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Connection formulas for general discrete Sobolev polynomials: Mehler–Heine asymptotics ☆

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ABSTRACT

In this paper the discrete Sobolev inner product

$$\langle p, q\rangle = \int p(x)q(x)\,d\mu + \sum_{i=0}^{r} M_i\, p^{(i)}(c)\, q^{(i)}(c)$$

is considered, where μ is a finite positive Borel measure supported on an infinite subset of the real line, $c \in \mathbb{R}$ and $M_i \geq 0, i = 0, 1, \ldots, r$.

Connection formulas for the orthonormal polynomials associated with $\langle ., .\rangle$ are obtained. As a consequence, for a wide class of measures μ, we give the Mehler–Heine asymptotics in the case of the point c is a hard edge of the support of μ. In particular, the case of a symmetric measure μ is analyzed. Finally, some examples are presented.

1. Introduction

Let $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to a finite positive Borel measure μ, supported on an infinite subset of $\mathbb{R}$. We denote by $\{q_n(x)\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to an inner product of the form

$$\langle p, q\rangle = \int p(x)q(x)\,d\mu + \sum_{i=0}^{r} M_i\, p^{(i)}(c)\, q^{(i)}(c), \quad c \in \mathbb{R} \tag{1}$$

where $M_i \geq 0, i = 0, \ldots, r-1$ and $M_r > 0$. Such inner products are called discrete Sobolev or Sobolev type and they have been considered in different contexts.

In this paper we focus our attention on Mehler–Heine asymptotics for discrete Sobolev orthogonal polynomials. This asymptotic gives us one of the main differences that can be established in order to show how the addition of the derivatives in the inner product influences the orthogonal system. The Mehler–Heine formulas describe the asymptotic behavior of orthogonal polynomials near the hard edge, i.e. those endpoints of the support of the zero distribution which are also endpoints of the support of the measure. Thus, our interest is to show how the presence of the masses in the inner product changes the asymptotic behavior around this point.

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To prove the Mehler–Heine formula for Jacobi and Laguerre polynomials, usually the explicit representation of these polynomials is used (see [18]). Although the situation is a bit different, we would like to mention that recently, in [19], for some classical multiple orthogonal polynomials, asymptotic formulas of Mehler–Heine type are obtained using the explicit expression of the polynomials and the Lebesgue's dominated convergence theorem. But there are many other polynomials for which we have not an explicit representation. For example, Aptekarev in [5] realizes that for certain classes of weight functions supported on $[-1, 1]$, the Mehler–Heine asymptotic formula depends on the local behavior at the endpoint of the interval of orthogonality. So, this formula has been extended to a broader class of measures belonging to the Nevai's class. This result has been applied to deduce the Mehler–Heine formula for the generalized Jacobi polynomials (see [7]). On the other hand, for exponential weights (see [3]), it has been proved the Mehler–Heine formula for the so-called Freud polynomials using the asymptotic formula given by Kriecherbauer and McLaughlin in [12] obtained by the Riemann–Hilbert method.

For discrete Sobolev orthogonal polynomials, it is difficult to apply the same method as Jacobi and Laguerre. This analytic idea was developed for the discrete Laguerre Sobolev orthogonal polynomials in [2]. There, the authors obtained a new and specific formula for the derivatives of q_n which leads to achieve a uniform bound in order to use the Lebesgue's dominated convergence theorem. But in a general case, this is quite complicated.

On the other hand, an important tool to get asymptotics is the knowledge of certain connection formulas for q_n in terms of standard polynomials related with p_n.

One of them can be deduced from the well known fact that $\{q_n(x)\}_{n\geq 0}$ is quasi-orthogonal of order $r+1$ and consequently we can express q_n as a linear combination of the standard orthogonal polynomials corresponding to the modified measure $(x-c)^{r+1}\, d\mu(x)$. This connection formula has proved to be fruitful, for example, in the study of relative asymptotics when μ has compact support, see [13] and [17] in a more general setting. However, the situation is quite different in the case of measures with unbounded support. So, in [2], it was shown that this connection formula is not the adequate to study neither relative asymptotics nor Mehler–Heine formula when μ is the Laguerre weight.

For discrete Laguerre Sobolev orthogonal polynomials, another connection formula was given by Koekoek in [10] with an arbitrary r and $c = 0$. This formula has turned out to be of great importance to generating Mehler–Heine asymptotics. So, in [4], using the explicit expression of the connection coefficients given in [11] for $r = 1$, the authors prove the Mehler–Heine asymptotic for the corresponding Laguerre Sobolev polynomials. This idea has also been used in [14] for $r = 1$ and $c < 0$. However, in an inner product with an arbitrary (finite) number of terms in the discrete part, the problem is that we have not the explicit expression of the coefficients. In spite of this, in [16] the authors achieve the Mehler–Heine formula for an arbitrary number of masses and $c = 0$.

In this paper we prove that for a wide class of measures with support bounded or not, an arbitrary number of masses in the inner product (1) and without taking into account the location of the point c with respect to the support of μ, there exists a connection formula for $q_n(x)$ in terms of some canonical transformation of the polynomials p_n, called Christoffel perturbations. More precisely, $q_n(x) = \sum_{j=0}^{r+1} \lambda_{j,n}(x-c)^j\, p_{n-j}^{[2j]}(x)$ where $\{p_n^{[j]}(x)\}_{n\geq 0}$ denotes the sequence of orthonormal polynomials with respect to the measure μ_j with $d\mu_j(x) = (x-c)^j\, d\mu(x)$. The main contribution is that we are able to give information of asymptotic behavior of the connection coefficients, without the explicit expression of them. This is a significant improvement compared with the previous works. Our interest is focused on the application of this connection formula for obtaining the Mehler–Heine asymptotics and so to prove that whenever the asymptotic behavior near the hard edge involves Bessel functions, the presence of positive masses in the inner product produces a convergence acceleration to this point of $r+1$ zeros of the Sobolev polynomials.

Finally, we would like to notice that the connection formula obtained has interest by itself because it may be used to get other results as Cohen type inequality and other asymptotic properties.

The structure of the paper is as follows. In Section 2, we establish a connection formula for orthonormal polynomials with respect to the inner product (1), where c is an arbitrary real number. Moreover, we show a technical lemma that besides giving us asymptotic behavior at the point c of the successive derivatives of the polynomials q_n and so of their kernels, also provides information about the asymptotic behavior of the coefficients in the connection formula. In Section 3, for a wide class of measures μ, we obtain Mehler–Heine asymptotics for the sequence $\{q_n(x)\}_{n\geq 0}$ where in (1) all the masses are positive and the point c is either an endpoint of the interval where the measure μ is supported or the origin if the measure is symmetric. As an application, we obtain an important information about the distribution of the zeros of the polynomials q_n. In the last section we present some examples to illustrate the theory given.

Throughout this paper we use the notation $x_n \cong y_n$ when the sequence x_n/y_n converges to 1.

2. Connection formulas

Let $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to the measure μ and $\{q_n(x)\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (1).

In this section we will establish a connection formula for the discrete Sobolev polynomials q_n in terms of the polynomials $p_n^{[2j]}(x)$ orthogonal with respect to the measure $d\mu_{2j}(x) = (x-c)^{2j}\, d\mu(x)$.

