Holomorphic Lipschitz functions and composition operators

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Joint works with R. Aron, V. Dimant, J. Guerrero-Viu, M. Maestre and A. Procházka

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- V. Dimant, L.C. García-Lirola, J. Guerrero-Viu, A. Procházka. Composition operators for holomorphic Lipschitz functions. Preprint

Outline

1) Lipschitz free space
2) Holomorphic free space
3) Holomorphic Lipschitz free space
4) Approximation properties
5) Extension of holomorphic Lipschitz functions
6) Multiplicative forms and operators
7) Compactness of composition operators

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Consider

$$\delta : M \to \mathrm{Lip}_0(M)^*$$
$$x \mapsto \delta(x) : \langle f, \delta(x)\rangle = f(x)$$

The **Lipschitz-free space** (Kadec (1985), Pestov (1986), Godefroy-Kalton (2003)) $\mathcal{F}(M)$ (a.k.a. *Arens-Eells space*, *transportation cost space*) is defined as

$$\mathcal{F}(M) = \overline{\mathrm{span}}\{\delta(x) : x \in M\} \subset \mathrm{Lip}_0(M)^*$$

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Theorem (Weaver (1999), Godefroy-Kalton, (2003))

The space $\mathcal{F}(M)$ satisfies the linearization properties in the diagrams:

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If X is a separable Banach space, then X is (linearly) isometric to a 1-complemented subspace of $\mathcal{F}(X)$.

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- **In computer science** ("earthmover distance").

Theorem (Naor-Schechtman, 2007)

$\mathcal{F}(\mathbb{R}^2)$ is not (linearly) isomorphic to a subspace of $L_1 = \mathcal{F}(\mathbb{R})$.

This provides lower bounds for the computation time of certain algorithms related to the similarity of images.


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A function $f\colon U \to Y$ is said to be **holomorphic** at $x_0 \in U$ if it is Fréchet differentiable at x_0: there is $df(x_0) \in \mathcal{L}(X, Y)$ with

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - df(x_0)(h)}{\|h\|} = 0$$

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Equivalently, there is a sequence $(P_k f(x_0))_k$ of continuous k-homogeneous polynomials such that

$$f(x) = \sum_{k=0}^{\infty} P_k f(x_0)(x - x_0)$$

uniformly in some neighbourhood of x_0.

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There is a recent survey by García Sánchez - De Hevia - Tradacete.

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- Indeed, we have shown that
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- Thus, $\mathcal{H}L_0(\mathbb{D}) \equiv \mathcal{H}^\infty(\mathbb{D})$ isometrically.

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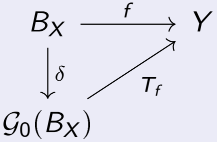

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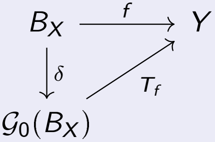

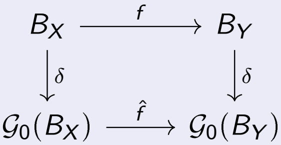

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As a consequence, X is (linearly) isometric to a 1-complemented subspace of $\mathcal{G}_0(B_X)$.

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($\mathcal{F}(B_X)$ =complex Lipschitz-free space, see Abbar-Coine-Petitjean)

Approximation properties

- X has the *Approximation Property* (AP) if the identity $I: X \to X$ can be approximated by finite-rank operators uniformly on compact sets.
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MAP for $\mathcal{G}_0(B_X)$

First we show:

a) Given $f \in \mathcal{H}L(B_X, Y)$ with $\|f\|_L \leqslant 1$, there are polinomials $P_n \colon X \to Y$ with $\|P_n|_{B_X}\|_L \leqslant 1$ and $P_n(x) \to f(x)$ for all $x \in B_X$.

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b) Assume that X has the MAP with $T_\alpha \to I$ pointwise. For each polynomial $P\colon X \to Y$ there are finite-type polinomials $P_\alpha = P \circ T_\alpha$ with $\|P_\alpha|_{B_X}\|_L \leqslant \|P|_{B_X}\|_L$ and $P_\alpha(x) \to P(x)$ for all $x \in B_X$.

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Then T_{P_α} has finite rank, $\|T_{P_\alpha}\| \leqslant 1$ and

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so $T_{P_\alpha} \to Id$ pointwise on $\operatorname{span}(\delta(x))$. Since (T_{P_α}) is bounded, the same holds for the closure.

Extension of holomorphic Lipschitz functions

Theorem (McShane, 1934)

Let M be a metric space, $N \subset M$ and $f : N \to \mathbb{R}$ be a Lipschitz map. Then there is an extension $\overline{f} : M \to \mathbb{R}$ with the same Lipschitz constant.

