The weak maximizing property and asymptotic geometry of Banach spaces

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Entangling Non-commutative Functional Analysis and Geometry of Banach Spaces
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This is a joint work with **Colin Petitjean** *(Université Gustave Eiffel a.k.a. Université Paris-Est Marne-la-Vallée)*

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- If X is reflexive, then every $x^*\colon X \to \mathbb{R}$ attains its norm (Hahn-Banach)
- If every $x^*\colon X \to \mathbb{R}$ attains its norm, then X is reflexive (James)
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Consider

$$\begin{aligned} T\colon \ell_2 &\to c_0 \\ x &\mapsto (\frac{1}{2}x_1, \frac{2}{3}x_2, \ldots, \ldots \frac{n}{n+1}x_n, \ldots) \end{aligned}$$

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- The "abundance" of norm-attaining operators depends on X and Y (Lindenstrauss, Bourgain, Huff, Schachermayer, Godun, Troyanski, Partington, Gowers, Zizler, Godefroy, Acosta, Martín, Kadets, Aron,...)

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Definition (Aron-García-Pellegrino-Teixeira, 2020)

(X, Y) has the **Weak Maximizing Property (WMP)** if for every $T \colon X \to Y$, the existence of a non-weakly null maximizing sequence for T implies that T attains its norm.

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Theorem (Pellegrino-Teixeira, 2009)

Let $1 < p < \infty$, $1 \leq q < \infty$. Then (ℓ_p, ℓ_q) has the WMP.

WMP and compact operators

Theorem (Aron-García-Pellegrino-Teixeira, 2020)

Assume (X, Y) has the WMP. Let $T, K \colon X \to Y$, K compact. If

$$\|T\| < \|T + K\|$$

Then $T + K$ attains its norm.

This extends a result of J. Kover (2005) for the case X, Y Hilbert spaces.

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Proof.

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- $\|T + K\| = \lim_n \|(T + K)x_n\| = \lim_n \|Tx_n\| \leq \|T\|$

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As a consequence,

$$(X, Y) \text{ WMP} \Rightarrow X \text{ is reflexive}$$

(if X is not reflexive, then there is $K\colon X \to Y$ not attaining the norm. Take $T = 0$, then $\|T\| < \|T + K\|$)

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(New) short proof:

- **Key fact:** If $(u_n) \subset \ell_p$ converges weakly to 0, then

$$\lim_{n\to\infty} \|x + u_n\|^p = \|x\|^p + \lim_{n\to\infty} \|u_n\|^p$$

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- We may assume that $x_n \xrightarrow{w} x \neq 0$, and that the sequences $(\|x - x_n\|)_n$, $(\|Tx_n\|)_n$ and $(\|Tx - Tx_n\|)_n$ are convergent.
$$\begin{aligned} 1 = \|T\| = \lim_{n\to\infty} \|Tx_n\| &= (\|Tx\|^q + \lim_{n\to\infty} \|Tx - Tx_n\|^q)^{1/q} \\ &\leq (\|Tx\|^q + \lim_{n\to\infty} \|x - x_n\|^q)^{1/q} \leq (\|Tx\|^p + \lim_{n\to\infty} \|x - x_n\|^p)^{1/p} \\ &= (\|Tx\|^p + 1 - \|x\|^p)^{1/p}, \qquad \text{and so } \|x\| \leq \|Tx\| \end{aligned}$$

A more general result

Theorem (G.L.-Petitjean, 2020)

Assume X is reflexive, and

- *for all $t > 0$, $\overline{\delta}_X(t) \geq \overline{\rho}_Y(t)$*
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$\overline{\delta}_X$ is the **modulus of asymptotic uniform convexity** of X:

$$\overline{\delta}_X(t) = \inf_{x \in S_X} \sup_{\substack{Z \subset X \\ \dim(X/Z) < \infty}} \inf_{z \in S_Z} \|x + tz\| - 1$$

$\overline{\rho}_Y$ is the **modulus of asymptotic uniform smoothness** of Y:

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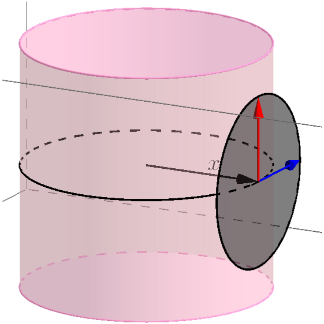