Theorem 1. *Assume that the polynomials $\{p_n(x)\}_{n\geq 0}$ satisfy*

$$p_n(c)\, p_{n-1}^{[2]}(c) \ldots p_{n-(r+1)}^{[2(r+1)]}(c) \neq 0 \tag{2}$$

then there exists a family of coefficients $(\lambda_{j,n})_{j=0}^{r+1}$, *not identically zero, such that the following connection formula holds*

$$q_n(x) = \sum_{j=0}^{r+1} \lambda_{j,n}\,(x-c)^j\,p_{n-j}^{[2j]}(x), \quad n \ge r+1. \tag{3}$$

Proof. We will show that there exists a family of coefficients $(\lambda_{j,n})_{j=0}^{r+1}$, not identically zero, such that the polynomial $r_n(x)$ defined by $r_n(x) = \sum_{j=0}^{r+1} \lambda_{j,n}\,(x-c)^j\,p_{n-j}^{[2j]}(x)$ satisfies

$$\langle r_n(x), (x-c)^k \rangle = 0, \quad 0 \le k \le n-1. \tag{4}$$

Indeed, for $0 \le j \le r+1 \le k \le n-1$

$$\begin{aligned}\langle (x-c)^j\,p_{n-j}^{[2j]}(x), (x-c)^k \rangle &= \int (x-c)^j p_{n-j}^{[2j]}(x)\,(x-c)^k\,d\mu(x)\\ &= \int (x-c)^{k-j} p_{n-j}^{[2j]}(x)\,d\mu_{2j}(x) = 0.\end{aligned}$$

Thus, if $0 \le k \le r$, (4) leads to the following system

$$\sum_{j=0}^{r+1} \lambda_{j,n}\,\langle (x-c)^j\,p_{n-j}^{[2j]}(x), (x-c)^k \rangle = 0$$

of $r+1$ equations on $r+2$ unknowns and then we can affirm that it has a non trivial solution $(\lambda_{j,n})_{j=0}^{r+1}$.

To assure that $r_n(x) = q_n(x)$ it is enough to prove that the polynomial r_n has degree n. Indeed, if $\deg r_n < n$, the condition (4) yields $r_n \equiv 0$, but this is in contradiction with the hypothesis (2), because if we denote by $\lambda_{j_0,n}$ the first coefficient non zero, then $r_n^{(j_0)}(c) = \lambda_{j_0,n}\,j_0!\,p_{n-j_0}^{[2j_0]}(c) \neq 0$. □

Remark 1. If c is not in the interior of the convex hull of the support of the measure μ, the condition (2) is always true.

In the next lemma, we get asymptotic estimates of the derivatives $q_n^{(k)}(c)$ from the ones of $p_n^{(k)}(c)$, which will play an important role along this paper.

Lemma 1. *Suppose that there exists a strictly increasing function f with* $2f(0)+1>0$ *and such that the polynomials* $\{p_n(x)\}_{n\ge 0}$ *satisfy the condition*

$$p_n^{(k)}(c) \cong C_k(-1)^n\,n^{f(k)}, \quad 0 \le k \le n. \tag{5}$$

Then the following statement holds:

$$\frac{q_n^{(k)}(c)}{p_n^{(k)}(c)} \cong \begin{cases} \dfrac{C_k}{n^{2f(k)+1}}, & \text{for } k \text{ such that } M_k > 0;\\ C_k, & \text{otherwise}, \end{cases} \tag{6}$$

where C_k *is a nonzero constant independent of n, but possibly different in each occurrence.*

Proof. We will prove the result by an induction process concerning the number of positive masses in the inner product (1).

We take the first mass which is positive, namely M_{j_1} ($j_1 \ge 0$), and consider the sequence of orthonormal polynomials $\{q_{n,1}\}_{n\ge 0}$ with respect to the inner product

$$(p,q)_1 = \int p(x)q(x)\,d\mu(x) + M_{j_1}\,p^{(j_1)}(c)\,q^{(j_1)}(c),$$

where $q_{n,1}(x) = \tilde{\gamma}_{n,1}\,x^n + \cdots$.

The Fourier expansion of the polynomial $q_{n,1}$ in the orthonormal basis $\{p_n\}_{n\ge 0}$ $(p_n(x) = \gamma_n\,x^n + \cdots)$ leads to

$$q_{n,1}(x) = \frac{\tilde{\gamma}_{n,1}}{\gamma_n} p_n(x) - M_{j_1}\,q_{n,1}^{(j_1)}(c)\,K_{n-1}^{(0,j_1)}(x,c),$$

and therefore

$$q_{n,1}(x) = \frac{\tilde{\gamma}_{n,1}}{\gamma_n}\left[p_n(x) - \frac{M_{j_1}\,p_n^{(j_1)}(c)}{1 + M_{j_1}K_{n-1}^{(j_1,j_1)}(c,c)} K_{n-1}^{(0,j_1)}(x,c)\right] \tag{7}$$

where, as usual, we denote by $K_n^{(k,h)}(x,y)$ the derivatives of the nth kernel for the sequence $\{p_n\}_{n\ge 0}$

$$K_n^{(k,h)}(x,y) = \frac{\partial^{k+h}}{\partial x^k\,\partial x^h} K_n(x,y) = \sum_{i=0}^{n} p_i^{(k)}(x)\,p_i^{(h)}(y), \quad k,h \in \mathbb{N}\cup\{0\}.$$

On the other hand, for $0 \le k \le n$, the hypothesis for the function f allows us to affirm that $f(k) + f(j_1) + 1 > 0$. So, applying Stolz criterion (see, e.g [9]) and the hypothesis (5) we obtain

$$\lim_n \frac{K_n^{(k,j_1)}(c,c)}{n^{f(k)+f(j_1)+1}} = \lim_n \frac{p_n^{(k)}(c)\,p_n^{(j_1)}(c)}{(f(k)+f(j_1)+1)\,n^{f(k)+f(j_1)}} = \frac{C_k C_{j_1}}{f(k)+f(j_1)+1} \neq 0, \tag{8}$$

and so

$$\frac{K_n^{(k,j_1)}(c,c)}{n\,p_n^{(k)}(c)\,p_n^{(j_1)}(c)} \cong \frac{1}{f(k)+f(j_1)+1}. \tag{9}$$

Moreover, it is easy to check that

$$\left(\frac{\gamma_n}{\tilde{\gamma}_{n,1}}\right)^2 = \left[1 + \frac{M_{j_1}(p_n^{(j_1)}(c))^2}{1 + M_{j_1} K_{n-1}^{(j_1,j_1)}(c,c)}\right]$$

and thus

$$\frac{\tilde{\gamma}_{n,1}}{\gamma_n} \cong 1. \tag{10}$$

Now, taking derivatives k times in (7) and evaluating at $x = c$, we have

$$\frac{q_{n,1}^{(k)}(c)}{p_n^{(k)}(c)} = \frac{\tilde{\gamma}_{n,1}}{\gamma_n}\left[1 - \frac{M_{j_1} K_{n-1}^{(k,j_1)}(c,c)}{1 + M_{j_1} K_{n-1}^{(j_1,j_1)}(c,c)} \frac{p_n^{(j_1)}(c)}{p_n^{(k)}(c)}\right].$$

Then, by (8) and (10), we get

$$\frac{q_{n,1}^{(j_1)}(c)}{p_n^{(j_1)}(c)} = \frac{\tilde{\gamma}_{n,1}}{\gamma_n} \frac{1}{1 + M_{j_1} K_{n-1}^{(j_1,j_1)}(c,c)} \cong \frac{C_{j_1}}{n^{2f(j_1)+1}},$$

and for $k \neq j_1$, from (9) and (10)

$$\frac{q_{n,1}^{(k)}(c)}{p_n^{(k)}(c)} \cong \left[1 - \frac{2f(j_1)+1}{f(k)+f(j_1)+1}\right] = \frac{f(k)-f(j_1)}{f(k)+f(j_1)+1} \neq 0.$$

If there are no more positive masses, since $q_{n,1} = q_n$ we have concluded the proof. Otherwise, suppose that the result holds for the sequence of orthonormal polynomials $\{q_{n,s-1}\}_{n \ge 0}$ orthogonal with respect to the inner product

$$(p,q)_{s-1} = \int p(x)q(x)\,d\mu(x) + \sum_{i=1}^{s-1} M_{j_i}\, p^{(j_i)}(c)\, q^{(j_i)}(c),$$

where $j_1 < j_2 < \ldots < j_{s-1}$ and all these masses are positive.

