# Holomorphic Lipschitz functions and composition operators

Luis C. García-Lirola

Joint works with R. Aron, V. Dimant, J. Guerrero-Viu, M. Maestre and A. Procházka

Universidad de Zaragoza

Composition operators and Banach spaces theory
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- V. Dimant, L.C. García-Lirola, J. Guerrero-Viu, A. Procházka. Composition operators for holomorphic Lipschitz functions. Preprint

# Outline

1) Lipschitz free space
2) Holomorphic free space

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4) Multiplicative operators
5) Compactness of composition operators
6) Iterates of composition operators

# Lipschitz-free spaces

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Consider

$$\delta \colon M \to \mathrm{Lip}_0(M)^*$$
$$x \mapsto \delta(x) : \langle f, \delta(x)\rangle = f(x)$$

The **Lipschitz-free space** (Kadec (1985), Pestov (1986), Godefroy-Kalton (2003)) $\mathcal{F}(M)$ (a.k.a. *Arens-Eells space*, *transportation cost space*) is defined as

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*The space $\mathcal{F}(M)$ satisfies the linearization properties in the diagrams:*

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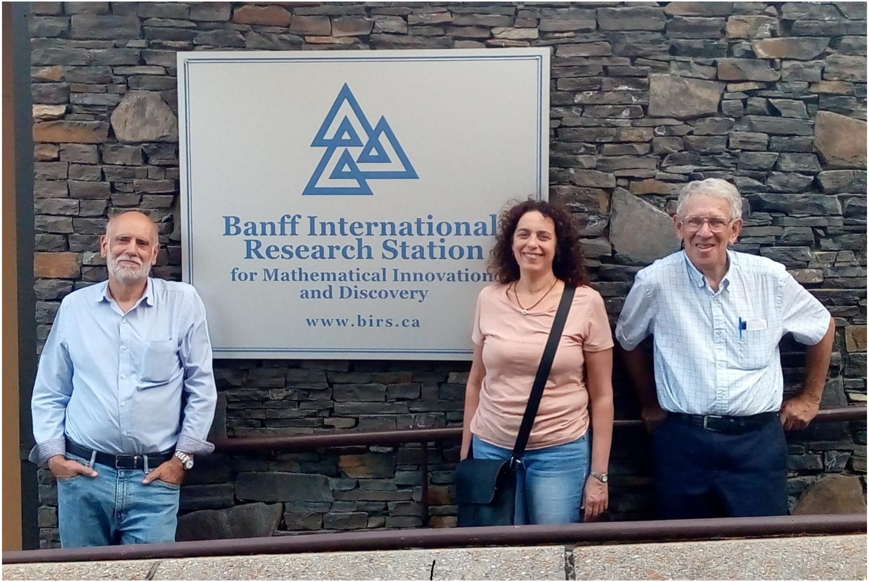
**Theorem (Godefroy-Kalton, 2003)**

*Let $X, Y$ be real Banach spaces, with $X$ separable. Assume there is a (non-linear) isometry $f \colon X \to Y$. Then there exists a linear isometry $T \colon X \to Y$. Also, the result does not hold for non-separable $X$.*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There is a recent survey by García Sánchez - De Hevia - Tradacete.

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- Thus, $\mathcal{H}L_0(\mathbb{D}) \equiv \mathcal{H}^\infty(\mathbb{D})$ isometrically.

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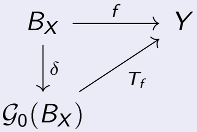


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  ($\mathcal{F}(B_X)$ =complex Lipschitz-free space, see Abbar-Coine-Petitjean)

# Approximation properties

- $X$ has the *Approximation Property* (AP) if the identity $I\colon X \to X$ can be approximated by finite-rank operators uniformly on compact sets.
- If the operators can be taken with norm $\leqslant \lambda$ then $X$ has the $\lambda$-*Bounded Approximation Property* ($\lambda$-BAP).
- If $\lambda = 1$ then $X$ has the *Metric Approximation Property* (MAP).

