Norm-attainment in projective tensor products

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Joint works with S. Dantas, J. Guerrero-Viu, M. Jung, A. Rueda Zoca

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- S. Dantas, L. C. García-Lirola, M. Jung, A. Rueda Zoca. On norm-attainment in (symmetric) tensor products. *Quaest. Math.* 46 (2023), no. 2, 393–409.
- L. C. García-Lirola, J. Guerrero-Viu, A. Rueda Zoca. Projective tensor products where every element is norm-attaining. *Banach J. Math. Anal.*, 19 (2025), article 19.

Outline

- Tensor products
- Projective tensor products where every tensor attains its norm
- Denseness of norm-attaining tensors

Tensor products

$X, Y, Z =$ vector spaces

A map $A\colon X \times Y \to Z$ is **bilinear** if

- $A(\alpha x + \beta x', y) = \alpha A(x, y) + \beta A(x', y)$
- $A(x, \alpha y + \beta y') = \alpha A(x, y) + \beta A(x, y')$

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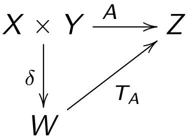
We would like to **linearize** bilinear maps. That is, to find a vector space W and a linear embedding $\delta\colon X \times Y \to W$ such that for any $A \in B(X \times Y, Z)$ there is a linear map $T_A\colon W \to Z$ such that this diagram commutes:



Tensor products

Given $x \in X$ and $y \in Y$, consider the linear functional

$$x \otimes y \colon B(X \times Y, \mathbb{K}) \to \mathbb{K}$$
$$A \mapsto A(x, y)$$

The **tensor product** $X \otimes Y$ is defined as

$$X \otimes Y = \text{span}\{x \otimes y : x \in X,\ y \in Y\} \subset B(X \times Y, \mathbb{K})^{\#}$$

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A **tensor** $u \in X \otimes Y$ has the form $u = \sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in X$ and $y_i \in Y$. This representation is **NOT UNIQUE**.
Indeed,

- $(\alpha x + \beta x') \otimes y = \alpha x \otimes y + \beta x' \otimes y$.
- $x \otimes (\alpha y + \beta y') = \alpha x \otimes y + \beta x \otimes y'$

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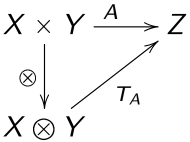
Define $\otimes\colon X \times Y \to X \otimes Y$ by $\otimes(x, y) = x \otimes y$.



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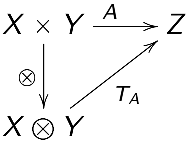

Given $A \in B(X \times Y, Z)$, define

$$T_A\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} A(x_i, y_i).$$

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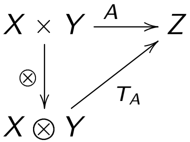

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Conversely, if $T\colon X \otimes Y \to Z$ is linear, define $A \in B(X \times Y, Z)$ by

$$A(x, y) = T(x \otimes y).$$

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Given $u \in X \otimes Y$, we define the **projective norm** as follows

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Important example: $L_1(\mu) \widehat{\otimes}_\pi X = L_1(\mu, X)$

The dual of $X\widehat{\otimes}_\pi Y$

A bilinear map $A\colon X \times Y \to Z$ is said to be **bounded** if there is $C > 0$ such that

$$\|A(x,y)\| \leqslant C\|x\|\|y\|, \quad \forall x \in X, y \in Y.$$

$\mathcal{B}(X \times Y, Z)$ denotes the Banach space of bounded bilinear mappings with the norm $\|A\| = \sup\{\|A(x,y)\| : \|x\| \leqslant 1, \|y\| \leqslant 1\}$.

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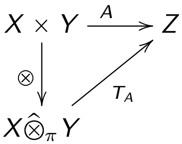

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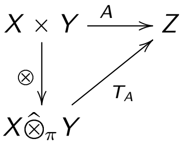

$$\mathcal{B}(X \times Y, Z) = \mathcal{L}(X\widehat{\otimes}_\pi Y, Z)$$

$$\mathcal{B}(X \times Y, \mathbb{R}) = \mathcal{L}(X, Y^*) = (X\widehat{\otimes}_\pi Y)^*$$

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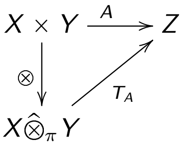

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Since $\|A\| = \sup\{|\langle A, x \otimes y\rangle| : x \in B_X, y \in B_Y\}$, it follows that

$$B_{X\hat{\otimes}_\pi Y} = \overline{\operatorname{conv}}(B_X \otimes B_Y)$$

Norm-attainment

Every $u \in X \widehat{\otimes}_\pi Y$ can be written as $u = \sum_{i=1}^{\infty} x_i \otimes y_i$ with $x_i \in X$, $y_i \in Y$.