Now, we have to prove the result for the orthonormal polynomials $q_{n,s}$ with respect to

$$(p,q)_s = (p,q)_{s-1} + M_{j_s}\, p^{(j_s)}(c)\, q^{(j_s)}(c),$$

where $M_{j_s} > 0$, and we can work as before. Then the Fourier expansion of the polynomial $q_{n,s}$ ($q_{n,s}(x) = \tilde{\gamma}_{n,s} x^n + \cdots$) in the orthonormal basis $\{q_{n,s-1}\}_{n \ge 0}$ leads to

$$q_{n,s}(x) = \frac{\tilde{\gamma}_{n,s}}{\tilde{\gamma}_{n,s-1}} q_{n,s-1}(x) - M_{j_s}\, q_{n,s}^{(j_s)}(c)\, K_{n-1,s-1}^{(0,j_s)}(x,c),$$

where $K_{n,s-1}$ denotes the corresponding nth kernel for the sequence $\{q_{n,s-1}\}$ and

$$K_{n,s-1}^{(k,h)}(x,y) = \sum_{i=0}^{n} q_{i,s-1}^{(k)}(x)\, q_{i,s-1}^{(h)}(y), \quad k,h \in \mathbb{N} \cup \{0\}.$$

Therefore,

$$q_{n,s}(x) = \frac{\tilde{\gamma}_{n,s}}{\tilde{\gamma}_{n,s-1}}\left[q_{n,s-1}(x) - \frac{M_{j_s}\, q_{n,s-1}^{(j_s)}(c)}{1 + M_{j_s} K_{n-1,s-1}^{(j_s,j_s)}(c,c)} K_{n-1,s-1}^{(0,j_s)}(x,c)\right], \tag{11}$$

and

$$\left(\frac{\tilde{\gamma}_{n,s-1}}{\tilde{\gamma}_{n,s}}\right)^2 = \left[1 + \frac{M_{j_s}(q_{n,s-1}^{(j_s)}(c))^2}{1 + M_{j_s} K_{n-1,s-1}^{(j_s,j_s)}(c,c)}\right]. \tag{12}$$

Applying Stolz criterion, the hypothesis for the function f and the hypothesis for $\{q_{n,s-1}\}_{n\geq 0}$, we can obtain

$$K_{n,s-1}^{(k,j_s)}(c,c) \cong \begin{cases} C_k\, n^{f(k)+f(j_s)+1}, & \text{if } k \neq j_1,\dots,j_{s-1}; \\ C_k\, n^{f(j_s)-f(k)}, & \text{if } k = j_1,\dots,j_{s-1}, \end{cases}$$

where C_k is a nonzero constant independent of n, but possibly different in each occurrence.

Indeed, for $k \neq j_1,\dots,j_{s-1}$,

$$\lim_n \frac{K_{n,s-1}^{(k,j_s)}(c,c)}{n^{f(k)+f(j_s)+1}} = \lim_n \frac{q_{n,s-1}^{(k)}(c)\, q_{n,s-1}^{(j_s)}(c)}{(f(k)+f(j_s)+1)\, n^{f(k)+f(j_s)}} = \lim_n \frac{q_{n,s-1}^{(k)}(c)}{p_n^{(k)}(c)} \lim_n \frac{q_{n,s-1}^{(j_s)}(c)}{p_n^{(j_s)}(c)} \lim_n \frac{p_n^{(k)}(c)\, p_n^{(j_s)}(c)}{(f(k)+f(j_s)+1)\, n^{f(k)+f(j_s)}} \neq 0, \tag{13}$$

and, for $k = j_1, \dots, j_{s-1}$,

$$\lim_n \frac{K_{n,s-1}^{(k,j_s)}(c,c)}{n^{f(j_s)-f(k)}} = \lim_n \frac{q_{n,s-1}^{(k)}(c)\, q_{n,s-1}^{(j_s)}(c)}{(f(j_s)-f(k))\, n^{f(j_s)-f(k)-1}} = \lim_n \frac{p_n^{(k)}(c)\, p_n^{(j_s)}(c)}{(f(j_s)-f(k))\, n^{f(k)+f(j_s)}} \lim_n n^{2f(k)+1} \frac{q_{n,s-1}^{(k)}(c)}{p_n^{(k)}(c)} \lim_n \frac{q_{n,s-1}^{(j_s)}(c)}{p_n^{(j_s)}(c)} \neq 0. \tag{14}$$

Then from (12)

$$\frac{\tilde{\gamma}_{n,s}}{\tilde{\gamma}_{n,s-1}} \cong 1. \tag{15}$$

Now, taking derivatives k times in (11) and evaluating at $x = c$, we obtain

$$\frac{q_{n,s}^{(k)}(c)}{p_n^{(k)}(c)} \cong \frac{q_{n,s-1}^{(k)}(c)}{p_n^{(k)}(c)} \left[1 - \frac{M_{j_s} K_{n-1,s-1}^{(k,j_s)}(c,c)}{1 + M_{j_s} K_{n-1,s-1}^{(j_s,j_s)}(c,c)} \frac{q_{n,s-1}^{(j_s)}(c)}{q_{n,s-1}^{(k)}(c)} \right]. \tag{16}$$

For $k = j_s$, the hypothesis for $q_{n,s-1}$ and the estimation of the kernel yield

$$\frac{q_{n,s}^{(j_s)}(c)}{p_n^{(j_s)}(c)} \cong \frac{q_{n,s-1}^{(j_s)}(c)}{p_n^{(j_s)}(c)} \frac{1}{1 + M_{j_s} K_{n-1,s-1}^{(j_s,j_s)}(c,c)} \cong \frac{C_{j_s}}{n^{2f(j_s)+1}},$$

with C_{j_s} a nonzero constant independent of n.

Moreover, for $k \neq j_s$, taking into account (13), (14) and the hypothesis for $q_{n,s-1}$, we can deduce

$$\frac{K_{n-1,s-1}^{(k,j_s)}(c,c)\, q_{n,s-1}^{(j_s)}(c)}{K_{n-1,s-1}^{(j_s,j_s)}(c,c)\, q_{n,s-1}^{(k)}(c)} \cong \begin{cases} \dfrac{2f(j_s)+1}{f(k)+f(j_s)+1}, & \text{if } k \neq j_1,\dots,j_{s-1}; \\[2mm] \dfrac{2f(j_s)+1}{f(j_s)-f(k)}, & \text{if } k = j_1,\dots,j_{s-1}. \end{cases} \tag{17}$$

Thus, taking limits in (16), we get for the polynomials $q_{n,s}$,

$$\frac{q_{n,s}^{(k)}(c)}{p_n^{(k)}(c)} \cong \begin{cases} \dfrac{C_k}{n^{2f(k)+1}}, & \text{if } k = j_1,\dots,j_s; \\ C_k, & \text{otherwise}, \end{cases}$$

where the hypothesis for f allows us to affirm that C_k is a nonzero constant independent of n, but possibly different in each occurrence. Hence the result follows. □

Corollary 1. *Under the same hypothesis of the previous lemma we have*

$$\frac{q_n^{(r+1)}(c)}{p_n^{(r+1)}(c)} \cong \prod_{i=1}^{s} \frac{f(r+1)-f(j_i)}{f(r+1)+f(j_i)+1} \tag{18}$$

where $(M_{j_i})_{i=1}^{s}$ *are the positive masses in the inner product (1).*

Proof. We obtain the result applying a recursive process concerning the number of positive masses in the inner product (1).

Indeed, using (16) and (17), it follows that

$$\frac{q_{n,s}^{(r+1)}(c)}{p_n^{(r+1)}(c)} \cong \frac{q_{n,s-1}^{(r+1)}(c)}{p_n^{(r+1)}(c)} \left[\frac{f(r+1)-f(j_s)}{f(r+1)+f(j_s)+1} \right]$$

and we get the result. □

Remark 2. Lemma 1 and hence Corollary 1 are also true if in the condition (5) the factor $(-1)^n$ is deleted.

Next, as a consequence of Lemma 1, we prove that whenever the polynomials $p_n^{[2j]}(x)$ satisfy a similar condition to (5), then there exists the connection formula (3) and moreover there exists limit of their connection coefficients $\lambda_{j,n}$, $j = 0, 1, \ldots, r+1$.

