On strongly norm attaining Lipschitz maps

Luis C. García-Lirola

Joint work with Bernardo Cascales, Rafael Chiclana, Miguel Martín and Abraham Rueda Zoca

Kent State University

Analysis Seminar
University of Illinois at Urbana-Champaign
December 17th, 2019

f SéNeCa (+)
Agencia de Ciencia y Tecnología
Región de Murcia

GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA, INDUSTRIA Y COMPETITIVIDAD

Cascales, B., R. Chiclana, L. García-Lirola, M. Martín, and A. Rueda Zoca. “On strongly norm attaining Lipschitz maps”. In: *J. of Funct. Anal.* 277 (2019), pp. 1677–1717.

Chiclana, R., L. C. García-Lirola, M. Martín, and A. Rueda Zoca. “Examples and applications of strongly norm attaining Lipschitz maps”. arXiv. 2019.

(M, d) complete metric space
Y real Banach space

(M, d) complete metric space
Y real Banach space

$$\mathrm{Lip}_0(M, Y) := \{f\colon M \to Y : f \text{ is Lipschitz} \qquad\qquad \}$$

$$\|f\|_L := \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y\right\}.$$

(M, d) complete metric space $\qquad 0 \in M$ distinguished point
Y real Banach space

$$\mathrm{Lip}_0(M, Y) := \{f \colon M \to Y : f \text{ is Lipschitz}, f(0) = 0\}$$

$$\|f\|_L := \sup\left\{ \frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y \right\}.$$

(M, d) complete metric space $\qquad 0 \in M$ distinguished point
Y real Banach space

$$\mathrm{Lip}_0(M, Y) := \{f \colon M \to Y : f \text{ is Lipschitz}, f(0) = 0\}$$

$$\|f\|_L := \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y\right\}.$$

$(\mathrm{Lip}_0(M, Y), \|\cdot\|_L)$ is a Banach space.

(M, d) complete metric space $\qquad 0 \in M$ distinguished point
Y real Banach space

$$\mathrm{Lip}_0(M, Y) := \{f\colon M \to Y : f \text{ is Lipschitz}, f(0) = 0\}$$

$$\|f\|_L := \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y\right\}.$$

$(\mathrm{Lip}_0(M, Y), \|\cdot\|_L)$ is a Banach space.
We say that f **strongly attains its norm** if

$$\|f\|_L = \frac{\|f(x) - f(y)\|}{d(x, y)}$$

for some $x, y \in M$. We denote $\mathrm{SNA}(M, Y)$ the set of such maps.

(M, d) complete metric space $\quad$ $0 \in M$ distinguished point
Y real Banach space

$$\mathrm{Lip}_0(M, Y) := \{f \colon M \to Y : f \text{ is Lipschitz}, f(0) = 0\}$$

$$\|f\|_L := \sup\left\{\frac{\|f(x) - f(y)\|}{d(x, y)} : x \neq y\right\}.$$

$(\mathrm{Lip}_0(M, Y), \|\cdot\|_L)$ is a Banach space.
We say that f **strongly attains its norm** if

$$\|f\|_L = \frac{\|f(x) - f(y)\|}{d(x, y)}$$

for some $x, y \in M$. We denote $\mathrm{SNA}(M, Y)$ the set of such maps.

Problem (Godefroy, 2015)

What are the couples (M, Y) such that $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$?

Negative results

Negative results

Theorem (Kadets-Martín-Soloviova, 2016)

If M is geodesic (in particular, $M = [0,1]$) then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$.

Negative results

Theorem (Kadets-Martín-Soloviova, 2016)

If M is geodesic (in particular, $M = [0,1]$) then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$.

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca, 2019)

$\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ *provided*

- *M is a length space (i.e. $d(x,y)$ is the infimum of the length of curves joining x and y, for every x,y).*

Negative results

Theorem (Kadets-Martín-Soloviova, 2016)

If M is geodesic (in particular, $M = [0,1]$) then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$.

