Linearization of holomorphic Lipschitz mappings

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Joint work with R. Aron, V. Dimant and M. Maestre

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Outline

1) Lipschitz functions and free spaces
2) Holomorphic functions and free spaces
3) Lipschitz holomorphic functions and free spaces
4) Extension of Lipschitz and holomorphic functions

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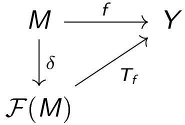

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Example

- $\mathcal{F}(\mathbb{N}) \equiv \ell_1$
- $\mathcal{F}(\mathbb{R}) \equiv L_1(\mathbb{R})$

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$\mathcal{F}(\mathbb{R}^2)$ is not isomorphic to a subspace of $L_1 = \mathcal{F}(\mathbb{R})$.

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Open question: Are $\mathcal{F}(\mathbb{R}^2)$ and $\mathcal{F}(\mathbb{R}^3)$ isomorphic?

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Theorem (GL-Procházka-Rueda Zoca, 2018)

TFAE:

i) $\mathcal{F}(M)$ has the Daugavet property.

ii) $\mathrm{Lip}_0(M)$ has the Daugavet property.

iii) M is a length space.

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$f: U \to Y$ is holomorphic $\Leftrightarrow y^* \circ f$ is holomorphic $\forall y^* \in Y^*$.

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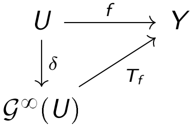

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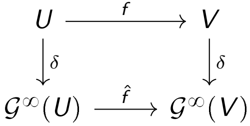

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In particular $\mathcal{H}L_0(B_X) = \mathcal{G}_0(B_X)^*$.

The holomorphic Lipschitz free space

- **Linearization property**

$$\begin{array}{ccc} B_X & \xrightarrow{\ f\ } & Y \\ \Big\downarrow{\scriptstyle \delta} & \nearrow_{T_f} & \\ \mathcal{G}_0(B_X) & & \end{array} \qquad\qquad \begin{array}{ccc} B_X & \xrightarrow{\ f\ } & B_Y \\ \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \delta} \\ \mathcal{G}_0(B_X) & \xrightarrow{\ \hat{f}\ } & \mathcal{G}_0(B_X) \end{array}$$

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Relation with $\mathcal{F}(B_X)$ and $\mathcal{G}^\infty(B_X)$

- The map

$$\mathcal{H}L_0(B_X) \to \mathrm{Lip}_0(B_X)$$
$$f \mapsto f$$

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where $e_{x,y}(f) = df(x)(y)$. We have $\mathcal{G}^\infty(\mathbb{D}) \equiv \mathcal{G}_0(\mathbb{D})$.

Approximation properties

- X has the Approximation Property (AP) if the identity $I\colon X \to X$ can be approximated by finite-rank operators in $\mathcal{L}(X, X)$ uniformly on compact sets.
- If the operators can be taken with norm $\leqslant \lambda$ then we say that X has the λ-Bounded Approximation Property (λ-BAP).
- If $\lambda = 1$ then we say that X has the Metric Approximation Property (MAP).

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Theorem (Mujica, 1991)

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MAP for $\mathcal{G}_0(B_X)$

First we show:

a) Given $f \in \mathcal{H}L(B_X, Y)$ with $\|f\|_L \leqslant 1$, there are polinomials $P_n \colon X \to Y$ with $\|P_n|_{B_X}\|_L \leqslant 1$ and $P_n(x) \to f(x)$ for all $x \in B_X$.

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b) Assume that X has the MAP with $T_\alpha \to I$ pointwise. For each polynomial $P \colon X \to Y$ there are finite-type polinomials $P_\alpha = P \circ T_\alpha$ with $\|P_\alpha|_{B_X}\|_L \leqslant \|P|_{B_X}\|_L$ and $P_\alpha(x) \to P(x)$ for all $x \in B_X$.

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Then T_{P_α} has finite rank, $\|T_{P_\alpha}\| \leqslant 1$ and

$$T_{P_\alpha}(\delta(x)) = P_\alpha(x) \to \delta(x) = Id(\delta(x))$$

so $T_{P_\alpha} \to Id$ pointwise on $\text{span}(\delta(x))$.

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so $T_{P_\alpha} \to Id$ pointwise on $\operatorname{span}(\delta(x))$. Since (T_{P_α}) is bounded, the same holds for the closure.

