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On Symmetric Differential Operators Associated with Sobolev Orthogonal Polynomials: A Characterization

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Abstract. Given the Sobolev bilinear form

$$(f, g)_S = \langle u_0, fg\rangle + \langle u_1, f'g'\rangle,$$

with u_0 and u_1 linear functionals, a characterization of the linear second–order differential operators with polynomial coefficients, symmetric with respect to $(\cdot,\cdot)_S$ in terms of u_0 and u_1 is obtained. In particular, several interesting functionals u_0 and u_1 are considered, recovering as particular cases of our study, results already known in the literature.

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1. Introduction

As is well known, polynomials orthogonal with respect to a Sobolev inner product, that is, an inner product involving derivatives, do not satisfy the same general properties as those orthogonal with respect to a standard inner product. The interest in studying these families of orthogonal polynomials lies not only in their connections with topics such as least squares data fitting, spectral theory of ordinary differential equations, Fourier expansions, but also in their applications to the theory of orthogonal polynomials. For instance, it has been shown that some families of classical polynomials as Laguerre ($L_n^{(\alpha)}(x)$) or Jacobi ($P_n^{(\alpha,\beta)}(x)$) polynomials which, for some values of their parameters, are not orthogonal in the standard sense. However, they are orthogonal with respect to a Sobolev inner product (see Alfaro *et al.*

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(1999), Álvarez de Morales *et al.* (1998b), Kwon *et al.* (1995, 1996, 1998), Pérez *et al.* (1996)).

One of these properties is the existence of a recurrence relation: standard orthogonal polynomials satisfy a three-term recurrence relation as a consequence of the symmetry of the operator associated to the multiplication by x. However, multiplications by polynomials are not symmetric operators for many Sobolev inner products (see Evans *et al.* (1995)) and therefore the corresponding orthogonal polynomials do not satisfy an algebraic recurrence relation. We can avoid this unpleasant situation looking for a linear operator $\mathcal{F}$ on the linear space $\mathbb{P}$ of real polynomials, symmetric with respect to a Sobolev bilinear form (for a preliminary comment, see Danese (1976)). So, $\mathcal{F} Q_n$ can be expressed as a linear combination of polynomials Q_h, beeing the number of terms independent of n and (Q_n) the sequence of monic Sobolev orthogonal polynomials. The derivatives in the bilinear form suggest the use of a differential operator. This has been done, for instance, in Alfaro *et al.* (1999), Álvarez de Morales *et al.* (1998a, 1998b), Evans *et al.* (1995), Marcellán *et al.* (1994b, 1995, 1996a, 1996b), Pérez *et al.* (1998), where differential operators with polynomial coefficients, symmetric with respect to Sobolev bilinear forms have been constructed. Because of the symmetric character of the operator, a differential recurrence relation for the Sobolev orthogonal polynomials can be deduced. The operator $\mathcal{F}$ has also been used to obtain several properties about zeros of the Sobolev polynomials in some particular cases (see Marcellán *et al.* (1994b, 1996a)).

The aim of this paper is to characterize the linear second-order differential operators with polynomial coefficients $\mathcal{F} = p_0 I + p_1 D + p_2 D^2$ which are symmetric with respect to the Sobolev bilinear form defined by $(f, g)_S = \langle u_0, fg\rangle + \langle u_1, f'g'\rangle$, where u_0 and u_1 are linear functionals on $\mathbb{P}$.

For the particular case when u_0 and u_1 are defined by positive Borel measures, and $p_1 = p_2 = 0$, that is, $\mathcal{F}$ is a multiplication operator, this problem has been solved in Evans *et al.* (1995), where it has been proved that $\mathcal{F}$ is symmetric if, and only if, u_1 is given by a discrete positive measure.