Theorem 2. *Suppose that there exists a strictly increasing function f with* $2f(0) + 1 > 0$ *and such that for all* $j = 0, 1, \ldots, r+1$, *the polynomials* $\{p_n^{[2j]}(x)\}$ *satisfy the condition*

$$\left(p_n^{[2j]}\right)^{(k)}(c) \cong C_{k,j}\,(-1)^n\, n^{f(k+j)}, \quad 0 \le k \le n, \tag{19}$$

where $C_{k,j}$ *is a nonzero constant independent of n.*

Then, there exists

$$\lim_n \lambda_{j,n} = \lambda_j \in \mathbb{R}, \quad j = 0, 1, \ldots, r+1,$$

where $\{\lambda_{j,n}\}_0^{r+1}$ *are the coefficients in the connection formula (3). Moreover, if all the masses in the inner product (1) are positive, we obtain*

$$\lim_n \lambda_{j,n} = 0, \quad j = 0, 1, \ldots, r$$

and

$$\lim_n \lambda_{r+1,n} \neq 0.$$

Proof. Notice that the existence of the connection formula (3) for n large enough is a straightforward consequence of (19).

So, taking derivatives k times in (3) and evaluating at $x = c$, we deduce

$$\frac{q_n^{(k)}(c)}{p_n^{(k)}(c)} = \sum_{j=0}^{k} \lambda_{j,n} \binom{k}{j} j!\, A_j(k,n), \quad 0 \le k \le r+1, \tag{20}$$

where $A_0(k,n) = 1$ and

$$A_j(k,n) = \frac{(p_{n-j}^{[2j]})^{(k-j)}(c)}{p_n^{(k)}(c)}. \tag{21}$$

From condition (19), we can deduce that there exists $\lim_n A_j(k,n) \neq 0$. Then, applying recursively (6) and (20), we can assure that there exists $\lim_n \lambda_{j,n} = \lambda_j$, $j = 0, 1, \ldots, r+1$. More precisely, for $k = 0$ we have

$$\lim_n \lambda_{0,n} = \lim_n \frac{q_n(c)}{p_n(c)} = \lambda_0 \begin{cases} = 0, & \text{if } M_0 > 0; \\ \neq 0, & \text{if } M_0 = 0. \end{cases}$$

Now, from (20) for $k = 1$ and (6) we get

$$\lim_n \lambda_{1,n} = \lim_n \frac{1}{A_1(1,n)} \left(\frac{q_n'(c)}{p_n'(c)} - \lambda_{0,n} \right) = \lambda_1.$$

Observe that

$$\lambda_1 \begin{cases} = 0, & \text{if } M_0 > 0 \quad \text{and } M_1 > 0; \\ \neq 0, & \text{if } M_0 > 0 \quad \text{and } M_1 = 0. \end{cases}$$

In this way, recursively, if $M_0 M_1 \ldots M_i > 0$ and $M_{i+1} = 0$, we can assure that

$$\lim_n \lambda_{j,n} = \lambda_j \begin{cases} = 0, & \text{if } 0 \le j \le i; \\ \neq 0, & \text{if } j = i+1, \end{cases}$$

and we obtain the result. □

Remark 3. Theorem 2 is also true if in the condition (19) the factor $(-1)^n$ is deleted.

2.1. Symmetric case

Next, if the measure μ is symmetric, by a symmetrization process, we will obtain similar results to those discussed above.

Let $\{p_n(x)\}_{n \ge 0}$ be the sequence of orthonormal polynomials with respect to a symmetric positive Borel measure μ on $(-d, d)$ where $0 < d \le \infty$. Let ν be the image of measure μ on $J = (0, d^2)$ under the mapping $\Phi(x) = x^2$, i.e. $\nu = \Phi(\mu)$.

A symmetrization process, see [6] for monic polynomials, yields

$$p_{2n}(x) = u_n(x^2), \quad p_{2n+1}(x) = x\, u_n^*(x^2), \tag{22}$$

where $\{u_n(x)\}_{n \ge 0}$ and $\{u_n^*(x)\}_{n \ge 0}$ are the sequences of orthonormal polynomials with respect to the measures $d\nu(x)$ and $x\, d\nu(x)$, respectively.

Now, we rename the inner product (1) as

$$\langle p, q\rangle = \int p(x)q(x)\,d\mu + \sum_{i=0}^{2r+1} M_i\, p^{(i)}(0)\, q^{(i)}(0), \tag{23}$$

where $M_i \geq 0, i = 0, 1, \ldots, 2r-1, M_{2r} > 0$ and $M_{2r+1} > 0$. This inner product has been already considered in [1].

First notice that if the initial measure μ is symmetric, then the Sobolev type polynomials q_n orthogonal with respect to (23) are also symmetric. Again, by the symmetrization process we can write

$$q_{2n}(x) = s_n(x^2), \quad q_{2n+1}(x) = x\, s_n^*(x^2), \tag{24}$$

where now the sequences of orthonormal polynomials $\{s_n(x)\}_{n\geq 0}$ and $\{s_n^*(x)\}_{n\geq 0}$ are orthogonal with respect to the Sobolev type inner products:

$$\langle p, q\rangle_1 = \int_J p(x)q(x)\,d\nu(x) + \sum_{i=0}^{r} \tilde{M}_{2i}\, p^{(i)}(0)\, q^{(i)}(0), \tag{25}$$

and

$$\langle p, q\rangle_2 = \int_J p(x)q(x)\,x d\nu(x) + \sum_{i=0}^{r} \tilde{M}_{2i+1}\, p^{(i)}(0)\, q^{(i)}(0), \tag{26}$$

respectively, where

$$\tilde{M}_{2i} = \left(\frac{(2i)!}{i!}\right)^2 M_{2i}, \quad \tilde{M}_{2i+1} = \left(\frac{(2i+1)!}{i!}\right)^2 M_{2i+1}, \quad i = 0, \ldots, r,$$

(see Theorem 2 in [1]).

Also, since the measure μ_{2j} is symmetric, then the polynomials $p_n^{[2j]}(x)$ are symmetric and it is easy to check that

$$p_{2n}^{[2j]}(x) = u_n^{[j]}(x^2), \quad p_{2n+1}^{[2j]}(x) = x\,(u_n^*)^{[j]}(x^2), \tag{27}$$

where $\{u_n^{[j]}(x)\}_{n\geq 0}$ and $\{(u_n^*)^{[j]}(x)\}_{n\geq 0}$ are the sequences of orthonormal polynomials with respect to the measures $x^j\,d\nu(x)$ and $x^{j+1}\,d\nu(x)$, respectively.

Theorem 3. *Let $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to a symmetric measure μ and $\{q_n(x)\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (23).*

Then, there exist two families of coefficients $(\lambda_{j,n})_{j=0}^{r+1}$ and $(\lambda_{j,n}^*)_{j=0}^{r+1}$ not identically zero, such that the following connection formulas hold

$$q_{2n}(x) = \sum_{j=0}^{r+1} \lambda_{j,n}\, x^{2j}\, p_{2n-2j}^{[4j]}(x), \quad n \geq r+1, \tag{28}$$

and

$$q_{2n+1}(x) = \sum_{j=0}^{r+1} \lambda_{j,n}^*\, x^{2j}\, p_{2n+1-2j}^{[4j]}(x), \quad n \geq r+1. \tag{29}$$

Proof. The result is a simple consequence of Theorem 1 and the symmetrization process described above. Indeed, from (27) and since $p_n^{[2j]}(x)$ are symmetric polynomials, we get

$$u_n(0)u_{n-1}^{[2]}(0)\ldots u_{n-(r+1)}^{[2(r+1)]}(0) = p_{2n}(0)p_{2n-2}^{[4]}(0)\ldots p_{2n-2(r+1)}^{[4(r+1)]}(0) \neq 0,$$

and

$$u_n^*(0)\,(u_{n-1}^*)^{[2]}(0)\ldots(u_{n-(r+1)}^*)^{[2(r+1)]}(0) = (p_{2n+1})'(0)\left(p_{2n-1}^{[4]}\right)'(0)\ldots\left(p_{2n+1-2(r+1)}^{[4(r+1)]}\right)'(0) \neq 0.$$