AP for $\mathcal{G}_0(B_X)$

$$\begin{array}{ccc} B_X & \xrightarrow{P_\alpha} & \mathcal{G}_0(B_X) \\ {\scriptstyle\delta}\downarrow & \nearrow{\scriptstyle T_{P_\alpha}} & \\ \mathcal{G}_0(B_X) & & \end{array}$$

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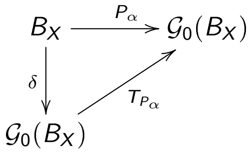

$$P_\alpha(x) \to \delta(x)\,\forall x \in B_X \not\Rightarrow$$
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In our case, we get:

Lemma

Let τ_γ be the locally convex topology on $\mathcal{H}L_0(B_X, Y)$ generated by the seminorms $p(f) = \sup_j \alpha_j \frac{\|f(x_j)-f(y_j)\|}{\|x_j-y_j\|}$ where $(\alpha_j) \in c_0$, $(x_j, y_j) \subset (B_X \times B_X)\backslash\Delta$ and $\alpha_j > 0$.
Then we have a homeomorphism:

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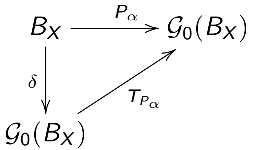

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Lemma

τ_K and τ_γ coincide on $\mathcal{P}(^m X, Y)$ for each $m \in \mathbb{N}$.

Extension of Lipschitz maps

Given metric spaces with $N \subset M$, we have a map

$$\rho\colon \mathcal{F}(N) \to \mathcal{F}(M)$$
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For $K = \mathbb{R}$, that extension exists! Just take

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So $\mathcal{F}(N) \subset \mathcal{F}(M)$ in a canonical way (isometrically for $K = \mathbb{R}$ and isomorphically in the complex case).

Extension of holomorphic Lipschitz functions

Given complex Banach spaces $X \subset Y$, we have a map

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There is no McShane's extension theorem!

Aron-Berner, 1978

Let $P\colon \ell_2 \to \mathbb{C}$ given by $P(x) = \sum_{n=1}^{\infty} x_n^2$ and consider an embedding $\ell_2 \hookrightarrow \ell_\infty$.
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Still, there are some cases where we know that ρ is an isometry. For instance, if X is 1-complemented in Y.

When $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$?

Theorem (Aron-Dimant-GL-Maestre, 2023)

Let $X \subset Y$. If there is an isometric extension operator $s\colon X^* \to Y^*$ and X is symmetrically regular, then $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$

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Recall that

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This is the case, for instance, if $Y = X^{**}$ (then $s\colon X^* \to X^{***}$ is just the inclusion map).

When $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$?

Let $P\colon X \to \mathbb{C}$ be an n-homogeneous polynomial. Then $P(x) = A(x, \ldots, x)$ for a multilinear symmetric map $A\colon X \times \cdots \times X \to \mathbb{C}$. Define

$$\overline{A}(x_1^{**}, \ldots x_n^{**}) = \lim_{\alpha_1} \cdots \lim_{\alpha_n} A(x_{\alpha_1}, \ldots x_{\alpha_n}) \quad (*)$$

where $x_{\alpha_i} \xrightarrow{w^*} x_i^{**}$.

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A similar argument works for the vector-valued case and

$$AB\colon \mathcal{H}^\infty(B_X, Y) \to \mathcal{H}^\infty(B_{X^{**}}, Y^{**})$$
$$f \mapsto \tilde{f}$$

is an isometry (Davie-Gamelin, 1989).

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Now, given $f \in \mathcal{H}^\infty(B_X)$, we can define $\tilde{f} \in \mathcal{H}^\infty(B_{X^{**}})$ extending f.
A similar argument works for the vector-valued case and

$$AB\colon \mathcal{H}^\infty(B_X, Y) \to \mathcal{H}^\infty(B_{X^{**}}, Y^{**})$$
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is an isometry (Davie-Gamelin, 1989).
X **symmetrically regular** means that we may interchange the limits in $(*)$.

When $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$?

Let $P\colon X \to \mathbb{C}$ be an n-homogeneous polynomial. Then $P(x) = A(x, \ldots, x)$ for a multilinear symmetric map $A\colon X \times \cdots \times X \to \mathbb{C}$. Define

$$\overline{A}(x_1^{**}, \ldots x_n^{**}) = \lim_{\alpha_1} \cdots \lim_{\alpha_n} A(x_{\alpha_1}, \ldots x_{\alpha_n}) \quad (*)$$

where $x_{\alpha_i} \stackrel{w^*}{\to} x_i^{**}$.
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This is the case of $X = c_0, C(K), \mathcal{H}^\infty(\mathbb{D})$ but fails for $X = \ell_1$.

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If X is symmetrically regular, then we get $d\widetilde{f} = \widetilde{df}$ for all $f \in \mathcal{H}L_0(B_X)$

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We also get that if X and Y are symmetrically regular and $X^* \equiv Y^*$, then $\mathcal{H}L_0(B_X) \equiv \mathcal{H}L_0(B_Y)$. This is based on a result by Lassalle-Zalduendo, 2000.

Thank you for your attention!