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca, 2019)

$\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ *provided*

- *M is a length space (i.e. $d(x,y)$ is the infimum of the length of curves joining x and y, for every x, y).*
- *$M \subset \mathbb{R}$ is closed and $\lambda(M) > 0$.*

Negative results

Theorem (Kadets-Martín-Soloviova, 2016)

If M is geodesic (in particular, $M = [0,1]$) then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$.

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca, 2019)

$\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ *provided*

- *M is a length space (i.e. $d(x,y)$ is the infimum of the length of curves joining x and y, for every x, y).*
- *$M \subset \mathbb{R}$ is closed and $\lambda(M) > 0$.*
- *$M = \mathbb{S}^1 \subset \mathbb{R}^2$.*

Negative results

Theorem (Kadets-Martín-Soloviova, 2016)

If M is geodesic (in particular, $M = [0,1]$) then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$.

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca, 2019)

$\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ *provided*

- *M is a length space (i.e. $d(x,y)$ is the infimum of the length of curves joining x and y, for every x, y).*
- *$M \subset \mathbb{R}$ is closed and $\lambda(M) > 0$.*
- *$M = \mathbb{S}^1 \subset \mathbb{R}^2$.*

Is there an equivalent distance d' on $[0,1]$ such that $\overline{\mathrm{SNA}([0,1],d')} = \mathrm{Lip}_0([0,1],d')$?

Negative results

Theorem (Kadets-Martín-Soloviova, 2016)

If M is geodesic (in particular, $M = [0,1]$) then $\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$.

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca, 2019)

$\overline{\mathrm{SNA}(M,\mathbb{R})} \neq \mathrm{Lip}_0(M,\mathbb{R})$ *provided*

- *M is a length space (i.e. $d(x,y)$ is the infimum of the length of curves joining x and y, for every x,y).*
- *$M \subset \mathbb{R}$ is closed and $\lambda(M) > 0$.*
- *$M = \mathbb{S}^1 \subset \mathbb{R}^2$.*

Is there an equivalent distance d' on $[0,1]$ such that $\overline{\mathrm{SNA}([0,1],d')} = \mathrm{Lip}_0([0,1],d')$?

Let $M \subset \mathbb{R}^n$ be a compact differential manifold, endowed with the metric inherited from $\mathbb{R}^n$. Do we have $\overline{\mathrm{SNA}(M)} \neq \mathrm{Lip}_0(M)$?

Lipschitz-free spaces

Given $x \in M$, we denote $\delta(x) \in \mathrm{Lip}_0(M, \mathbb{R})^*$ the evaluation functional:

$$\langle \delta(x), f \rangle = f(x).$$

Lipschitz-free spaces

Given $x \in M$, we denote $\delta(x) \in \mathrm{Lip}_0(M,\mathbb{R})^*$ the evaluation functional:

$$\langle \delta(x), f \rangle = f(x).$$

The *Lipschitz-free* space over M is defined as

$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(x) : x \in M\} \subset \mathrm{Lip}_0(M,\mathbb{R})^*$$

Lipschitz-free spaces

Given $x \in M$, we denote $\delta(x) \in \mathrm{Lip}_0(M, \mathbb{R})^*$ the evaluation functional:

$$\langle \delta(x), f \rangle = f(x).$$

The *Lipschitz-free* space over M is defined as

$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(x) : x \in M\} \subset \mathrm{Lip}_0(M, \mathbb{R})^*$$

Example

- $\mathcal{F}(\mathbb{N}) = \ell_1$ $(\delta(n) \mapsto e_1 + \ldots + e_n)$.
- $\mathcal{F}([0,1]) = L_1([0,1])$ $(\delta(x) \mapsto \chi_{(0,x)})$.

