On strongly norm attaining Lipschitz maps

Luis C. García-Lirola

Joint work with Bernardo Cascales, Rafael Chiclana, Miguel Martín and Abraham Rueda Zoca

Kent State University

Banach spaces and optimization: Conference on the occasion of Robert Deville's 60th birthday
June, 2019

f SéNeCa (+)
Agencia de Ciencia y Tecnología
Región de Murcia

GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD


HARVARD
John A. Paulson
School of Engineering
and Applied Sciences

Given a **complete** metric space (M, d) with a distinguished point $0 \in M$ and a Banach space Y, we consider the space

$$\mathrm{Lip}_0(M, Y) := \{f\colon M \to Y : f \text{ is Lipschitz}, f(0) = 0\}$$

which is a Banach space when equipped with the norm

$$\|f\|_L := \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y\right\}.$$

We say that f **strongly attains its norm** if

$$\|f\|_L = \frac{\|f(x) - f(y)\|}{d(x, y)}$$

for some $x, y \in M$. We denote $\mathrm{SNA}(M, Y)$ the set of such maps.

Problem (Godefroy, 2015)

What are the couples (M, Y) such that $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$?

Negative results

Theorem (Kadets-Martín-Solodiova, 2016)

Let $A \subset [0,1]$ be a closed set with empty interior and $\lambda(A) > 0$. Then, the Lipschitz function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(t) = \int_0^t \chi_A(s)ds$$

is not in $\overline{\mathrm{SNA}([0,1],\mathbb{R})}$.
Indeed, $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ whenever M is a geodesic space.

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca, 2018)

$\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ *provided*

- *M is a length space (i.e. $d(x,y)$ is the infimum of the length of curves joining x and y, for every x, y).*
- *$M \subset \mathbb{R}$ is compact and $\lambda(M) > 0$.*

Positive results

By the fundamental property of Lipschitz free spaces,

$$\mathrm{Lip}_0(M, Y) = \mathcal{L}(\mathcal{F}(M), Y)$$

Note that, if f strongly attains its norm at $x, y \in M$, then

$$\left\| f\left(\frac{\delta(x) - \delta(y)}{d(x, y)}\right) \right\| = \|f\|_L$$

Therefore

$$\mathrm{SNA}(M, Y) \subset \mathrm{NA}(\mathcal{F}(M), Y)$$

Theorem (Godefroy, 2015)

Assume M is a compact metric space and $\mathrm{lip}_0(M)^* = \mathcal{F}(M)$. Then $\mathrm{SNA}(M, Y) = \mathrm{NA}(\mathcal{F}(M), Y)$ for all Y. Moreover, if Y is finite-dimensional, then $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$.

Positive results

Recall that

$$\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y) \Longrightarrow \overline{\mathrm{NA}(\mathcal{F}(M), Y)} = \mathrm{Lip}_0(M, Y)$$

Therefore, it is reasonable to discuss the known sufficient conditions for a Banach space X to have $\overline{\mathrm{NA}(X, Y)} = \mathcal{L}(X, Y)$ for every Y:

- RNP
- Property α
- the existence of a norming set which is uniformly strongly exposed.

The space $\mathcal{F}(M)$ has the RNP in the following cases:

- M is uniformly discrete (Kalton, 2004)
- M is compact countable (Dalet, 2015)
- M is compact Hölder (Weaver, 1999)
- M is a closed subset of $\mathbb{R}$ with measure 0 (Godard, 2010)

Positive results

Theorem (GL-Petitjean-Procházka-Rueda Zoca, 2018)

If $\mathcal{F}(M)$ has the RNP, then $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for every Y.

Proof.

- Bourgain, 1977: the set of operators in $\mathcal{L}(\mathcal{F}(M), Y)$ which are absolutely strongly exposing is a G_δ dense.
- Every absolutely strongly exposing operator attains its norm at a strongly exposed point.
- Weaver, 1999: every strongly exposed point of $B_{\mathcal{F}(M)}$ is of the form $\frac{\delta(x)-\delta(y)}{d(x,y)}$.

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

There exists a compact metric space M such that $\mathcal{F}(M)$ fails the RNP and $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for every Y.