Asymptotic moduli of classical spaces

$\max\{0, t-1\} = \overline{\rho}_{c_0}(t) \leq \overline{\rho}_X(t), \qquad \overline{\delta}_X(t) \leq \overline{\delta}_{\ell_1}(t) \leq t, \qquad \overline{\delta}_X(t) \leq \overline{\rho}_X(t)$

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- Let $1 \leq p < \infty$. If $X = (\sum_{n=1}^{\infty} E_n)_p$, where $\dim(E_n) < \infty$, then $\overline{\delta}_X(t) = \overline{\rho}_X(t) = (1+t^p)^{1/p} - 1$.
- $\overline{\delta}_{\ell_p(\Gamma)}(t) = \overline{\rho}_{\ell_p(\Gamma)}(t) = (1+t^p)^{1/p} - 1$.
- $\overline{\delta}_{c_0(\Gamma)}(t) = \overline{\rho}_{c_0(\Gamma)}(t) = \max\{0, t-1\}$.
- Let J be the James space. Then $\overline{\delta}_J(t) = (1+t^2)^{1/2} - 1$.
- The Lorentz sequence space $d(w,p)$ satisfies $\overline{\rho}_{d(w,p)}(t) = (1+t^p)^{1/p} - 1$.
- For a reflexive Orlicz space ℓ_φ,

$$(1+t^{q_\varphi})^{1/q_\varphi} - 1 \leq \overline{\delta}_{\ell_\varphi}(t), \quad \overline{\rho}_{\ell_\varphi}(t) \leq (1+t^{p_\varphi})^{1/p_\varphi} - 1,$$
$$p_\varphi = \sup\{p > 0 : u^{-p}\varphi(u) \text{ is non-decreasing for all } 0 < u \leq \varphi^{-1}(1)\}$$
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If $\overline{\rho}_X(t) < \overline{\delta}_Y(t)$ for some $t > 0$, then every $T\colon X \to Y$ is compact (Johnson-Lindenstrauss-Preiss-Schechtman, 2002).

We have

$$\liminf_n \|x + x_n\| \geq \|x\| \left(1 + \overline{\delta}_X \left(\frac{\liminf_n \|x_n\|}{\|x\|}\right)\right) \quad \forall x \neq 0, x_n \xrightarrow{w} 0 \text{ in } X.$$

Also,

$$\limsup_n \|y + y_n\| \leq \|y\| \left(1 + \overline{\rho}_Y \left(\frac{\limsup_n \|y_n\|}{\|y\|}\right)\right) \quad \forall y \neq 0, y_n \xrightarrow{w} 0 \text{ in } Y.$$

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Lemma

$$N(s,t) = \begin{cases} |s| + |s|\overline{\rho}_Y(|t|/|s|) & \text{if } s \neq 0, \\ |t| & \text{if } s = 0. \end{cases}$$

defines an absolute norm in $\mathbb{R}^2$ that satisfies

$$\limsup_n \|y + y_n\| \leq N(\|y\|, \limsup_n \|y_n\|) \quad \forall y, y_n \in Y, y_n \xrightarrow{w} 0$$

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- Let $T\colon X \to Y, \|T\| = 1$ and (x_n) be maximizing for T and not weak-null.
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- We may assume $x_n \xrightarrow{w} x \neq 0$, and the sequences $(\|x_n - x\|)_n, (\|Tx_n - Tx\|)_n$ are convergent.
- Since $Tx_n - Tx \xrightarrow{w} 0$,

$$1 = \|T\| = \lim_n \|Tx_n\| = \lim_n \|Tx + Tx_n - Tx\| \leq N(\|Tx\|, \lim_n \|Tx_n - Tx\|)$$

$$1 \leq N(\|Tx\|, \lim_n \|Tx_n - Tx\|) \leq N(\|Tx\|, \lim_n \|x_n - x\|) \leq N(\|x\|, \lim_n \|x_n - x\|)$$

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Since $x \neq 0$,

$$N(\|x\|, \lim_n \|x_n - x\|) = \|x\| + \|x\|\overline{\rho}_Y \left(\frac{\lim_n \|x_n - x\|}{\|x\|} \right)$$

$$\leq \|x\| + \|x\|\overline{\delta}_X \left(\frac{\lim_n \|x_n - x\|}{\|x\|} \right) \leq \lim_n \|x + x_n - x\| = 1.$$

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Thus, all the previous inequalities are in fact equalities:

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If $\|Tx\| < \|x\|$, we get $\lim_n \|x_n - x\| = 1$, and so $\overline{\delta}_X(\frac{1}{\|x\|}) = \overline{\rho}_Y(\frac{1}{\|x\|}) = \frac{1}{\|x\|} - 1$, a contradiction.

