Linearization of holomorphic Lipschitz mappings

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Joint work with R. Aron, V. Dimant and M. Maestre

Universidad de Zaragoza

ÆSY TO DEFINE, HARD TO ANALYSE
Besançon
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Outline

1) Holomorphic functions and the holomorphic free space
2) The holomorphic Lipschitz free space
3) Approximation properties
4) Extension of holomorphic Lipschitz functions

Holomorphic functions

- X, Y = complex Banach spaces
- $U \subset X$ open subset
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A function $f \colon U \to Y$ is said to be **holomorphic** at $x_0 \in U$ if it is Fréchet differentiable at x_0: there is $df(x_0) \in \mathcal{L}(X, Y)$ with

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - df(x_0)(h)}{\|h\|} = 0$$

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Equivalently, there is a sequence $(P_k f(x_0))_k$ of continuous k-homogeneous polynomials such that

$$f(x) = \sum_{k=0}^{\infty} P_k f(x_0)(x - x_0)$$

uniformly in some neighbourhood of x_0.

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$f: U \to Y$ is holomorphic $\Leftrightarrow$ $y^* \circ f$ is holomorphic $\forall y^* \in Y^*$.

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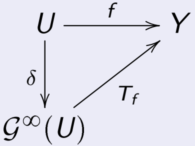
There is a Banach space $\mathcal{G}^\infty(U)$ and a holomorphic bounded map $\delta\colon U \to \mathcal{G}^\infty(U)$ satisfying the linearization property of the diagram.



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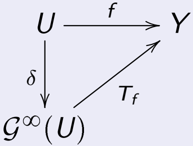

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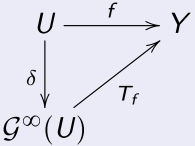

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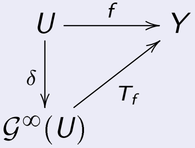

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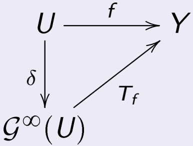

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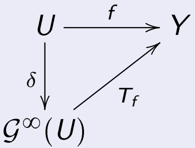

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There is a recent survey by García Sánchez - De Hevia - Tradacete.

The holomorphic Lipschitz free space

$$\begin{aligned}\mathcal{H}L_0(B_X, Y) &= \{f\colon B_X \to Y : f \text{ is holomorphic and Lipschitz}, f(0) = 0\} \\ &= \{f \in \mathcal{H}(B_X, Y) : df \in \mathcal{H}^\infty(B_X, \mathcal{L}(X, Y)), f(0) = 0\}\end{aligned}$$

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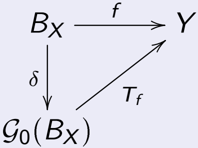

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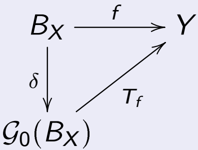

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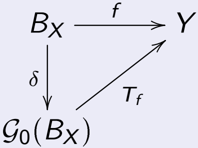

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The unit ball of $\mathcal{G}_0(B_X)$

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- About the extreme points...
  - The unit ball of $\mathcal{G}_0(\mathbb{D}) \equiv \mathcal{G}^\infty(\mathbb{D})$ does not have extreme points.
  - $e_{x,y}$ is not a extreme point.

Relation with $\mathcal{F}(B_X)$ and $\mathcal{G}^\infty(B_X)$

- The map

$$\mathcal{H}L_0(B_X) \to \mathrm{Lip}_0(B_X)$$
$$f \mapsto f$$

is an into isometry. It is the adjoint of the quotient operator

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- The map

$$\mathcal{H}L_0(B_X) \to \mathcal{H}^\infty(B_X, X^*)$$
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$$\Psi\colon \mathcal{G}^\infty(B_X)\widehat{\otimes}_\pi X \to \mathcal{G}_0(B_X)$$
$$\delta(x) \otimes y \mapsto e_{x,y}$$

Approximation properties

- X has the Approximation Property (AP) if the identity $I\colon X \to X$ can be approximated by finite-rank operators in $\mathcal{L}(X, X)$ uniformly on compact sets.
- If the operators can be taken with norm $\leqslant \lambda$ then we say that X has the λ-Bounded Approximation Property (λ-BAP).
- If $\lambda = 1$ then we say that X has the Metric Approximation Property (MAP).