Lipschitz-free spaces

Given $x \in M$, we denote $\delta(x) \in \mathrm{Lip}_0(M, \mathbb{R})^*$ the evaluation functional:

$$\langle \delta(x), f \rangle = f(x).$$

The *Lipschitz-free* space over M is defined as

$$\mathcal{F}(M) := \overline{\mathrm{span}}\{\delta(x) : x \in M\} \subset \mathrm{Lip}_0(M, \mathbb{R})^*$$

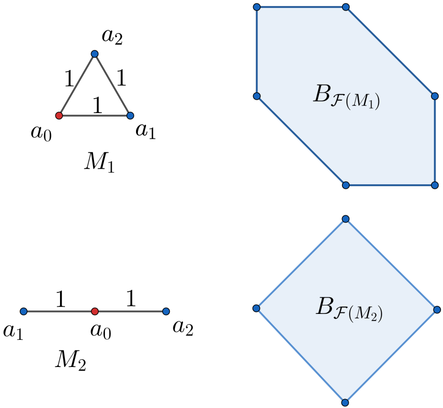
Example

- $\mathcal{F}(\mathbb{N}) = \ell_1$ $(\delta(n) \mapsto e_1 + \ldots + e_n)$.
- $\mathcal{F}([0,1]) = L_1([0,1])$ $(\delta(x) \mapsto \chi_{(0,x)})$.

Lipschitz-free spaces are also called *Arens-Eells spaces*, *transportation cost spaces* and *Wassertein 1 spaces*.

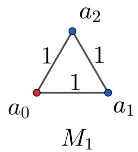
Extremal structure of Lipschitz-free spaces

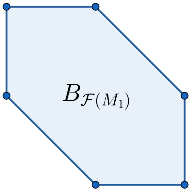
$$B_{\mathcal{F}(M)} = \overline{\text{conv}} \left\{ \frac{\delta(x) - \delta(y)}{d(x,y)} : i \neq j \right\}$$



Extremal structure of Lipschitz-free spaces

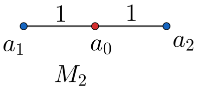
$$B_{\mathcal{F}(M)} = \overline{\text{conv}} \left\{ \frac{\delta(x) - \delta(y)}{d(x,y)} : i \neq j \right\}$$

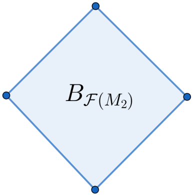




Theorem (Aliaga-Pernecká, 2019)

$\frac{\delta(x)-\delta(y)}{d(x,y)}$ *is an extreme point* $B_{\mathcal{F}(M)}$ *if and only if* $d(x,y) < d(x,z) + d(z,y)$ *for all* $z \in M \setminus \{x,y\}$





Propaganda: Volume product of metric spaces

In a preprint with M. Alexander, M. Fradelizi and A. Zvavitch, we introduce $\mathcal{P}(M) := \mathrm{vol}_n(B_{\mathcal{F}(M)}) \cdot \mathrm{vol}_n(B_{\mathrm{Lip}_0(M)})$ where M is a metric space with $n+1$ points.
$\mathcal{P}(M)$ measures, in some sense, how far is M from being a tree.

Propaganda: Volume product of metric spaces

In a preprint with M. Alexander, M. Fradelizi and A. Zvavitch, we introduce $\mathcal{P}(M) := \mathrm{vol}_n(B_{\mathcal{F}(M)}) \cdot \mathrm{vol}_n(B_{\mathrm{Lip}_0(M)})$ where M is a metric space with $n+1$ points.
$\mathcal{P}(M)$ measures, in some sense, how far is M from being a tree.

Let M be a finite metric space with minimal volume product such that $B_{\mathcal{F}(M)}$ is a simplicial polytope. Then M is a tree (and so $\mathcal{P}(M) = \mathcal{P}(B_1^n)$).

Let M be a finite metric space such that $\mathcal{P}(M)$ is maximal among the metric spaces with the same number of elements. Then

- $d(x,y) < d(x,z) + d(z,y)$ for all different points $x, y, z \in M$.
- $B_{\mathcal{F}(M)}$ is a simplicial polytope.