Now, taking into account (24)–(26), we can apply Theorem 1 to the polynomials s_n and s_n^* and so there exist two families of coefficients $(\lambda_{j,n})_{j=0}^{r+1}$ and $(\lambda_{j,n}^*)_{j=0}^{r+1}$, not identically zero, such that

$$s_n(x) = \sum_{j=0}^{r+1} \lambda_{j,n} x^j\, u_{n-j}^{[2j]}(x), \quad s_n^*(x) = \sum_{j=0}^{r+1} \lambda_{j,n}^* x^j\, \left(u_{n-j}^*\right)^{[2j]}(x). \tag{30}$$

To conclude it is enough to use again (24) and (27). □

Lemma 2. *Suppose that there exists a strictly increasing function f with $2f(0)+1>0$ and such that the polynomials $\{p_n\}$ satisfy the conditions*

$$p_{2n}^{(2k)}(0) \cong C_k(-1)^n\, n^{f(2k)}, \qquad 0 \le k \le n,$$
$$p_{2n+1}^{(2k+1)}(0) \cong C_k(-1)^n\, n^{f(2k+1)}, \quad 0 \le k \le n.$$

Then the following statements hold:

$$\frac{q_{2n}^{(2k)}(0)}{p_{2n}^{(2k)}(0)} \cong \begin{cases} \dfrac{C_k}{n^{2f(2k)+1}}, & \text{for } k \text{ such that } M_{2k}>0; \\ C_k, & \text{otherwise}, \end{cases}$$

$$\frac{q_{2n+1}^{(2k+1)}(0)}{p_{2n+1}^{(2k+1)}(0)} \cong \begin{cases} \dfrac{C_k}{n^{2f(2k+1)+1}}, & \text{for } k \text{ such that } M_{2k+1}>0; \\ C_k, & \text{otherwise}, \end{cases}$$

where C_k is a nonzero constant independent of n, but possibly different in each occurrence.

Proof. Again we will apply the symmetrization process.

From (27), with $j=0$, we have that

$$u_n^{(k)}(0) = \frac{k!}{(2k)!}\, p_{2n}^{(2k)}(0) \cong C_k(-1)^n\, n^{f(2k)} = C_k(-1)^n\, n^{g(k)}$$

and

$$(u_n^*)^{(k)}(0) = \frac{k!}{(2k+1)!}\, p_{2n+1}^{(2k+1)}(0) \cong C_k(-1)^n\, n^{f(2k+1)} = C_k(-1)^n\, n^{g^*(k)}$$

where g, g^* are strictly increasing functions satisfying $2g(0)+1>0, 2g^*(0)+1>0$. Then we can apply Lemma 1 and so we obtain:

$$\frac{s_n^{(k)}(0)}{u_n^{(k)}(0)} \cong \begin{cases} \dfrac{C_k}{n^{2g(k)+1}}, & \text{for } k \text{ such that } \tilde{M}_{2k}>0; \\ C_k, & \text{otherwise}, \end{cases}$$

$$\frac{(s_n^*)^{(k)}(0)}{(u_n^*)^{(k)}(0)} \cong \begin{cases} \dfrac{C_k}{n^{2g^*(k)+1}}, & \text{for } k \text{ such that } \tilde{M}_{2k+1}>0; \\ C_k, & \text{otherwise}, \end{cases}$$

where C_k is a nonzero constant independent of n, but possibly different in each occurrence.

Thus the result is proved. □

Theorem 4. *Assume that there exists a strictly increasing function f* with $2f(0)+1>0$ *and such that for all $j=0,1,\ldots,r+1$ the polynomials $\{p_n^{[4j]}(x)\}$ satisfy the conditions*

$$\left(p_{2n}^{[4j]}\right)^{(2k)}(0) \cong C_{k,j}\,(-1)^n\, n^{f(2k+2j)}, \quad 0 \le k \le n, \tag{31}$$

$$\left(p_{2n+1}^{[4j]}\right)^{(2k+1)}(0) \cong C_{k,j}\,(-1)^n\, n^{f(2k+2j+1)}, \quad 0 \le k \le n, \tag{32}$$

where $C_{k,j}$ is a nonzero constant independent of n, but possibly different in each occurrence.

Then, there exists

$$\lim_n \lambda_{j,n} = \lambda_j \in \mathbb{R}, \quad \text{and} \quad \lim_n \lambda_{j,n}^* = \lambda_j^* \in \mathbb{R}, \qquad j = 0,1,\ldots,r+1,$$

where $\{\lambda_{j,n}\}_0^{r+1}$, $\{\lambda_{j,n}^*\}_0^{r+1}$ are the families of coefficients in formulas (28) and (29). Moreover, if all the masses in the inner product (23) are positive we obtain

$$\lim_n \lambda_{j,n} = \lim_n \lambda_{j,n}^* = 0, \quad j = 0,1,\ldots,r$$

and

$$\lim_n \lambda_{r+1,n} \ne 0, \qquad \lim_n \lambda_{r+1,n}^* \ne 0.$$

Proof. From (27), (31) and (32) we have

$$\left(u_n^{[2j]}\right)^{(k)}(0) = \frac{k!}{(2k)!}\,(p_{2n}^{[4j]})^{(2k)}(0)$$
$$\cong C_{k,j}(-1)^n\, n^{f(2k+2j)} = C_{k,j}(-1)^n\, n^{g(k+j)},$$

and

$$\left((u_n^*)^{[2j]}\right)^{(k)}(0) = \frac{k!}{(2k+1)!}\left(p_{2n+1}^{[4j]}\right)^{(2k+1)}(0)$$
$$\cong C_{k,j}(-1)^n\, n^{f(2k+2j+1)} = C_{k,j}(-1)^n\, n^{g^*(k+j)},$$

where g, g^* are strictly increasing functions satisfying $2g(0)+1>0, 2g^*(0)+1>0$, and $C_{k,j}$ is a nonzero constant independent of n.

Thus the result follows, taking into account the connection formulas for $s_n(x)$ and $s_n^*(x)$ given in (30) and Theorem 2. □

3. Mehler–Heine type formulas

Let $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to a positive measure μ supported on an interval I. Let $\{q_n(x)\}_{n\geq 0}$ be the sequence of Sobolev type orthonormal polynomials with respect to the inner product (1) where all the masses are positive and c is an endpoint of the interval I. We assume without loss of generality that $c = \inf I,\ c\in\mathbb{R}$.

In this section we obtain Mehler–Heine asymptotics for $\{q_n(x)\}_{n\geq 0}$. To do this we will use essentially the connection formulas and the asymptotic estimates of their connection coefficients given in Section 2.