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Thus, all the previous inequalities are in fact equalities:

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If $\|Tx\| < \|x\|$, we get $\lim_n \|x_n - x\| = 1$, and so $\overline{\delta}_X(\frac{1}{\|x\|}) = \overline{\rho}_Y(\frac{1}{\|x\|}) = \frac{1}{\|x\|} - 1$, a contradiction.

Corollary

If $\overline{\delta}_X(t) > \overline{\rho}_Y(t)$ for any $t > 0$, then every non weakly null maximizing sequence for T has a norm-convergent subsequence.

Example

Consider $T : \ell_2 \to \ell_\infty$, $Tx = \Big(x_1\ ,\ x_1 + \Big(1 - \frac{1}{2}\Big)x_2\ ,\ \dots\ ,\ x_1 + \Big(1 - \frac{1}{n}\Big)x_n\ ,\ \dots\Big)$

- $\|T\| = \sqrt{2}$. Indeed,

$$\begin{aligned}\|Tx\|_\infty &= \sup\big\{|x_1 + (1 - 1/n)x_n| : n \geq 1\big\}\\ &\leq \sup\big\{(|x_1|^2 + |x_n|^2)^{1/2}(1 + (1 - 1/n)^2)^{1/2} : n \geq 2\big\} \leq \sqrt{2}\,\|x\|_2\,.\end{aligned}$$

- T does not attain its norm: if $\|Tx\|_\infty = \sqrt{2}$ for some x with $\|x\|_2 = 1$, then $\limsup_n(|x_1|^2 + |x_n|^2)^{1/2} = 1$. Thus $|x_1| = 1$, which implies that $x = \pm e_1$, a contradiction.
- $\Big\|T\Big(\frac{1}{\sqrt{2}}(e_1 + e_n)\Big)\Big\| \to \sqrt{2}$, so $\|T\| = \sqrt{2}$ the sequence $\big(\frac{1}{\sqrt{2}}(e_1 + e_n)\big)_n$ is maximizing and not weakly null.

Thus, (ℓ_2, ℓ_∞) fails the WMP.

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What about the pair (L_p, L_q)?

Partial negative answers by S. Dantas, M. Jung, G. Martínez Cervantes, and J. Rodríguez Abellán (still open for the case $p < 2$, $p < q$).

The non-reflexive case

Recall: if (X, Y) has the WMP, then X is reflexive.

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- $(X, \mathbb{R})$ has the WMP for any reflexive space X. However, there is X (isomorphic to c_0) such that $(X^*, \mathbb{R})$ fails the W*MP.
- (X, ℓ_1) has the WMP for any reflexive space X. However, there is X (isomorphic to c_0) such that (X^*, ℓ_1) fails the w*-to-w*MP.
- (ℓ_1, Y) has the W*MP for any Y.

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Consider the James space $\mathcal{J}_p$.

$$\|x\|_{\mathcal{J}_p} = \sup\Big\{\big(\sum_{i=1}^{k-1}|x(p_{i+1}) - x(p_i)|^p\big)^{1/p} \,:\, 1 \leq p_1 < p_2 < \ldots < p_k\Big\}.$$

Theorem (G.L.-Petitjean, 2020)

- *If* $1 < p \leq q < \infty$ *then there is an equivalent norm* $\tilde{\mathcal{J}}_q = (\mathcal{J}_q, \|\cdot\|)$ *such that* $(\mathcal{J}_p, \widetilde{\mathcal{J}_q})$ *has the w*-to-w*MP.*
- *If* $1 < q < p < \infty$ *then every* $T : \mathcal{J}_p \to \mathcal{J}_q$ *is compact. In particular, the pair* $(\mathcal{J}_p, \mathcal{J}_q)$ *has the w*-to-w*MP.*
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Thank you for your attention!