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Taking adjoints in the square diagram

$$\begin{array}{ccc} B_Y & \xrightarrow{\quad \phi \quad} & B_X \\ \Big\downarrow \delta & & \Big\downarrow \delta \\ \mathcal{G}_0(B_Y) & \xrightarrow{\ \hat{\phi}\ } & \mathcal{G}_0(B_X) \end{array}$$

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There are composition operators with arbitrarily large norm: just take $\phi(y) = y^*(y)^n x$. Then $\phi(B_Y) \subset B_X$ and $\|\phi\|_L = n$.

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A key fact is that the homogeneous polynomials on $X$ separate points in $\mathcal{G}_0(B_X)$. If $X$ has BAP, then it suffices to consider finite-type polynomials.

# Multiplicative operators

## Theorem (Dimant-G.L.-GuerreroViu-Procházka)

*Assume $X$ has the BAP. Let $A\colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y)$ be a multiplicative linear map (i.e. an algebra homomorphism). TFAE:*

i) *$A$ is a composition operator.*

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If $C_\phi\colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y)$ is an onto isomorphism and $X$ has BAP, then $\phi$ is the restriction of a linear isometry.

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Proof of *ii*) $\Rightarrow$ *i*). Assume that $A = T^*$, $T\colon \mathcal{G}_0(B_Y) \to \mathcal{G}_0(B_X)$.

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- By the maximum modulus theorem, $\phi(B_Y) \subset B_X$.

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**Theorem (Kamowitz-Scheinberg (1990), JiménezVargas - VillegasVallecillos (2013), Abbar-Coine-Petitjean (2023))**

*Let* $\phi\colon N \to M$ *be a Lipschitz map with* $\phi(0) = 0$. *TFAE:*

i) $C_\phi\colon \mathrm{Lip}_0(M) \to \mathrm{Lip}_0(N)$ *is compact.*

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**Theorem (Aron-Galindo-Lindström (1997))**

*Let $\phi \in \mathcal{H}^\infty(B_Y, X)$ such that $\phi(B_Y) \subset B_X$. TFAE*

i) $C_\phi\colon \mathcal{H}^\infty(B_X) \to \mathcal{H}^\infty(B_Y)$ *is compact.*

ii) $C_\phi\colon \mathcal{H}^\infty(B_X) \to \mathcal{H}^\infty(B_Y)$ *is weakly compact and* $\phi(B_Y)$ *is relatively compact.*

iii) $\|\phi\|_\infty < 1$ *and* $\phi(B_Y)$ *is relatively compact.*

# The scalar case

Let $\phi \in \mathcal{H}L_0(\mathbb{D})$ such that $\phi(\mathbb{D}) \subseteq \mathbb{D}$. Shapiro (1987) proved that if $C_\phi \colon \mathcal{H}L_0(\mathbb{D}) \to \mathcal{H}L_0(\mathbb{D})$ is compact, then $\|\phi\|_\infty < 1$.

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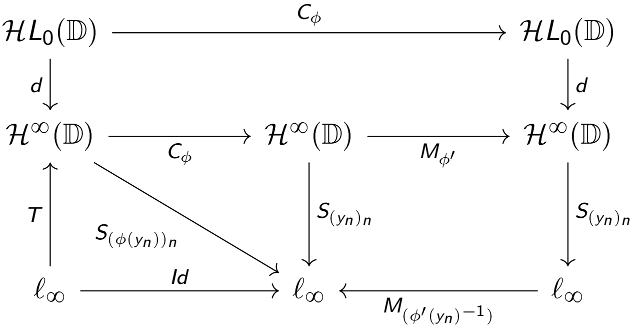

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iii)$\Rightarrow$ ii) follows also from a result of Bourgain for $T \colon \mathcal{H}^\infty(\mathbb{D}) \to Y$.

