Convexidad uniforme y cotipos en espacios de Banach

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- Kadec, 1956. If $\sum_{n=1}^{\infty} x_n$ converges unconditionally in X then

$$\sum_{n=1}^{\infty} \delta_X(\|x_n\|) < +\infty$$

where δ_X is the *modulus of uniform convexity* of X.

Uniform convexity

The **modulus of uniform convexity of** X is given by

$$\delta_X(t) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1, \|x-y\| \geq t\}$$

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- A space is superreflexive if and only if it has a uniformly convex renorming (Enflo), moreover one can get $\delta_X(t) \geq ct^p$ for some $p \geq 2$, $c > 0$ (Pisier).

Generalized cotypes

$$\text{Kadec:} \quad \sup_{\varepsilon_k=\pm 1} \sum_{k=1}^{n} \varepsilon_k x_k \Rightarrow \sum_{k=1}^{n} \delta_X(\|x_k\|) < +\infty$$

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Definition (Figiel)

A function ϕ satisfying

$$\exists C > 0 : \int_0^1 \| \sum_{k=1}^{n} r_k(t) x_k \| dt \leq 1 \Rightarrow \sum_{k=1}^{n} \phi(\|x_k\|) \leq C$$

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Clearly, X has Rademacher cotype p if and only if $t \mapsto t^p$ is a generalized cotype.

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YES if X is a UMD space.

- X has the *unconditional martingale difference property* (UMD) if for some (eq. for every) $1 < p < \infty$, for every X-valued martingale (f_n) bounded in $L_p(X)$, the series $\sum_{n=1}^{\infty}(f_{n+1} - f_n)$ converges unconditionally.

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- L_p, $1 < p < \infty$ and reflexive Orlicz spaces are UMD.
- UMD $\Rightarrow$ superreflexive, but not conversely (first examples by Pisier and Bourgain; $L_p(L_q(L_p(L_q(\ldots))))$ (Qiu, 2012)).

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Some ideas about the proof...

- The first task was to find an expression for the generalized cotype almost equivalent to the homogeneity of the classic cotype.

$$\sum_{k=1}^{n} \phi(\|x_k\|) \leq \Phi\left(\int_0^1 \|\sum_{k=1}^{n} r_k(t)x_k\|\, dt\right)$$

  where Φ can be taken convex and such that $\Phi(t^{1/q})$ is concave for some $q \geq 2$ thanks to the work of Figiel.

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- From that we get that

$$\sum_{n=1}^{\infty} \int_0^1 \phi(\|f_n(s)\|)\, ds < +\infty$$

  whenever the series $\sum_{n=1}^{\infty} f_n$ is unconditionally convergent in $L^q(X)$.

Best modulus of uniform convexity

Consider the order $\phi \preceq \psi$ if there is a constant $c > 0$ such that $\phi(t) \leq c\,\psi(t)$ for all $t \in (0, 1]$. If $\phi \preceq \psi$ and $\psi \preceq \phi$, then we say that ϕ and ψ are *equivalent*.

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Let X be a superreflexive Banach space. There exists a positive decreasing submultiplicative function $\mathfrak{N}_X(t)$ defined on $(0,1]$ such that $\mathfrak{N}_X(t)^{-1}$ is the supremum, up to equivalence, with respect to the order $\preceq$ of the set

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Indeed the argument also works for $L_p(X)$. As a byproduct we obtain:

Corollary (GL-Raja, 2021)

Let $1 < r, p < \infty$. Assume that $L_r(X)$ has an AUC renorming with power type p. Then X has a UC renorming of power type p.

This answers a question from C.L. García-W.B. Johnson (2003).

Figiel, T. “Uniformly convex norms on Banach lattices”. In: *Studia Math.* 68.3 (1980), pp. 215–247.

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Thank you for your attention!