Positive results

Positive results

Note that, if f strongly attains its norm at $x, y \in M$, then

$$\|f\|_L = \frac{\|f(x) - f(y)\|}{d(x,y)} = \left\| \hat{f}\left(\frac{\delta(x) - \delta(y)}{d(x,y)}\right) \right\|,$$

that is, $\hat{f}$ attains its operator norm. Therefore

$$\mathrm{SNA}(M, Y) \subset \mathrm{NA}(\mathcal{F}(M), Y)$$

Positive results

Note that, if f strongly attains its norm at $x, y \in M$, then

$$\|f\|_L = \frac{\|f(x) - f(y)\|}{d(x,y)} = \left\| \hat{f}\left(\frac{\delta(x) - \delta(y)}{d(x,y)} \right) \right\|,$$

that is, $\hat{f}$ attains its operator norm. Therefore

$$\mathrm{SNA}(M, Y) \subset \mathrm{NA}(\mathcal{F}(M), Y)$$

Note that $\mathrm{SNA}([0,1], \mathbb{R}) \neq \mathrm{NA}(\mathcal{F}([0,1]), \mathbb{R})$.

Positive results

Note that, if f strongly attains its norm at $x, y \in M$, then

$$\|f\|_L = \frac{\|f(x) - f(y)\|}{d(x,y)} = \left\| \hat{f}\left(\frac{\delta(x) - \delta(y)}{d(x,y)} \right) \right\|,$$

that is, $\hat{f}$ attains its operator norm. Therefore

$$\mathrm{SNA}(M, Y) \subset \mathrm{NA}(\mathcal{F}(M), Y)$$

Note that $\mathrm{SNA}([0,1], \mathbb{R}) \neq \mathrm{NA}(\mathcal{F}([0,1]), \mathbb{R})$.

Theorem (Godefroy, 2015)

Assume M is a compact metric space and $\mathrm{lip}_0(M)^* = \mathcal{F}(M)$. Then $\mathrm{SNA}(M, Y) = \mathrm{NA}(\mathcal{F}(M), Y)$ for all Y. Moreover, if Y is finite-dimensional, then $\overline{\mathrm{SNA}(M, Y)} = \mathrm{Lip}_0(M, Y)$.

Positive results

Theorem (GL-Petitjean-Procházka-Rueda Zoca, 2018)

If $\mathcal{F}(M)$ *has the RNP, then* $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ *for every* Y.

Positive results

Theorem (GL-Petitjean-Procházka-Rueda Zoca, 2018)

If $\mathcal{F}(M)$ has the RNP, then $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ *for every Y.*

Proof.

- Bourgain, 1977: the set of operators in $\mathcal{L}(\mathcal{F}(M),Y)$ which are absolutely strongly exposing is a G_δ dense.

Positive results

Theorem (GL-Petitjean-Procházka-Rueda Zoca, 2018)

If $\mathcal{F}(M)$ has the RNP, then $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ *for every Y.*

Proof.

- Bourgain, 1977: the set of operators in $\mathcal{L}(\mathcal{F}(M), Y)$ which are absolutely strongly exposing is a G_δ dense.
- Every absolutely strongly exposing operator attains its norm at a strongly exposed point.

Positive results

Theorem (GL-Petitjean-Procházka-Rueda Zoca, 2018)

If $\mathcal{F}(M)$ has the RNP, then $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for every Y.

Proof.

- Bourgain, 1977: the set of operators in $\mathcal{L}(\mathcal{F}(M), Y)$ which are absolutely strongly exposing is a G_δ dense.
- Every absolutely strongly exposing operator attains its norm at a strongly exposed point.
- Weaver, 1995: every strongly exposed point of $B_{\mathcal{F}(M)}$ is of the form $\frac{\delta(x)-\delta(y)}{d(x,y)}$.