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MAP for $\mathcal{G}_0(B_X)$

First we show:

a) Given $f \in \mathcal{H}L(B_X, Y)$ with $\|f\|_L \leqslant 1$, there are polinomials $P_n \colon X \to Y$ with $\|P_n|_{B_X}\|_L \leqslant 1$ and $P_n(x) \to f(x)$ for all $x \in B_X$.

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b) Assume that X has the MAP with $T_\alpha \to I$ pointwise. For each polynomial $P \colon X \to Y$ there are finite-type polinomials $P_\alpha = P \circ T_\alpha$ with $\|P_\alpha|_{B_X}\|_L \leqslant \|P|_{B_X}\|_L$ and $P_\alpha(x) \to P(x)$ for all $x \in B_X$.

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b) Assume that X has the MAP with $T_\alpha \to I$ pointwise. For each polynomial $P \colon X \to Y$ there are finite-type polinomials $P_\alpha = P \circ T_\alpha$ with $\|P_\alpha|_{B_X}\|_L \leqslant \|P|_{B_X}\|_L$ and $P_\alpha(x) \to P(x)$ for all $x \in B_X$.

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Then T_{P_α} has finite rank, $\|T_{P_\alpha}\| \leqslant 1$ and

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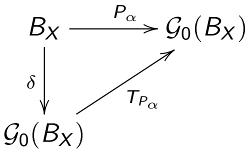

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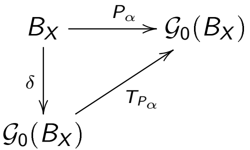

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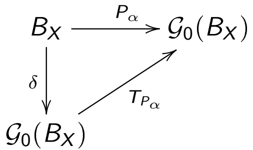

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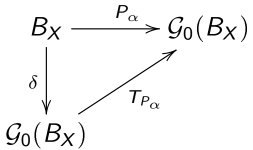

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To get the corresponding result for $\mathcal{G}_0(B_X)$, first we identify the compact sets:

If $K \subset \mathcal{G}_0(B_X)$ is norm-compact, then $K \subset \overline{\text{aconv}}(\{\alpha_j m_{x_j y_j}\})$ for some $(\alpha_j) \in c_0$ and $(x_j, y_j) \subset (B_X \times B_X) \backslash \Delta$.

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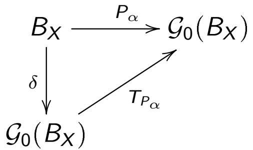

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Let τ_γ be the locally convex topology on $\mathcal{H}L_0(B_X, Y)$ generated by the seminorms $p(f) = \sup_j \alpha_j \frac{\|f(x_j) - f(y_j)\|}{\|x_j - y_j\|}$ where $(\alpha_j) \in c_0$, $(x_j, y_j) \subset (B_X \times B_X)\backslash\Delta$ and $\alpha_j > 0$. Then we have a homeomorphism:

$$\begin{array}{ccc} (\mathcal{H}L_0(B_X, Y), \tau_\gamma) & \to & (\mathcal{L}(\mathcal{G}_0(B_X), Y), \tau_0) \\ f & \mapsto & T_f \end{array}$$

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Our goal is to find a net (P_α), $P_\alpha\colon X \to \mathcal{G}_0(B_X)$ of finite-type polynomials with $P_\alpha \overset{\tau_\gamma}{\to} \delta$.

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Hence, we just need to show:

Lemma

τ_0 and τ_γ coincide on $\mathcal{P}(^m X, Y)$.

Given one of the seminorms p in the definition of τ_γ, we'll see there are $C > 0$ and a compact K such that $p(P) \leqslant C \sup_{x \in K} \|P(x)\| \ \forall P \in \mathcal{P}(^m X, Y)$.

$$p(P) \quad = \quad \sup_j \alpha_j \frac{\|P(x_j) - P(y_j)\|}{\|x_j - y_j\|} = \sup_j \frac{\|P(\alpha_j^{1/m} x_j) - P(\alpha_j^{1/m} y_j)\|}{\|x_j - y_j\|}$$

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for some compact sets K_1, K_2. Now,

$$\check{P}(a^k, b^{m-k}) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_m P\left(\left(\sum_{i=1}^k \varepsilon_i \right) a + \left(\sum_{i=k+1}^m \varepsilon_i \right) b \right).$$

So there is a compact set K such that $p(P) \leqslant \frac{2^m - 1}{m!} \sup_{x \in K} \|P(x)\|$.