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. For simplicity, let's take $Y = \mathbb{R}$. Let

$$A = \{f \in \mathrm{Lip}_0(M, \mathbb{R}) : \sup_{d(x,y)<\varepsilon} \frac{f(x) - f(y)}{d(x, y)} < \|f\|_L \text{ for some } \varepsilon > 0\}$$

Clearly, A is open. Let us see that $\mathrm{SNA}(M, \mathbb{R}) \subset \overline{A}$.
Take $\varepsilon > 0$ and f such that $\frac{f(x)-f(y)}{d(x,y)} = \|f\|_L = 1$ for some $x, y \in M$.
By Aliaga-Pernecká, we may assume that $\frac{\delta(x)-\delta(y)}{d(x,y)} \in \mathrm{ext}(B_{\mathcal{F}(M)})$.
Now, by Aliaga-Guirao and GL-Petitjean-Procházka-Rueda Zoca,

$$\frac{\delta(x) - \delta(y)}{d(x, y)} \in \mathrm{ext}(B_{\mathcal{F}(M)^{**}}) \cap \mathcal{F}(M) = \mathrm{dent}(B_{\mathcal{F}(M)})$$

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M,Y)$ *contains an* **open** *dense subset.*

Proof. Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1-\beta$ and $\operatorname{diam}\{\mu \in B_{\mathcal{F}(M)} : g(\mu) > 1-\beta\} < \varepsilon$.
Take $h = f + \varepsilon g$. Then $\|f-h\| = \varepsilon$. We claim that $h \in A$.
Note that

$$\|h\|_L \geq 1 + \varepsilon \frac{g(x)-g(y)}{d(x,y)} > 1 + \varepsilon(1-\beta).$$

Assume that

$$\frac{h(u)-h(v)}{d(u,v)} > 1 + \varepsilon(1-\beta)$$

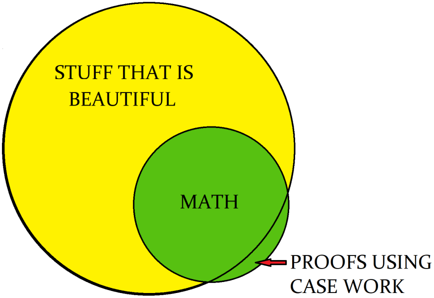
Then $1+\varepsilon(1-\beta) < 1 + \varepsilon\frac{g(u)-g(v)}{d(u,v)}$.
So, $g\left(\frac{\delta(u)-\delta(v)}{d(u,v)}\right) > 1-\beta$ and thus $\left\|\frac{\delta(u)-\delta(v)}{d(u,v)} - \frac{\delta(x)-\delta(y)}{d(x,y)}\right\| < \varepsilon$.
This implies that $d(u,v) \geq (1-2\varepsilon)d(x,y)$, that is, $h \in A$.

Weak density

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca)

$\mathrm{SNA}(M, \mathbb{R})$ *is weakly sequentially dense in* $\mathrm{Lip}_0(M, \mathbb{R})$

- This extends a result by Kadets-Martín-Soloviova, who proved that the same holds when M is a length space.
- The tool: $(f_n)_n \subset \mathrm{Lip}_0(M)$ bounded with pairwise disjoint supports $\Rightarrow$ $(f_n)_n$ is weakly null.



Aliaga, R. and A. Guirao. "On the preserved extremal estructure of Lipschitz-free spaces". In: *Studia Math.* 1.245 (2019), pp. 1–14.

Cascales, B., R. Chiclana, L. C. García-Lirola, M. Martín, A., and A. Rueda Zoca. "On strongly norm attaining Lipschitz maps". To appear in J. of Funct. Anal. 2018.

Chiclana, R., L. C. García-Lirola, M. Martín, and A. Rueda Zoca. "Examples and applications of strongly norm attaining Lipschitz maps". To appear. 2019.

García-Lirola, L. C., C. Petitjean, A. Procházka, and A. Rueda Zoca. "Extremal structure and duality of Lipschitz free spaces". In: *Mediterr. J. Math.* 15.2 (2018), Art. 69, 23.

Godefroy, G. "A survey on Lipschitz-free Banach spaces". In: *Comment. Math.* 55.2 (2015), pp. 89–118.

Kadets, V., M. Martín, and M. Soloviova. "Norm-attaining Lipschitz functionals". In: *Banach J. Math. Anal.* 10.3 (2016), pp. 621–637.

Thank you for your attention