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However, in $\mathcal{H}^\infty$ and $\mathcal{H}L_0$ there is **no lattice structure**.

Aron-Berner, 1978

Let $P : \ell_2 \to \mathbb{C}$ given by $P(x) = \sum_{n=1}^\infty x_n^2$ and consider an embedding $\ell_2 \hookrightarrow \ell_\infty$. There does not exists $f : B_{\ell_\infty} \to \mathbb{C}$ holomorphic extending $P|_{B_{\ell_2}}$.

When $\mathcal{G}_0(B_X) \subset \mathcal{G}_0(B_Y)$?

If $X \subset Y$, we have a map

$$\begin{aligned} \rho\colon \mathcal{G}_0(B_X) &\to \mathcal{G}_0(B_Y) \\ \varphi &\mapsto \widehat{\varphi}, \end{aligned}$$

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The proof relies on the **Aron-Berner extension**.

The Aron-Berner extension

Let $P\colon X \to \mathbb{C}$ be an n-homogeneous polynomial. Then $P(x) = A(x, \dots, x)$ for a multilinear symmetric map $A\colon X \times \cdots \times X \to \mathbb{C}$. Define

$$\overline{A}(x_1^{**}, \dots x_n^{**}) = \lim_{\alpha_1} \cdots \lim_{\alpha_n} A(x_{\alpha_1}, \dots x_{\alpha_n})$$

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If X is symmetrically regular (e.g. c_0, $C(K)$, $\mathcal{H}^\infty(\mathbb{D})$), then we get $d\tilde{f} = \widetilde{df}$ for all $f \in \mathcal{H}L_0(B_X)$.



Composition operators

Taking adjoints in the square diagram

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There are composition operators with arbitrarily large norm: just take $\phi(y) = y^*(y)^n x$. Then $\phi(B_Y) \subset B_X$ and $\|\phi\|_L = n$.

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Note that $\mathcal{H}L_0(B_X)$ is a **Gelfand algebra**: $\|f \cdot g\|_L \leqslant 2\,\|f\|_L\,\|g\|_L$
Composition operators are multiplicative:

$$C_\phi(f \cdot g) = (f \cdot g) \circ \phi = C_\phi(f) \cdot C_\phi(g)$$

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Let's focus first on the case where $\varphi\colon \mathcal{H}L_0(B_X) \to \mathbb{C}$ be a multiplicative linear form. We have shown:

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- φ is the restriction of a multiplicative element in $\mathcal{A}_u(B_X)^*$.

Multiplicative operators

Theorem (Dimant-G.L.-GuerreroViu-Procházka)

Assume X has the BAP. Let $A\colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y)$ be a multiplicative linear map (i.e. an algebra homomorphism). TFAE:

i) *A is a composition operator.*

ii) *There is a linear continuous map $T\colon \mathcal{G}_0(B_Y) \to \mathcal{G}_0(B_X)$ such that $A = T^*$.*

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We do not know if these results hold for spaces without BAP.

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Proof of *ii*) $\Rightarrow$ *i*). Assume that $A = T^*$, $T\colon \mathcal{G}_0(B_X) \to \mathcal{G}_0(B_Y)$.

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- By the maximum modulus theorem, $\phi(B_Y) \subset B_X$.

Compactness of composition operators

Theorem (Kamowitz-Scheinberg (1990), JiménezVargas - VillegasVallecillos (2013), Abbar-Coine-Petitjean (2023))

Let $\phi\colon N \to M$ be a Lipschitz map with $\phi(0) = 0$. TFAE:

i) $C_\phi\colon \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(N)$ *is compact.*

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Theorem (Aron-Galindo-Lindström (1997))

Let $\phi \in \mathcal{H}^\infty(B_Y, X)$ such that $\phi(B_Y) \subset B_X$. TFAE

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iii) $\|\phi\|_\infty < 1$ *and* $\phi(B_Y)$ *is relatively compact in* B_X.

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Let $\phi \in \mathcal{H}L_0(\mathbb{D})$ *such that* $\phi(\mathbb{D}) \subseteq \mathbb{D}$. *TFAE:*

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$$\begin{array}{ccccc}
\mathcal{H}L_0(B_X) & & \xrightarrow{\quad C_\phi \quad} & & \mathcal{H}L_0(B_Y) \\
\Big\downarrow d & & & & \Big\downarrow d \\
\mathcal{H}^\infty(B_X, X^*) & \xrightarrow[C_\phi^{X^*}]{} & \mathcal{H}^\infty(B_Y, X^*) & \xrightarrow[M]{} & \mathcal{H}^\infty(B_Y, Y^*)
\end{array}$$

Thank you for your attention!