Positive results

Theorem (GL-Petitjean-Procházka-Rueda Zoca, 2018)

If $\mathcal{F}(M)$ has the RNP, then $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ *for every* Y.

Proof.

- Bourgain, 1977: the set of operators in $\mathcal{L}(\mathcal{F}(M),Y)$ which are absolutely strongly exposing is a G_δ dense.
- Every absolutely strongly exposing operator attains its norm at a strongly exposed point.
- Weaver, 1995: every strongly exposed point of $B_{\mathcal{F}(M)}$ is of the form $\frac{\delta(x)-\delta(y)}{d(x,y)}$.

The space $\mathcal{F}(M)$ has the RNP in the following cases:

- M is uniformly discrete (Kalton, 2004)
- M is compact countable (Dalet, 2015)
- M is compact Hölder (Weaver, 1999)
- M is a closed subset of $\mathbb{R}$ with measure 0 (Godard, 2010)

Question (Godefroy, 2015)

Assume that M is compact and $\mathrm{SNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Does it follow that $\mathcal{F}(M) = \mathrm{lip}_0(M)^*$?

Question (Godefroy, 2015)

Assume that M is compact and $\mathrm{SNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Does it follow that $\mathcal{F}(M) = \mathrm{lip}_0(M)^*$?

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

There exists a compact metric space M such that $\mathcal{F}(M)$ fails the RNP and $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for every Y.

Question (Godefroy, 2015)

Assume that M is compact and $\mathrm{SNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Does it follow that $\mathcal{F}(M) = \mathrm{lip}_0(M)^*$?

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

There exists a compact metric space M such that $\mathcal{F}(M)$ fails the RNP and $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for every Y.

Idea: $\mathcal{F}(M)$ has property α, that is, there exist $\rho > 0$, $\{\mu_\gamma\}_{\gamma\in\Gamma} \subset S_{\mathcal{F}(M)}$, and $\{f_\gamma\}_{\gamma\in\Gamma} \subset S_{\mathrm{Lip}_0(M)}$ such that

- $|\langle \hat{f}_\gamma, \mu_\gamma\rangle| = 1$ for all γ,
- $|\langle \hat{f}_\gamma, \mu_{\gamma'}\rangle| \leq \rho$ if $\gamma' \neq \gamma$,
- $B_{\mathcal{F}(M)} = \overline{\mathrm{aconv}}(\{\mu_\gamma\}_{\gamma\in\Gamma})$.

By a result of Schachermayer (1983) we get that the set of operators in $\mathcal{L}(\mathcal{F}(M), Y)$ attaining their norm on $\overline{\{\mu_\gamma\}_{\gamma\in\Gamma}}$ is dense.

Question (Godefroy, 2015)

Assume that M is compact and $\mathrm{SNA}(M,\mathbb{R})$ is dense in $\mathrm{Lip}_0(M,\mathbb{R})$. Does it follow that $\mathcal{F}(M) = \mathrm{lip}_0(M)^*$?

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

There exists a compact metric space M such that $\mathcal{F}(M)$ fails the RNP and $\overline{\mathrm{SNA}(M,Y)} = \mathrm{Lip}_0(M,Y)$ for every Y.

Idea: $\mathcal{F}(M)$ has property α, that is, there exist $\rho > 0$, $\{\mu_\gamma\}_{\gamma\in\Gamma} \subset S_{\mathcal{F}(M)}$, and $\{f_\gamma\}_{\gamma\in\Gamma} \subset S_{\mathrm{Lip}_0(M)}$ such that

- $|\langle \hat{f}_\gamma, \mu_\gamma\rangle| = 1$ for all γ,
- $|\langle \hat{f}_\gamma, \mu_{\gamma'}\rangle| \leq \rho$ if $\gamma' \neq \gamma$,
- $B_{\mathcal{F}(M)} = \overline{\mathrm{aconv}}(\{\mu_\gamma\}_{\gamma\in\Gamma})$.

