On strongly norm attaining Lipschitz maps

Luis C. García-Lirola

Joint work with Bernardo Cascales, Rafael Chiclana, Miguel Martín and Abraham Rueda Zoca

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GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD

Cascales, B., R. Chiclana, L. García-Lirola, M. Martín, and A. Rueda Zoca. "On strongly norm attaining Lipschitz maps". In: *J. of Funct. Anal.* 277 (2019), pp. 1677–1717.

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for some $x, y \in M$. We denote $\mathrm{SNA}(M)$ the set of such maps.

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Problem (Godefroy, 2015)

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Equivalently, we want $\mathrm{SNA}(M)$ to be dense in the Banach space $(\mathrm{Lip}_0(M), \|\cdot\|_L)$.

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$\overline{\mathrm{SNA}(M)} \neq \mathrm{Lip}_0(M)$ *provided*

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Let $M \subset \mathbb{R}^n$ be a compact differential manifold, endowed with the metric inherited from $\mathbb{R}^n$. Do we have $\overline{\mathrm{SNA}(M)} \neq \mathrm{Lip}_0(M)$?

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- M is a length space if and only if $B_{\mathcal{F}(M)}$ does not have strongly exposed points. (Ivakhno-Kadets-Werner '09 + GL-Procházka-Rueda Zoca '18 + Avilés-Martínez Cervantes '19).

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Conjecture

M is a geodesic space if and only if $B_{\mathcal{F}(M)}$ does not have extreme points

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The above condition holds if M is compact and countable, or compact and Hölder.

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Theorem (GL-Petitjean-Procházka-Rueda Zoca, 2018)

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The space $\mathcal{F}(M)$ has the RNP in the following cases:

- M is uniformly discrete (Kalton, 2004)
- M is compact countable (Dalet, 2015)
- M is compact Hölder (Weaver, 1999)
- M is a closed subset of $\mathbb{R}$ with measure 0 (Godard, 2010)

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However, if $M = \mathbb{S}^1 \subset \mathbb{R}^2$ then $B_{\mathcal{F}(M)} = \overline{\mathrm{conv}}(\operatorname{strexp} B_{\mathcal{F}(M)})$!

Weak density

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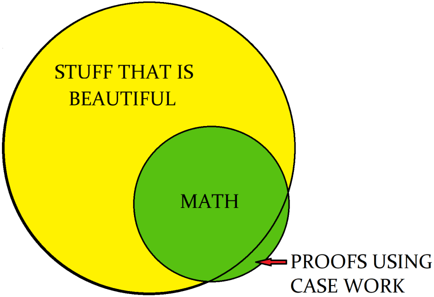
- This extends a result by Kadets-Martín-Soloviova, who proved that the same holds when M is a length space.
- The tool: $(f_n)_n \subset \mathrm{Lip}_0(M)$ bounded with pairwise disjoint supports $\Rightarrow$ $(f_n)_n$ is weakly null.



Thank you for your attention