Remind that the Bessel functions of the first kind J_ν of order ν, $\nu > -1$ are defined by

$$J_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2n+\nu}, \quad z\in\mathbb{C}.$$

Theorem 5. *Suppose that the sequence $\{p_n^{[2j]}(x)\}_{n\geq 0}$ satisfies uniformly on compact subsets of $\mathbb{C}$, for all $j = 0, 1, \ldots, r+1$, the following Mehler–Heine asymptotics*

$$\lim_n (-1)^n \frac{a_n^{1/2}}{b_n^j}\, p_n^{[2j]}\left(c+\frac{z^2}{b_n}\right) = z^{-(\nu+2j)} J_{\nu+2j}(2z), \tag{33}$$

where

$$a_n^{-1/2}\cong An^a, \quad b_n\cong Bn^b, \quad A,\ B,\ b>0, \quad \nu>-1, \tag{34}$$

and

$$2a+1 = b(\nu+1). \tag{35}$$

Then

$$\lim_n (-1)^n a_n^{1/2}\, q_n\left(c+\frac{z^2}{b_n}\right) = (-1)^{r+1} z^{-\nu} J_{\nu+2r+2}(2z), \tag{36}$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. From hypothesis (33) and using the Taylor expansion for the polynomials $p_n^{[2j]}(c+\frac{z^2}{b_n})$ at the point $z=c$, we have

$$(-1)^n \frac{a_n^{1/2}}{b_n^j}\sum_{k=0}^{n}\frac{\left(p_n^{[2j]}\right)^{(k)}(c)}{k!\, b_n^k} z^{2k} \longrightarrow \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+\nu+2j+1)} z^{2k}$$

uniformly on compact subsets of $\mathbb{C}$, which implies

$$\left(p_n^{[2j]}\right)^{(k)}(c) \cong (-1)^{n+k}\frac{a_n^{-1/2} b_n^{k+j}}{\Gamma(k+\nu+2j+1)}, \quad 0\leq k\leq n.$$

So, writing $f(x) = bx+a$, we have that f is a strictly increasing function with $2f(0)+1 = b(\nu+1)>0$ and so the polynomials $p_n^{[2j]}$ satisfy

$$\left(p_n^{[2j]}\right)^{(k)}(c) \cong (-1)^{n+k}\frac{A\,B^{k+j}}{\Gamma(k+\nu+2j+1)} n^{f(k+j)}, \quad 0\leq k\leq n. \tag{37}$$

Thus, since all the masses in the inner product are positive, from Theorem 2, we have

$$\lim_n \lambda_{j,n} = 0, \quad j = 0, 1, \ldots, r,$$
$$\lim_n \lambda_{r+1,n} = \lim_n \frac{q_n^{(r+1)}(c)}{p_n^{(r+1)}(c)}\frac{1}{A_{r+1}(r+1,n)\,(r+1)!} \neq 0.$$

Now, we will prove that $\lim_n \lambda_{r+1,n} = 1$. Indeed, from Corollary 1 and taking into account (35), we obtain

$$\frac{q_n^{(r+1)}(c)}{p_n^{(r+1)}(c)} \cong \prod_{j=0}^{r} \frac{(r+1-j)b}{(r+1+j)b+2a+1}$$
$$= \prod_{j=0}^{r} \frac{r+1-j}{r+j+\nu+2} = (r+1)!\,\frac{\Gamma(\nu+r+2)}{\Gamma(\nu+2r+3)}.$$

Then, from (21) and (37)

$$\lim_n \lambda_{r+1,n} = \frac{\Gamma(\nu+r+2)}{\Gamma(\nu+2r+3)} \lim_n \frac{p_n^{(r+1)}(c)}{p_{n-(r+1)}^{[2(r+1)]}(c)} = 1.$$

Finally, it is enough to use the connection formula (3) for the polynomials q_n to get their Mehler–Heine asymptotic. □

Remark 4. This theorem is also true when the point $c = \sup I$, if we change $(c + z^2/b_n)$ by $(c - z^2/b_n)$ and delete the factor $(-1)^n$ in formulas (33) and (36).

Remark 5. It is important to note that under the hypothesis of Theorem 5 we have that there exists the connection formula (3) and moreover all their coefficients tend to zero except the last one which tends to one.

Next, we prove that the hypothesis (33) for all $j = 0, 1, \ldots, r+1$ in the above theorem can be simplified by certain initial conditions involving only $j = 0, 1$. To do this, we will use the following well known formulas:

$$(x-c)p_{n-1}^{[j+2]}(x) = \frac{\gamma_{n-1}^{[j+2]}}{\gamma_n^{[j]}}\left[p_n^{[j]}(x) - \frac{p_n^{[j]}(c)}{K_{n-1}^{[j]}(c,c)}K_{n-1}^{[j]}(x,c)\right], \tag{38}$$

$$\left(\frac{\gamma_n^{[j]}}{\gamma_{n-1}^{[j+2]}}\right)^2 = 1 + \frac{(p_n^{[j]}(c))^2}{K_{n-1}^{[j]}(c,c)}, \tag{39}$$

(see for instance, [8]) and

$$p_n^{[j+1]}(x) = \frac{\gamma_n^{[j+1]}}{\gamma_n^{[j]}}\frac{K_n^{[j]}(x,c)}{p_n^{[j]}(c)}, \tag{40}$$

(see [6]), where $\{K_n^{[j]}(x,y)\}_{n\geq 0}$ denotes the sequence of kernels relative to μ_j.

Proposition 1. *Assume that the sequence $\{p_n^{[j]}(x)\}_{n\geq 0}$ satisfies the asymptotic formulas:*

$$\lim_n (-1)^n \frac{a_n^{1/2}}{b_n^{j/2}} p_n^{[j]}\left(c + \frac{z^2}{b_n}\right) = z^{-(\nu+j)} J_{\nu+j}(2z), \quad j = 0, 1, \tag{41}$$

uniformly on compact subsets of $\mathbb{C}$, where (34), (35) and the conditions

$$\lim_n \frac{\gamma_n}{\gamma_n^{[1]}}\frac{b_n^{1/2}}{n} = \frac{1}{b} \tag{42}$$

$$\lim_n \frac{\gamma_n^{[1]}}{\gamma_{n+1}}\frac{b_n^{1/2}}{n} = \frac{1}{b} \tag{43}$$

hold. Then the sequence $\{p_n^{[j]}(x)\}_{n\geq 0}$ satisfies the Mehler–Heine type formula (41) for all j.

Proof. The idea of the proof is to apply a recursive process such that, whenever we have a Mehler–Heine type formula for two consecutive indices j and $j+1$, and moreover the following condition

$$\lim_n \frac{\gamma_n^{[j]}}{\gamma_n^{[j+1]}}\frac{b_n^{1/2}}{n} = \frac{1}{b} \tag{44}$$

holds, then we get a Mehler–Heine type formula for $j+2$.

Indeed, suppose that we have (41) for j and $j+1$. Then, it can be derived that

$$\left(p_n^{[j]}\right)(c) \cong (-1)^n \frac{A\,B^{j/2}}{\Gamma(\nu+j+1)} n^{f(j/2)},$$

where $f(x) = b\,x + a$. Hence we have

$$\lim_n \frac{p_n^{[j]}(c)}{p_{n-1}^{[j]}(c)} = -1, \tag{45}$$

and, since $2f(j/2)+1=b(\nu+j+1)$ by Stolz criterion, we obtain

$$K_n^{[j]}(c,c)\cong\frac{n\left(p_n^{[j]}(c)\right)^2}{b(\nu+j+1)} \tag{46}$$

and from (39) we get

$$\gamma_n^{[j]}\cong\gamma_{n-1}^{[j+2]}. \tag{47}$$

Now, taking into account (38) evaluated at $c+z^2/b_n$ and (40), we get

$$\frac{(-1)^{n-1}a_n^{1/2}}{b_n^{(j+2)/2}}z^2\,p_{n-1}^{[j+2]}\left(c+\frac{z^2}{b_n}\right)=\frac{\gamma_{n-1}^{[j+2]}}{\gamma_n^{[j]}}\frac{(-1)^{n-1}a_n^{1/2}}{b_n^{j/2}}\left[p_n^{[j]}\left(c+\frac{z^2}{b_n}\right)-\frac{p_n^{[j]}(c)\,p_{n-1}^{[j]}(c)}{K_{n-1}^{[j]}(c,c)}\frac{\gamma_{n-1}^{[j]}}{\gamma_{n-1}^{[j+1]}}p_{n-1}^{[j+1]}\left(c+\frac{z^2}{b_n}\right)\right], \tag{48}$$

and besides, from (44)–(46) we have

$$-\frac{p_n^{[j]}(c)\,p_{n-1}^{[j]}(c)}{K_{n-1}^{[j]}(c,c)}\frac{\gamma_{n-1}^{[j]}}{\gamma_{n-1}^{[j+1]}}b_n^{1/2}\cong b(\nu+1+j)\frac{\gamma_{n-1}^{[j]}}{\gamma_{n-1}^{[j+1]}}\frac{b_n^{1/2}}{n}\longrightarrow\nu+1+j.$$