Extension of holomorphic Lipschitz functions

If $X \subset Y$, we have a map

$$\rho\colon \mathcal{G}_0(B_X) \to \mathcal{G}_0(B_Y)$$
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where $\langle f, \widehat{\varphi}\rangle = \langle f|_{B_X}, \varphi\rangle$.

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Still, there are some cases where we know that ρ is an isometry. For instance, if X is 1-complemented in Y.

When $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$?

Theorem (Aron-Dimant-GL-Maestre, 2023)

Let $X \subset Y$. If there is an isometric extension operator $s \colon X^* \to Y^*$ and X is symmetrically regular, then $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$

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This is the case, for instance, if $Y = X^{**}$ (then $s\colon X^* \to X^{***}$ is just the inclusion map).

The Aron-Berner extension

Let $P\colon X \to \mathbb{C}$ be an n-homogeneous polynomial. Then $P(x) = A(x, \ldots, x)$ for a multilinear symmetric map $A\colon X \times \cdots \times X \to \mathbb{C}$. Define

$$\overline{A}(x_1^{**}, \ldots x_n^{**}) = \lim_{\alpha_1} \cdots \lim_{\alpha_n} A(x_{\alpha_1}, \ldots x_{\alpha_n})$$

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A similar argument works for the vector-valued case and

$$AB\colon \mathcal{H}^\infty(B_X, Y) \to \mathcal{H}^\infty(B_{X^{**}}, Y^{**})$$
$$f \mapsto \tilde{f}$$

is an isometry (Davie-Gamelin, 1989).

Symmetric regularity

X is **Arens regular** if

$$\lim_{\alpha_1}\lim_{\alpha_2} A(x_{\alpha_1}, x_{\alpha_2}) = \lim_{\alpha_2}\lim_{\alpha_1} A(x_{\alpha_1}, x_{\alpha_2}) \qquad (*)$$

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$$\lim_{\alpha_1}\lim_{\alpha_2} A(x_{\alpha_1}, x_{\alpha_2}) = \lim_{\alpha_2}\lim_{\alpha_1} A(x_{\alpha_1}, x_{\alpha_2}) \qquad (*)$$

for all $x_{\alpha_i} \xrightarrow{w^*} x_i^{**}$ and for all continuous bilinear maps $A\colon X \times X \to \mathbb{C}$.
Equivalently, every $T\colon X \to X^*$ is weakly compact.
X is **symmetrically regular** if $(*)$ holds for all symmetric A.

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- (Leung, 1996) There is a symmetrically regular space that is not Arens regular.

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We also get that if X and Y are symmetrically regular and $X^* \equiv Y^*$, then $\mathcal{H}L_0(B_X) \equiv \mathcal{H}L_0(B_Y)$. This is based on a result by Lassalle-Zalduendo, 2000.

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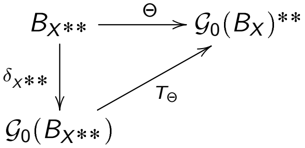

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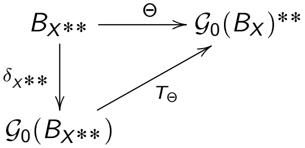

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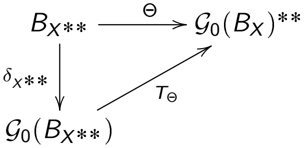

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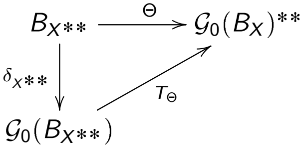

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Assume X^{**} has BAP. Is $\mathcal{H}^\infty(B_{X^{**}}) \overset{c}{\hookrightarrow} \mathcal{H}^\infty(B_X)^{**}$?

Unique norm-preserving extensions

Lemma (Godefroy, 1981)

Let $x \in S_{X*}$. *TFAE:*

(i) x^* *has a unique norm preserving extension to a functional on* X^{**}.

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Assume X^{**} *has the MAP. For* $P \in \mathcal{P}({}^n X)$ *with* $\|P\| = 1$*, TFAE:*

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Thank you for your attention!