The paper is organized as follows: In Section 2, we obtain the main result, namely the characterization of the symmetry of $\mathcal{F} = p_0 I + p_1 D + p_2 D^2$ in terms of the functionals u_0 and u_1. As a consequence, a substitute of the recurrence relation is derived. Section 3 is devoted to analyze several situations for different functionals u_0 and u_1; all the results known in the literature are recovered as particular cases of our analysis. In Section 4, the case when $\mathcal{F}$ is a degree preserving operator is considered.

2. The Main Result

Let $\mathbb{P}$ be the linear space of real polynomials, u_0, u_1 linear functionals on $\mathbb{P}$, and

$$(f, g)_S = \langle u_0, fg\rangle + \langle u_1, f'g'\rangle,$$

a bilinear form on $\mathbb{P}$. We define a second-order differential operator

$$\mathcal{F} = p_0 I + p_1 D + p_2 D^2, \tag{1}$$

where p_0, p_1, p_2 are arbitrary polynomials and I, D denotes the identity and the derivative operator, respectively.

We say that $\mathcal{F}$ is *symmetric* with respect to $(\cdot, \cdot)_S$ if

$$(\mathcal{F} f, g)_S = (f, \mathcal{F} g)_S, \quad \text{for all} \quad f, g \in \mathbb{P}. \tag{2}$$

Notice that if c is a constant, the operator $\mathcal{F} = cI$ is trivially symmetric. Moreover, if $\mathcal{F}$ is symmetric, then $c\mathcal{F}$ and $\mathcal{F} + cI$ are symmetric too. Then, we will consider expression (1) for p_1 and p_2 given, and p_0 up to an additive constant.

The goal of this section is to characterize the symmetry of such a operator $\mathcal{F}$ in terms of u_0, u_1.

Let us recall the definition of some useful operations for linear functionals u on $\mathbb{P}$ (see, for instance, Marcellán *et al.* (1994a)):

- Given $p \in \mathbb{P}$, we define the *left multiplication* of the functional u by the polynomial p as the functional such that $\langle pu, f\rangle = \langle u, pf\rangle$, for all $f \in \mathbb{P}$.
- The (distributional) *derivative* of the functional u, is the functional Du such that $\langle Du, f\rangle = -\langle u, f'\rangle$ for all $f \in \mathbb{P}$.

THEOREM 1. *Let us consider the Sobolev bilinear form*

$$(f, g)_S = \langle u_0, fg\rangle + \langle u_1, f'g'\rangle, \tag{3}$$

where u_0, u_1 are nonzero linear functionals. Let p_0, p_1, p_2 be polynomials and $\mathcal{F} = p_0 I + p_1 D + p_2 D^2$ a linear differential operator, nontrivially symmetric. Then $\mathcal{F}$ is symmetric with respect to (3) *if and only if the linear functionals u_0, u_1 satisfy*

$$p_2 D u_0 + (p_2' - p_1) u_0 + p_0' u_1 = 0, \tag{4}$$

and

$$p_2 D u_1 = p_1 u_1. \tag{5}$$

Moreover, in this situation the functional

$$p_2 u_0 + p_0 u_1, \tag{6}$$

is a solution of Equation (5).

Proof. Let assume that the operator $\mathcal{F} = p_0 I + p_1 D + p_2 D^2$ is symmetric, that is, it satisfies (2). Then, we have

$$\begin{aligned}&\langle u_0, (p_1 f' + p_2 f'')g\rangle + \langle u_1, (p_0' f + p_1 f'' + p_2' f'' + p_2 f''')g'\rangle \\ &\quad = \langle u_0, f(p_1 g' + p_2 g'')\rangle + \langle u_1, f'(p_0' g + p_1 g'' + p_2' g'' + p_2 g''')\rangle\end{aligned} \tag{7}$$

for all $f, g \in \mathbb{P}$. Equation (7) for $f = 1$ and $g \in \mathbb{P}$ gives

$$\langle u_1, p_0' g' \rangle = \langle u_0, p_1 g' + p_2 g'' \rangle,$$

and we obtain (4). Observe that if a linear functional u satisfies $Du = 0$, then $u = 0$ (see Lemma 2.3 in Kwon *et al.* (1996)).