Norm-attainment

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$$\|u\|_\pi = \inf\left\{ \sum_{i=1}^{\infty} \|x_i\|\|y_i\| : \sum_{i=1}^{\infty} \|x_i\|\|y_i\| < \infty, u = \sum_{i=1}^{\infty} x_i \otimes y_i \right\}.$$

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Dantas-Jung-Roldán-Rueda Zoca (2022)

We say that $u \in X\hat{\otimes}_\pi Y$ **attains its projective norm** if

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The set of those u is denoted by $\mathrm{NA}_\pi(X\hat{\otimes}_\pi Y)$.

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Most of the results can be translated to other settings:

- Nuclear operators (tensors in $X^*\widehat{\otimes}_\pi Y$ correspond to operators $X \to Y$)
- Symmetric tensors ($\widehat{\otimes}_{\pi,s,N} X$ is a predual of $\mathcal{P}(^N X)$)

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Definition (Dantas-Jung-Roldán-Rueda Zoca, 2022)

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In the finite-dimensional case, it follows from Caratheodory theorem that every u has an optimal representation with at most $\dim(X)\dim(Y)$ terms (or $2\dim(X)\dim(Y)$ for $\mathbb{K} = \mathbb{C}$).

Theorem (Pełczyński - Tomczak-Jaegermann, 1988)

Given n, m, there are spaces X and Y with $\dim(X) = n$ and $\dim(Y) = m$ and $u \in X\hat{\otimes}_\pi Y$ such that all the optimal representations of u have nm terms (resp. $2nm$ for $\mathbb{K} = \mathbb{C}$).

When does every tensor attain its projective norm?

Definition (Dantas, Jung, Roldán, Rueda Zoca, 2022)

$\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ in the following cases:

a) X and Y are finite dimensional.

b) $X = \ell_1(I)$ and Y is any Banach space.

c) X and Y are complex Hilbert spaces.

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Sketch of the proof.

a) Compactness

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b) $\ell_1(I)\widehat{\otimes}_\pi Y = \ell_1(I, Y)$

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a) Compactness

b) $\ell_1(I)\widehat{\otimes}_\pi Y = \ell_1(I, Y)$

c) Diagonalization

When does every tensor attain its projective norm?

- Given $u \in X\widehat{\otimes}_\pi Y$ with $\|u\| = 1$, we have that $u \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ if and only if $u = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$ with $\|x_n\| = 1 = \|y_n\|$, $\lambda_n \geqslant 0$, $\sum_n \lambda_n = 1$.

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- Now, given $T \in (X\widehat{\otimes}_\pi Y)^* \equiv \mathcal{L}(X, Y^*)$ with $\langle T, u\rangle = 1 = \|T\|$, it follows that $T(x_n)(y_n) = \|x_n\| \, \|y_n\| \ \ \forall n$.

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Thus, there are tensors not attaining its projective norm in $L_1(\mathbb{T})\widehat{\otimes}_\pi \ell_2^2$, $L_1[0,1]\widehat{\otimes}_\pi L_1[0,1]$, $\ell_p\widehat{\otimes}_\pi G$ $(1 < p < +\infty)$,...

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Rueda Zoca (2023)

$\mathrm{NA}_\pi(c_0\widehat{\otimes}_\pi \ell_2) \subset c_0 \otimes \ell_2$.

When does every tensor attain its projective norm?

Dantas, G.L., Jung, Rueda Zoca (2023)

Let X be a finite-dimensional polyhedral Banach space and Y is any dual Banach space. Then $\mathrm{NA}_{\pi}(X \widehat{\otimes}_{\pi} Y) = X \widehat{\otimes}_{\pi} Y$

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$$\begin{aligned} B_{X\widehat{\otimes}_\pi Y} &= \overline{\mathrm{conv}}(B_X \otimes B_Y) \\ &= \overline{\mathrm{conv}}\Big(\bigcup_{i=1}^{n}\{x_i\} \otimes B_Y\Big) \end{aligned}$$

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Note that B_X is a polytope if and only if B_{X^*} is a polytope.

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Therefore, X is a subspace of Z with $Z^* = \ell_1^n$.

