.

Finally, a curious consequence of Theorem 3 is the following:

The norm of $\mathrm{Lip}_0(M)$ is Gâteaux differentiable at f if and only if it is Fréchet differentiable at f.

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