By a result of Schachermayer (1983) we get that the set of operators in $\mathcal{L}(\mathcal{F}(M), Y)$ attaining their norm on $\overline{\{\mu_\gamma\}_{\gamma\in\Gamma}}$ is dense. Each μ_γ is strongly exposed by $\hat{f}_\gamma$, so

$$\overline{\{\mu_\gamma\}_{\gamma\in\Gamma}} \subset \overline{\left\{\frac{\delta(x)-\delta(y)}{d(x,y)} : x \neq y\right\}} = \left\{\frac{\delta(x)-\delta(y)}{d(x,y)} : x \neq y\right\}$$

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M *is compact and* $\mathcal{F}(M)$ *has the RNP. Then* $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. For simplicity, let's take $Y = \mathbb{R}$. Let

$$A = \{f \in \mathrm{Lip}_0(M, \mathbb{R}) : \sup_{d(x,y)<\varepsilon} \frac{f(x) - f(y)}{d(x, y)} < \|f\|_L \text{ for some } \varepsilon > 0\}$$

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. For simplicity, let's take $Y = \mathbb{R}$. Let

$$A = \{f \in \mathrm{Lip}_0(M, \mathbb{R}) : \sup_{d(x,y)<\varepsilon} \frac{f(x) - f(y)}{d(x,y)} < \|f\|_L \text{ for some } \varepsilon > 0\}$$

Clearly, A is open and $A \subset \mathrm{SNA}(M, \mathbb{R})$. Let us see that $\mathrm{SNA}(M, \mathbb{R}) \subset \overline{A}$.

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. For simplicity, let's take $Y = \mathbb{R}$. Let

$$A = \{f \in \mathrm{Lip}_0(M, \mathbb{R}) : \sup_{d(x,y)<\varepsilon} \frac{f(x) - f(y)}{d(x, y)} < \|f\|_L \text{ for some } \varepsilon > 0\}$$

Clearly, A is open and $A \subset \mathrm{SNA}(M, \mathbb{R})$. Let us see that $\mathrm{SNA}(M, \mathbb{R}) \subset \overline{A}$. Take $\varepsilon > 0$ and f such that $\frac{f(x)-f(y)}{d(x,y)} = \|f\|_L = 1$ for some $x, y \in M$.

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. For simplicity, let's take $Y = \mathbb{R}$. Let

$$A = \{f \in \mathrm{Lip}_0(M, \mathbb{R}) : \sup_{d(x,y)<\varepsilon} \frac{f(x) - f(y)}{d(x, y)} < \|f\|_L \text{ for some } \varepsilon > 0\}$$

Clearly, A is open and $A \subset \mathrm{SNA}(M, \mathbb{R})$. Let us see that $\mathrm{SNA}(M, \mathbb{R}) \subset \overline{A}$. Take $\varepsilon > 0$ and f such that $\frac{f(x)-f(y)}{d(x,y)} = \|f\|_L = 1$ for some $x, y \in M$. By Aliaga-Pernecká, we may assume that $\frac{\delta(x)-\delta(y)}{d(x,y)} \in \mathrm{ext}(B_{\mathcal{F}(M)})$.

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. For simplicity, let's take $Y = \mathbb{R}$. Let

$$A = \{f \in \mathrm{Lip}_0(M, \mathbb{R}) : \sup_{d(x,y)<\varepsilon} \frac{f(x) - f(y)}{d(x, y)} < \|f\|_L \text{ for some } \varepsilon > 0\}$$