To get the Mehler–Heine type formula for $j+2$, we only need to take limits in (48) and use the relation satisfied by the Bessel functions (see [18])

$$J_{\nu-1}(z)+J_{\nu+1}(z)=2\,\nu\,z^{-1}J_\nu(z).$$

To conclude the proof, it remains to observe that the hypothesis of the proposition is enough to carry out the whole process. Indeed, the condition (47) obtained in each step leads to

$$\gamma_n^{[2j]}\cong\gamma_{n+j},\qquad\gamma_n^{[2j+1]}\cong\gamma_{n+j}^{[1]}.$$

So, from conditions (42) and (43), we have

$$\lim_n\frac{\gamma_n^{[2j]}}{\gamma_n^{[2j+1]}}\frac{b_n^{1/2}}{n}=\lim_n\frac{\gamma_{n+j}}{\gamma_{n+j}^{[1]}}\frac{b_n^{1/2}}{n}=\frac{1}{b},$$

and

$$\lim_n\frac{\gamma_n^{[2j+1]}}{\gamma_n^{[2j+2]}}\frac{b_n^{1/2}}{n}=\lim_n\frac{\gamma_{n+j}^{[1]}}{\gamma_{n+j+1}}\frac{b_n^{1/2}}{n}=\frac{1}{b}.$$

Therefore the required condition (44) is satisfied and we conclude the proof. □

3.1. Symmetric case

Let $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthonormal polynomials with respect to a symmetric measure μ and $\{q_n\}_{n\geq 0}$ the sequence of orthonormal polynomials with respect to the inner product (23) where all the masses are positive. Now, we will show Mehler–Heine asymptotics for $\{q_n\}_{n\geq 0}$, using again a symmetrization process.

As we can see in the following lemma, there is a remarkable difference with the previous case. We only need to have the Mehler–Heine formula for $\{p_n\}$ and an additional condition on the leading coefficients of $\{p_n\}$ to achieve a Mehler–Heine formula for $\{p_n^{[2j]}\}$ for all j.

Lemma 3. *Suppose that the sequence $\{p_n(x)\}_{n\geq 0}$ satisfies the Mehler–Heine type formulas:*

$$\lim_n(-1)^n a_n^{1/2}\,p_{2n}\left(\frac{z}{b_n}\right)=z^{-\nu}J_\nu(2z)$$
$$\lim_n(-1)^n a_n^{1/2}\,p_{2n+1}\left(\frac{z}{b_n}\right)=z^{-\nu}J_{\nu+1}(2z) \tag{49}$$

uniformly on compact subsets of $\mathbb{C}$, where

$$a_n^{-1/2}\cong An^a,\quad b_n\cong Bn^b,\quad A,\ B,\ b>0,\quad \nu>-1$$

with

$$2a+1=2b(\nu+1) \tag{50}$$

and moreover the following conditions hold

$$\lim_n\frac{\gamma_{2n}}{\gamma_{2n+1}}\frac{b_n}{n}=\frac{1}{2b}\qquad\text{and}\qquad\lim_n\frac{\gamma_{2n+1}}{\gamma_{2n+2}}\frac{b_n}{n}=\frac{1}{2b}. \tag{51}$$

Then, for all j,

$$\lim_n (-1)^n \frac{a_n^{1/2}}{b_n^j} p_{2n}^{[2j]}\left(\frac{z}{b_n}\right) = z^{-(\nu+j)} J_{\nu+j}(2z)$$
$$\lim_n (-1)^n \frac{a_n^{1/2}}{b_n^j} p_{2n+1}^{[2j]}\left(\frac{z}{b_n}\right) = z^{-(\nu+j)} J_{\nu+1+j}(2z)$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. We will use again the symmetrization process. From the hypothesis and the relation given in (22) we have

$$\lim_n (-1)^n a_n^{1/2} u_n\left(\frac{z^2}{b_n^2}\right) = z^{-\nu} J_\nu(2z)$$
$$\lim_n (-1)^n \frac{a_n^{1/2}}{(b_n^2)^{1/2}} u_n^{[1]}\left(\frac{z^2}{b_n^2}\right) = z^{-(\nu+1)} J_{\nu+1}(2z).$$

If we write $u_n^{[j]}(x) = \overline{\gamma}_n^{[j]} x^n + \cdots$ the hypothesis (51) reads as

$$\lim_n \frac{\overline{\gamma}_n}{\overline{\gamma}_n^{[1]}} \frac{(b_n^2)^{1/2}}{n} = \frac{1}{2b} \quad \text{and} \quad \lim_n \frac{\overline{\gamma}_n^{[1]}}{\overline{\gamma}_{n+1}} \frac{(b_n^2)^{1/2}}{n} = \frac{1}{2b}.$$

Besides, observe that the condition (50) is now the appropriate to apply Proposition 1 to $\{u_n(x)\}_{n\geq 0}$. Therefore, we obtain for all j

$$\lim_n (-1)^n \frac{a_n^{1/2}}{(b_n^2)^{j/2}} u_n^{[j]}\left(\frac{z^2}{b_n^2}\right) = z^{-(\nu+j)} J_{\nu+j}(2z),$$

uniformly on compact subsets of $\mathbb{C}$.

Thus, the result follows from (27). □

Theorem 6. *With the hypothesis of* Lemma 3, *we have the following Mehler–Heine type formulas for* $\{q_n(x)\}_{n\geq 0}$

$$\lim_n (-1)^n a_n^{1/2} q_{2n}\left(\frac{z}{b_n}\right) = (-1)^{r+1} z^{-\nu} J_{\nu+2r+2}(2z)$$
$$\lim_n (-1)^n a_n^{1/2} q_{2n+1}\left(\frac{z}{b_n}\right) = (-1)^{r+1} z^{-\nu} J_{\nu+2r+3}(2z) \tag{52}$$

both uniformly on compact subsets of $\mathbb{C}$.

Proof. From Lemma 3 and using the following relations described in the symmetrization process

$$p_{2n}^{[4j]}(x) = u_n^{[2j]}(x^2), \quad p_{2n+1}^{[4j]}(x) = x\,(u_n^*)^{[2j]}(x^2),$$

we get

$$\lim_n (-1)^n \frac{a_n^{1/2}}{(b_n^2)^j} u_n^{[2j]}\left(\frac{z^2}{b_n^2}\right) = z^{-(\nu+2j)} J_{\nu+2j}(2z).$$

and

$$\lim_n (-1)^n \frac{a_n^{1/2}/b_n}{(b_n^2)^j} (u_n^*)^{[2j]}\left(\frac{z^2}{b_n^2}\right) = z^{-(\nu+1+2j)} J_{\nu+1+2j}(2z).$$

To conclude the proof it is enough to check that we can apply Theorem 5 to the sequences $\{u_n(x)\}_{n\geq 0}$ and $\{u_n^*(x)\}_{n\geq 0}$. In this way, we deduce a Mehler–Heine formulas for s_n and s_n^* and therefore, using (24) for q_{2n} and q_{2n+1}, respectively. □

Remark 6. Note that under the hypothesis of Theorem 6 we have that there exist the two connection formulas (28) and (29) where all their coefficients tend to zero except the last one in each case that tends to one.

3.2. Asymptotic zero distribution

The results above allow us to deduce some additional information about the asymptotic zero distribution of $\{q_n(x)\}_{n\geq 0}$ in terms of the zeros of the known special functions, more precisely the Bessel functions.

Let $\{p_n(x)\}_{n\geq 0}$ be the sequence of polynomials orthonormal with respect to the measure μ supported on an interval I. Assume they satisfy, in a neighborhood of the point $c = \inf I$, the following Mehler–Heine formula

$$\lim_n (-1)^n a_n^{1/2} p_n\left(c + \frac{z^2}{b_n}\right) = z^{-\nu} J_\nu(2z), \quad \nu > -1,$$

uniformly on compact subsets of $\mathbb{C}$. This asymptotic behavior, by Hurwitz's theorem, gives an additional information of the zeros of $\{p_n(x)\}_{n\geq 0}$. More precisely, if we denote by $x_{k,n}, 1 \leq k \leq n$ the zeros of the polynomial $p_n(x)$ in increasing order, and taking into account that the entire function $z^{-\nu} J_\nu(2z)$ does not vanish at the origin, we can deduce that for all k

$$\lim_n b_n(x_{k,n} - c) = \left(\frac{1}{2} j_k^\nu\right)^2,$$

where j_k^ν is the kth positive zero of J_ν. Similar result can be obtained if we take $c = \sup I$ where we rename the zeros in decreasing order.