# The vector-valued case

Consider $\delta\colon B_Y \to \mathcal{G}_0(B_Y)$. For $f \in \mathcal{H}L_0(B_Y)$, we have

$$L(f) = \sup_{y_1 \in B_Y, y_2 \in S_Y} |df(y_1)(y_2)| = \sup_{y_1 \in B_Y, y_2 \in S_Y} |\langle f, d\delta(y_1)(y_2)\rangle|$$

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$$\begin{aligned} & C_\phi\colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y) \text{ is (weakly) compact} \\ \Leftrightarrow \quad & \widehat{\phi}(d\delta(B_Y)(S_Y)) \text{ is relatively (weakly) compact in } \mathcal{G}_0(B_X) \end{aligned}$$

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$$\begin{aligned} & C_\phi\colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y) \text{ is (weakly) compact} \\ \Leftrightarrow \quad & \widehat{\phi}(d\delta(B_Y)(S_Y)) \text{ is relatively (weakly) compact in } \mathcal{G}_0(B_X) \end{aligned}$$

But

$$\widehat{\phi}(d\delta(y_1)(y_2)) = d(\widehat{\phi} \circ \delta)(y_1)(y_2) = d(\delta \circ \phi)(y_1)(y_2) = d\delta(\phi(y_1))(d\phi(y_1)(y_2))$$

# The vector-valued case

This leads to:

Assume that $\overline{\phi(B_Y)} \subseteq B_X$. Then:

a) $C_\phi \colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y)$ is compact if and only if $\phi(B_Y)$ and $d\phi(B_Y)(S_Y)$ are relatively compact.

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Let $X = \ell_2$ and $\phi \colon X \to X$ be given by $\phi(x) = (\frac{c}{n} x_n^n)$ with $c > 0$ small enough. Then $\phi(B_X)$ is relatively compact but $d\phi(B_X)(S_X)$ is only relatively weakly compact.
Thus $C_\phi \colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_X)$ is weakly compact but not compact.

# The vector-valued case

In some cases, we can get rid of the hypothesis $\overline{\phi(B_Y)} \subseteq B_X$.

We obtain that for $\phi\colon Y \to X$ **linear operator** with $\|\phi\| \leqslant 1$, TFAE:

- $C_\phi\colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_Y)$ is compact.
- $\phi$ is a compact operator and $\|\phi\| < 1$.

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Let $\phi\colon \ell_2 \to \ell_2$ be given by $\phi(x) = x/2$. Then $\phi$ is a weakly compact operator but $C_\phi$ fixes the subspace

$$Z = \{f_a(x) = \sum_{n=1}^{\infty} a_n x_n^2 : a \in \ell_\infty\} \subset \mathcal{H}L_0(B_{\ell_2})$$

that is isometric to $\ell_\infty$.

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Let $X = T^*$ be the original Tsirelson space and $\phi(x) = x/2$. Then $C_\phi\colon \mathcal{H}L_0(B_X) \to \mathcal{H}L_0(B_X)$ is weakly compact but not compact.

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The key point is that every continuous polynomial $P\colon X \to X^*$ is weakly uniformly continuous on $B_X$ (Gonzalo-Jaramillo). Thanks to that, we can show that $C_\phi\colon \mathcal{H}^\infty(B_X, X^*) \to \mathcal{H}^\infty(B_X, X^*)$ is weakly compact.

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We have the commutative diagram:

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where $M_{d\phi}(g)(y) = g(y) \circ d\phi(y)$

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## Theorem (Dimant-G.L.-GuerreroViu-Procházka)

*Let $\phi \in \mathcal{H}L_0(B_Y, X)$ such that $\phi(B_Y) \subseteq B_X$.*
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Thus, if $\|\phi\|_\infty = r < 1$, then $\left\|\phi^{(2)}\right\|_\infty < r^2$ and

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The following are equivalent:

i) There is some $n \in \mathbb{N}$ such that $\left\|\phi^{(n)}\right\|_\infty < 1$.

ii) $\lim_{n\to\infty} \left\|\phi^{(n)}\right\|_L = 0$.

If $\dim(X) < \infty$, this is also equivalent to:

iii) $\phi(C) \nsubseteq C$ for all $\varnothing \neq C \subseteq S_X$.

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Thank you for your attention!