Clearly, A is open and $A \subset \mathrm{SNA}(M, \mathbb{R})$. Let us see that $\mathrm{SNA}(M, \mathbb{R}) \subset \overline{A}$.
Take $\varepsilon > 0$ and f such that $\frac{f(x)-f(y)}{d(x,y)} = \|f\|_L = 1$ for some $x, y \in M$.
By Aliaga-Pernecká, we may assume that $\frac{\delta(x)-\delta(y)}{d(x,y)} \in \mathrm{ext}(B_{\mathcal{F}(M)})$.
Now, by Aliaga-Guirao and GL-Petitjean-Procházka-Rueda Zoca,

$$\frac{\delta(x) - \delta(y)}{d(x, y)} \in \mathrm{ext}(B_{\mathcal{F}(M)^{**}}) \cap \mathcal{F}(M) = \mathrm{dent}(B_{\mathcal{F}(M)})$$

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1 - \beta$ and $\operatorname{diam}\{\mu \in B_{\mathcal{F}(M)} : \hat{g}(\mu) > 1 - \beta\} < \varepsilon$.

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1-\beta$ and $\mathrm{diam}\{\mu \in B_{\mathcal{F}(M)} : \hat{g}(\mu) > 1-\beta\} < \varepsilon$.
Take $h = f + \varepsilon g$. Then $\|f-h\| = \varepsilon$. We claim that $h \in A$.

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1 - \beta$ and $\mathrm{diam}\{\mu \in B_{\mathcal{F}(M)} : \hat{g}(\mu) > 1 - \beta\} < \varepsilon$.
Take $h = f + \varepsilon g$. Then $\|f - h\| = \varepsilon$. We claim that $h \in A$.
Note that

$$\|h\|_L \geq 1 + \varepsilon \frac{g(x) - g(y)}{d(x, y)} > 1 + \varepsilon(1 - \beta).$$

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1-\beta$ and $\mathrm{diam}\{\mu \in B_{\mathcal{F}(M)} : \hat{g}(\mu) > 1-\beta\} < \varepsilon$.
Take $h = f + \varepsilon g$. Then $\|f - h\| = \varepsilon$. We claim that $h \in A$.
Note that

$$\|h\|_L \geq 1 + \varepsilon \frac{g(x)-g(y)}{d(x,y)} > 1 + \varepsilon(1-\beta).$$

Assume that

$$\frac{h(u)-h(v)}{d(u,v)} > 1 + \varepsilon(1-\beta)$$

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1-\beta$ and $\mathrm{diam}\{\mu \in B_{\mathcal{F}(M)} : \hat{g}(\mu) > 1-\beta\} < \varepsilon$.
Take $h = f + \varepsilon g$. Then $\|f - h\| = \varepsilon$. We claim that $h \in A$.
Note that

$$\|h\|_L \geq 1 + \varepsilon \frac{g(x)-g(y)}{d(x,y)} > 1 + \varepsilon(1-\beta).$$

Assume that

$$\frac{h(u)-h(v)}{d(u,v)} > 1 + \varepsilon(1-\beta)$$

Then $1 + \varepsilon(1-\beta) < 1 + \varepsilon \frac{g(u)-g(v)}{d(u,v)}$.

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1-\beta$ and $\operatorname{diam}\{\mu \in B_{\mathcal{F}(M)} : \hat{g}(\mu) > 1-\beta\} < \varepsilon$.
Take $h = f + \varepsilon g$. Then $\|f-h\| = \varepsilon$. We claim that $h \in A$.
Note that

$$\|h\|_L \geq 1 + \varepsilon \frac{g(x)-g(y)}{d(x,y)} > 1 + \varepsilon(1-\beta).$$

Assume that

$$\frac{h(u)-h(v)}{d(u,v)} > 1 + \varepsilon(1-\beta)$$

Then $1 + \varepsilon(1-\beta) < 1 + \varepsilon \frac{g(u)-g(v)}{d(u,v)}$.
So, $\hat{g}\left(\frac{\delta(u)-\delta(v)}{d(u,v)}\right) > 1-\beta$ and thus $\left\| \frac{\delta(u)-\delta(v)}{d(u,v)} - \frac{\delta(x)-\delta(y)}{d(x,y)} \right\| < \varepsilon.$

Theorem (Chiclana-GL-Martín-Rueda Zoca, 2019)