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Let Z be such that $Z^* = \ell_1(I)$, $X \subset Z$ and Y be any dual space. If either X^* or Y has the approximation property, then $\mathrm{NA}_\pi(X^* \widehat{\otimes}_\pi Y) = X^* \widehat{\otimes}_\pi Y$.

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We can take X^* the **Lipschitz-free space** $\mathcal{F}(M)$ for a metric space M satisfying one of the following conditions:

a) M is countable and compact.

b) M is uniformly discrete, countable, and there is a compact Hausdorff topology τ on M such that d is τ-lower semicontinuous, and $V = \{d(x, y) \; : \; (x, y) \in M^2\} \subseteq \mathbb{R}_0^+$ is a compact set.

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- Say $Y = E^*$.
- $T\colon X \widehat{\otimes}_\varepsilon E \to Z \widehat{\otimes}_\varepsilon E$ given by $T(u) = u$ is a linear (into) isometry.

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- $\tilde{u}$ admits an optimal representation $\Rightarrow$ the same holds for u.

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Assume that X is separable and $\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y) = X^*\widehat{\otimes}_\pi Y$ for any Y.
Then $B_X = \overline{\mathrm{conv}}(\exp B_X)$.

Denseness of norm-attaining tensors

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Let $f = \sum_{n=1}^{N} \chi_{E_n} \cdot y_n \in L_1(\mu, Y) = L_1(\mu)\widehat{\otimes}_\pi Y$ be a simple function with (E_n) pairwise disjoint sets.

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That is, $f \in \mathrm{NA}_\pi(L_1(\mu)\widehat{\otimes}_\pi Y)$. It follows that

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There exist X, Y such that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is **NOT** dense in $X\widehat{\otimes}_\pi Y$.

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Given $u \in B_{X \hat{\otimes}_\pi Y}$ with $\|u\|_\pi = 1$, take $A \in (X \hat{\otimes}_\pi Y)^*$ with $\langle A, u \rangle = 1$.

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Under these hypotheses, $\mathrm{ext}(B_{X \hat{\otimes}_\pi Y}) \subset B_X \otimes B_Y$ (Collins-Ruess (1983))

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Thus,

$$u \in \overline{\text{conv}}\{x \otimes y \in S_X \otimes S_Y : A(x, y) = 1\}$$

Denseness of norm-attaining tensors

A space X is said to have the **metric π-property** if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset S_X$, we can find a finite-dimensional 1-complemented subspace $M \subset X$ and points $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon \ \forall i$.

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Key point: If M is 1-complemented in X then $M \widehat{\otimes}_\pi Y$ is (isometrically) a subspace of $X \widehat{\otimes}_\pi Y$.

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Let X be a space with the metric π-property. $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)} = X\widehat{\otimes}_\pi Y$ if

- Y has the metric π-property or Y is uniformly convex (Dantas-Jung-Roldán-Rueda Zoca, 2022).
- X is polyhedral and Y is a dual space (Dantas-G.L.-Jung-Rueda Zoca, 2023).

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These results are implied by the following theorem:

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Suppose that for every $\varepsilon > 0$ and $x_1, \ldots, x_n \in X$, there exists a finite dimensional 1-complemented subspace $M \subseteq X$ and $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$ for each $i = 1, \ldots, n$. Assume that $\overline{\mathrm{NA}_\pi(M\hat{\otimes}_\pi Y)} = M\hat{\otimes}_\pi Y$. Then,

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As a consequence, $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)} = X\widehat{\otimes}_\pi Y$ in the following cases:

a) $X^* = L_1(\mu)$ and Y is 1-complemented in Y^{**}.

b) X has the metric π-property and Y is a dual space with the RNP.

Some related questions

Is every extreme point of $B_{X\hat{\otimes}_\pi Y}$ of the form $x \otimes y$ with $x \in B_X$, $y \in B_Y$?

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Are there X, Y such that the set of operators attaining its nuclear norm is **NOT** dense in $\mathcal{N}(X, Y)$?

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Is every extreme point of $B_{X\hat{\otimes}_\pi Y}$ of the form $x \otimes y$ with $x \in B_X$, $y \in B_Y$?

Are there X, Y such that the set of operators attaining its nuclear norm is **NOT** dense in $\mathcal{N}(X, Y)$?

Is there X such that the set of symmetric tensors attaining its projective norm is **NOT** dense in $\widehat{\otimes}_{\pi,s,N} X$?

Thank you for your attention!