Concerning the zeros of the discrete Sobolev orthogonal polynomials q_n, we know that all of them are real and simple and at least $n-(r+1)$ are in the interior of the interval I. Although in a similar way as before the zeros converge to c, in the following proposition we show a remarkable difference on the convergence acceleration of the zeros to c.

Proposition 2. *Let $\{\xi_{k,n}\}_{k=1}^n$ be the zeros of $q_n(x)$ in increasing order. Then, under the hypothesis of Theorem 5, we have*

$$\lim_n b_n(\xi_{k,n} - c) = 0, \qquad 1 \leq k \leq r+1,$$
$$\lim_n b_n(\xi_{k,n} - c) = \left(\frac{1}{2} j_{k-r-1}^{\nu+2r+2}\right)^2, \qquad k \geq r+2.$$

Proof. It is sufficient to observe that the limit function in (36) has a zero in the origin of multiplicity $r+1$. □

Remark 7. In the case that the measure μ is symmetric, since the zeros of the polynomials q_n are symmetric, then analogous information about convergence acceleration of the zeros to $c = 0$ can be obtained from formula (52).

4. Examples

Let $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ be the sequences of orthonormal polynomials with respect to the measure $d\mu(x)$ and the discrete Sobolev inner product (1) (or (23) if μ is symmetric) with all the masses are positive real numbers.

As we have seen in Section 3, under the hypothesis of Theorem 5 or 6 for μ symmetric, we can assure that there exist the connection formulas (3) or (28) and (29) where all the coefficients tend to zero except the last one which tends to one. Moreover we have the Mehler–Heine asymptotic (36) or (52), respectively.

Here, we show some examples of discrete Sobolev polynomials for which the hypothesis of Theorem 5 or 6 hold and therefore we get their Mehler–Heine asymptotic in a neighborhood of the point c.

4.1. Laguerre weight

Let $d\mu(x) = x^\alpha e^{-x}\, dx, \alpha > -1, I = [0, \infty), c = 0$ and $p_n(x) = l_n^\alpha(x)$ the Laguerre orthonormal polynomials. Since $p_n^{[2j]}(x) = l_n^{\alpha+2j}(x)$, formula (33) holds with

$$a_n^{-1/2} = n^{\alpha/2}, \quad b_n = n, \quad \nu = \alpha$$

(see [18]), and therefore we get (36).

4.2. Nevai's class

Let μ be in the well known Nevai's class, $M(0, 1)$ and $p_n(x)$ the orthonormal polynomials with respect to μ. We would like to remark (see, for example, Theorem 10 of [15]) that every measure μ with $\operatorname{supp} \mu = [-1, 1]$ and $\mu' > 0$ a.e. (μ' denotes the absolutely continuous part of μ) belongs to $M(0, 1)$.

In Theorem 1 of [5], for $\mu \in M(0, 1)$, a sufficient condition is given to obtain a Mehler–Heine asymptotic for $p_n(x)$ in a neighborhood of the point 1 involving Bessel functions. Notice that, μ_{2j} also belongs to $M(0, 1)$ and so, if the polynomials $p_n^{[2j]}(x)$ satisfy, for all $j = 0, 1, \ldots, r+1$, the following condition given in [5] for $\nu > -1$

$$\frac{p_{n+1}^{[2j]}(1)}{p_n^{[2j]}(1)} \cong 1 + \frac{\nu + 2j + 1/2}{n} + o(1/n), \quad n \to \infty,$$

then we have that formula (33) holds with

$$a_n^{-1/2} = 2^{-\nu} n^{\nu+1/2}, \quad b_n = n^2/2. \tag{53}$$

Thus, by Theorem 5 we obtain the Mehler–Heine asymptotic formula similar to (36) with $c = 1$. Similar results can be obtained for $c = -1$.

4.2.1. Modified Jacobi weight

Let $d\mu(x) = h(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx, \alpha, \beta > -1, I = [-1, 1]$ and $h(x)$ a real analytic and positive function on I.

Recently in [7] the author, using Theorem 1 in [5], gives a Mehler–Heine asymptotics for orthonormal polynomials $p_n(x) = p_n^{\alpha,\beta}(x)$ with respect to $d\mu(x)$ with the restrictions $\alpha > 0$ and $\beta > 0$.

Thus, for $c = 1$, since $p_n^{[2j]}(x) = p_n^{\alpha+2j,\beta}(x)$ formula (33) holds with $\nu = \alpha$ and the values of a_n and b_n are given by (53). Then, we get the analogous formula to (36) with the corresponding changes due to $c = \sup I$, see Remark 4.

On the other hand, for $c = -1$, since $p_n^{[2j]}(x) = p_n^{\alpha,\beta+2j}(x)$ formula (33) holds with the same values of a_n and b_n but now with $\nu = \beta$. Therefore, we can assure that (36) holds.

4.2.2. Jacobi weight

Let $d\mu(x) = 2^{\alpha-\beta}(1-x)^{\alpha}(1+x)^{\beta}\,dx, \alpha > -1, \beta > -1, I = [-1, 1]$. The same asymptotics are true for $c = \pm 1$ but now there are no restrictions on the parameters α and β, that is $\alpha, \beta > -1$ (see [18]).

4.3. Generalized Freud weight

Let $d\mu(x) = x^{2m}\exp(-2|x|^{\alpha})\,dx, \alpha > 1, m \in \mathbb{N} \cup \{0\}, I = \mathbb{R}, c = 0$ and $p_n^{[2m]}(x)$ the generalized Freud orthonormal polynomials with respect to $d\mu(x)$. Note that for $m = 0$ we have the Freud polynomials.

A Mehler–Heine asymptotic near to 0 for $p_n^{[2m]}$ was obtained in Theorem 2 of [3]. From there, with a change of notation, we have that for all j

$$\lim_n (-1)^n \frac{a_n^{1/2}}{b_n^{m+j}}\, p_{2n}^{[2m+2j]}\left(\frac{z}{b_n}\right) = z^{-(\nu+j)} J_{\nu+j}(2z)$$
$$\lim_n (-1)^n \frac{a_n^{1/2}}{b_n^{m+j}}\, p_{2n+1}^{[2m+2j]}\left(\frac{z}{b_n}\right) = z^{-(\nu+j)} J_{\nu+1+j}(2z)$$

uniformly on compact subsets of $\mathbb{C}$ with $\nu = m - 1/2$ and

$$a_n^{-1/2} = \sqrt{2}\,(2c_{\alpha})^{-1/2\alpha}\, n^{-1/2\alpha}, \quad b_n = \frac{\alpha}{\alpha-1}(2c_{\alpha})^{-1/\alpha}\, n^{1-1/\alpha},$$

where $c_{\alpha} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha/2)}{\Gamma((\alpha+1)/2)}$.

Thus, in a similar way of Theorem 6, we get the following Mehler–Heine formulas for $\{q_n(x)\}_{n\geq 0}$,

$$\lim_n (-1)^n \frac{a_n^{1/2}}{b_n^{m}}\, q_{2n}\left(\frac{z}{b_n}\right) = (-1)^{r+1} z^{-\nu} J_{\nu+2r+2}(2z)$$
$$\lim_n (-1)^n \frac{a_n^{1/2}}{b_n^{m}}\, q_{2n+1}\left(\frac{z}{b_n}\right) = (-1)^{r+1} z^{-\nu} J_{\nu+2r+3}(2z)$$

uniformly on compact subsets of $\mathbb{C}$.

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