Assume that M is compact and $\mathcal{F}(M)$ has the RNP. Then $\mathrm{SNA}(M, Y)$ *contains an* **open** *dense subset.*

Proof. Therefore, there is $g \in S_{\mathrm{Lip}_0(M)}$ and $\beta > 0$ such that $\frac{g(x)-g(y)}{d(x,y)} > 1-\beta$ and $\mathrm{diam}\{\mu \in B_{\mathcal{F}(M)} : \hat{g}(\mu) > 1-\beta\} < \varepsilon$.
Take $h = f + \varepsilon g$. Then $\|f-h\| = \varepsilon$. We claim that $h \in A$.
Note that

$$\|h\|_L \geq 1 + \varepsilon \frac{g(x)-g(y)}{d(x,y)} > 1 + \varepsilon(1-\beta).$$

Assume that

$$\frac{h(u)-h(v)}{d(u,v)} > 1 + \varepsilon(1-\beta)$$

Then $1 + \varepsilon(1-\beta) < 1 + \varepsilon \frac{g(u)-g(v)}{d(u,v)}$.
So, $\hat{g}\left(\frac{\delta(u)-\delta(v)}{d(u,v)}\right) > 1-\beta$ and thus $\left\|\frac{\delta(u)-\delta(v)}{d(u,v)} - \frac{\delta(x)-\delta(y)}{d(x,y)}\right\| < \varepsilon$.
This implies that $d(u,v) \geq (1-2\varepsilon)d(x,y)$, that is, $h \in A$.

Weak density

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca)

$\mathsf{SNA}(M, \mathbb{R})$ *is weakly sequentially dense in* $\mathrm{Lip}_0(M, \mathbb{R})$

Weak density

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca)

SNA$(M, \mathbb{R})$ *is weakly sequentially dense in* $\mathrm{Lip}_0(M, \mathbb{R})$

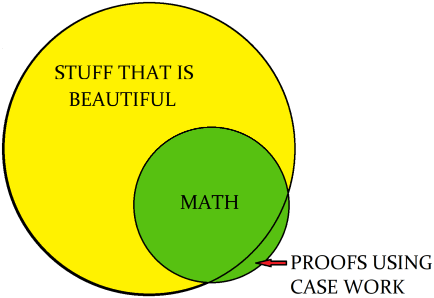
- This extends a result by Kadets-Martín-Soloviova, who proved that the same holds when M is a length space.

Weak density

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca)

$\mathrm{SNA}(M, \mathbb{R})$ *is weakly sequentially dense in* $\mathrm{Lip}_0(M, \mathbb{R})$

- This extends a result by Kadets-Martín-Soloviova, who proved that the same holds when M is a length space.
- The tool: $(f_n)_n \subset \mathrm{Lip}_0(M)$ bounded with pairwise disjoint supports $\Rightarrow (f_n)_n$ is weakly null.

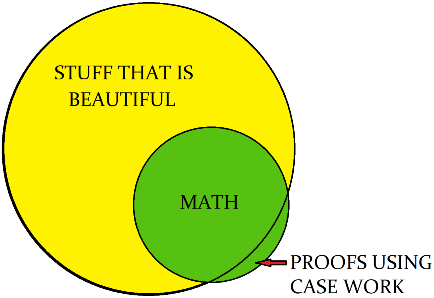


Weak density

Theorem (Cascales-Chiclana-GL-Martín-Rueda Zoca)

$\mathrm{SNA}(M,\mathbb{R})$ *is weakly sequentially dense in* $\mathrm{Lip}_0(M,\mathbb{R})$

- This extends a result by Kadets-Martín-Soloviova, who proved that the same holds when M is a length space.
- The tool: $(f_n)_n \subset \mathrm{Lip}_0(M)$ bounded with pairwise disjoint supports $\Rightarrow (f_n)_n$ is weakly null.



Thank you for your attention