Lipschitz operators which preserve injectivity

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Ongoing joint work with Colin Petitjean and Tony Procházka

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Lipschitz-free spaces

Let (M, d) be a complete metric space, $0 \in M$.

$$\begin{aligned} \mathrm{Lip}_0(M) &= \{f\colon M \to \mathbb{R}, f(0) = 0\} \\ \|f\|_L &= \sup\left\{\frac{f(x) - f(y)}{d(x,y)} : x \neq y\right\} \end{aligned}$$

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$$\delta \colon M \to \mathrm{Lip}_0(M)^*$$
$$x \mapsto \delta(x) : \langle f, \delta(x) \rangle = f(x)$$

The **Lipschitz-free space** (Kadec (1985), Pestov (1986), Godefroy-Kalton (2003)) $\mathcal{F}(M)$ over M (a.k.a. *Arens-Eells space*, *transportation cost space*) is defined as

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$M \overset{\delta}{\hookrightarrow} \mathcal{F}(M)$ is an isometric embedding and $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$.

Lipschitz-free spaces

Fundamental property: for every Lipschitz function from M to N with $f(0) = 0$ there is a unique bounded linear operator $\hat{f} \colon \mathcal{F}(M) \to \mathcal{F}(N)$ such that $\|\hat{f}\| = \|f\|_L$ and the following diagram commutes:

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There is a lot of recent work relating properties of M and of $\mathcal{F}(M)$...

Theorem (Aliaga-Gartland-Petitjean-Procházka, 2021)

The following are equivalent:

i) *$\mathcal{F}(M)$ has the Radon-Nikodym Property.*

ii) *$\mathcal{F}(M)$ has the Schur property.*

iii) *M is purely 1-unrectifiable.*

M purely 1-unrectifiable means that it contains no *curve fragment* ($\gamma\colon K \to M$ bi-Lipschitz embedding with $K \subset \mathbb{R}$ compact with $\lambda(K) > 0$).

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If you don't like free spaces... consider $\hat{f}^* = C_f \colon \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M)$

f injective $\Rightarrow$ C_f has weak*-dense range?

Counterexamples

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That also shows that

$$f \text{ injective} + \text{locally bi-Lipschitz} \not\Rightarrow \hat{f} \text{ injective}$$

A positive example

Every biLipschitz map!

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Let $0 < \alpha < 1$ and $Id\colon ([0,1], |\cdot|^\alpha) \to ([0,1], |\cdot|)$ (which is Lipschitz and injective). Then $\hat{Id}$ is injective.

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Proof.

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$$\hat{Id}^* = C_{Id}\colon \mathrm{Lip}_0([0,1], |\cdot|) \to \mathrm{Lip}_0([0,1], |\cdot|^\alpha)$$

- Easy exercise: $T\colon X \to Y$ injective if and only if $\overline{T^*(Y^*)}^{w^*} = X^*$.

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- Claim': $\exists c > 0\ \forall x \neq y \in [0,1]\ \exists g \in \operatorname{Lip}_0([0,1], |\cdot|)$ such that

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- Fix $x \neq y \in [0,1]$. Consider $\omega_n(t) := \min\{t^\alpha, nt\} \to t^\alpha$ and

$$g_n(z) := \omega_n(|z-y|) - \omega_n(|y|)) \quad \forall z \in [0,1]$$

- One can check $\|g_n\|_{\operatorname{Lip}_0([0,1],|\cdot|)} \leq n$, $\|g_n \circ Id\|_{\operatorname{Lip}_0([0,1],|\cdot|^\alpha)} \leq 1$ and

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Metric spaces where injectivity is always preserved

Definition. We say M is **Lip-lin injective** (or *OTOTOTO*) if for every N and every Lipschitz injective function $f\colon M \to N$ with $f(0) = 0$, it follows that $\hat{f}\colon \mathcal{F}(M) \to \mathcal{F}(N)$ is injective.

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Proof. Let $f\colon L \to N$. Extend f to $\tilde{f}\colon M \to \ell_\infty(N)$. Let $\rho = \min\{1, d\}$ and

$$\begin{aligned} g\colon M &\to \ell_\infty(N) \times \mathcal{F}(M, \rho) \\ x &\mapsto (\tilde{f}(x), \rho(L, x)\delta(x)) \end{aligned}$$

Then g is Lipschitz, injective, and $\ker(\hat{f}) \subset \ker(\hat{g})$.

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However, the converse statement does not hold.

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The following spaces are Lip-lin injective:

a) M is uniformly discrete.

b) M is compact and scattered.

c) M is compact and $\mathcal{H}^1(M) = 0$.

d) M is compact and there is $\rho > 1$ such that for every $\varepsilon > 0$, M can be covered by finitely many balls $B(x_i, r)$ of radius $r \leq \varepsilon$ such that the balls $B(x_i, \rho r)$ are pairwise disjoint (for instance, the Cantor dust).

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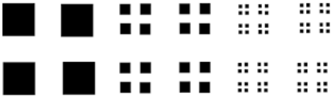

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Assume M is compact. The following are equivalent:

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Idea for a) and b): look at the support of elements in $\mathcal{F}(M)$.

Lipschitz functions preserving supports

Aliaga-Pernecká-Petitjean-Procházka, 2020

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- If there are $r, \rho > 0$ such that $f|_{B(x,r)}$ is bi-Lipschitz and $f^{-1}(B(f(x), \rho)) \subset B(x, r)$, then $f(x) \in \mathrm{supp}\hat{f}(\mu)$ whenever $x \in \mathrm{supp}\mu$.

Lipschitz functions preserving supports

Aliaga-Pernecká-Petitjean-Procházka, 2020

Given $0 \neq \mu \in \mathcal{F}(M)$, its support $\mathrm{supp}\mu$ is the intersection of closed subsets $0 \in L \subset M$ with $\mu \in \mathcal{F}(L)$.

If $\mu = \sum_{n=1}^{\infty} a_n \delta(x_n)$, $(a_n) \in \ell_1$ and $a_n \neq 0$, then $\mathrm{supp}\mu = \overline{\{x_n : n \in \mathbb{N}\}}$.

We say that $f : M \to N$ **preserve supports** if

$$\mathrm{supp}\hat{f}(\mu) = \overline{f(\mathrm{supp}\mu)} \quad \forall \mu \in \mathcal{F}(M)$$

Let $f : M \to N$ be a Lipschitz injective map with $f(0) = 0$.

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- If M is uniformly discrete then f preserves supports.
- If there are $r, \rho > 0$ such that $f|_{B(x,r)}$ is bi-Lipschitz and $f^{-1}(B(f(x), \rho)) \subset B(x, r)$, then $f(x) \in \mathrm{supp}\hat{f}(\mu)$ whenever $x \in \mathrm{supp}\mu$.
- If f is closed and x is in the closure of isolated points of $\mathrm{supp}\mu$, then $f(x) \in \mathrm{supp}\mu$.

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Thank you for your attention!