To deduce (5), it suffices to replace $f = x$ in (7) and from (4) we have, for all $g \in \mathbb{P}$, $\langle p_1 u_1 - p_2 D u_1, g'' \rangle = 0$. Conversely, to derive the symmetry of $\mathcal{F}$ from (4) and (5), previously we need to obtain the expressions $(p_0 f, g)_S$, $(p_1 f, g)_S$ and $(p_2 f, g)_S$ in terms of u_0, u_1 and $\mathcal{F}$. In fact, from (4), we have

$$\begin{aligned}
(p_0 f, g)_S &= \langle u_0, p_0 f g \rangle + \langle u_1, p_0' f g' \rangle + \langle u_1, p_0 f' g' \rangle \\
&= \langle u_0, p_0 f g \rangle + \langle p_1 u_0 - D(p_2 u_0), f g' \rangle + \langle u_1, p_0 f' g' \rangle \\
&= \langle u_0, p_0 f g \rangle + \langle u_0, p_1 f g' \rangle + \langle p_2 u_0, (f g')' \rangle + \langle p_0 u_1, f' g' \rangle \\
&= \langle u_0, f \mathcal{F} g \rangle + \langle p_2 u_0 + p_0 u_1, f' g' \rangle.
\end{aligned}$$

In a similar way, (4) and (5) give

$$(p_1 f, g)_S = -\langle p_2 u_0 + p_0 u_1, (fg)' \rangle - \langle D u_1, f \mathcal{F} g \rangle.$$

Using (5), we obtain

$$(p_2 f, g)_S = \langle p_2 u_0 + p_0 u_1, fg \rangle - \langle u_1, f \mathcal{F} g \rangle.$$

Thus, by straightforward calculations, we get

$$(\mathcal{F} f, g)_S = (p_0 f + p_1 f' + p_2 f'', g)_S = (f, \mathcal{F} g)_S.$$

Finally, a simple computation shows that the functional $p_2 u_0 + p_0 u_1$ is a solution of the distributional equation $p_2 D v = p_1 v$. □

Remark. Sobolev bilinear forms like (3) are usually called *diagonal*. The nondiagonal case (see Álvarez de Morales *et al.* (1998a), Marcellán *et al.* (1996b)) can be expressed as

$$(f, g)_S = \langle u_{0,0}, fg \rangle + \langle u_{0,1}, f g' \rangle + \langle u_{1,0}, f' g \rangle + \langle u_{1,1}, f' g' \rangle,$$

where $u_{0,1} = u_{1,0}$, in order to preserve the symmetry of the bilinear form. Therefore, we get

$$(f, g)_S = \langle u_{0,0} - D u_{0,1}, fg \rangle + \langle u_{1,1}, f' g' \rangle,$$

and the nondiagonal Sobolev bilinear form reduces to the diagonal one.

In the sequel, we will assume that the bilinear form (3) is regular, that is, all the principal minors of the associated Gram matrix with respect to the canonical basis $\{x^n; n \geqslant 0\}$, are nonzero. Then, there exists a sequence of monic polynomials, namely $\{Q_n\}_n$, orthogonal with respect to (3). In this situation, we can deduce some consequences of the previous theorem.

Let assume that u_0, u_1 satisfy (4) and (5), and $\mathcal{F} = p_0 I + p_1 D + p_2 D^2$ is the symmetric operator associated with (3). We can ask about the degree of the polynomial $\mathcal{F} x^n$, $n \in \mathbb{N}$. Clearly it depends on the polynomials p_0, p_1 and p_2.

COROLLARY 1. *If the Sobolev bilinear form* (3) *is regular, then*

(i) *There exists* $n_0 \in \mathbb{N}$ *such that* $\deg \mathcal{F} x^{n_0} \geqslant n_0$.
(ii) *For all* $n \in \mathbb{N}$ *except at most for two values of* n, $\deg \mathcal{F} x^n = n + r$ *where* $r = \max\{\deg p_0, \deg p_1 - 1, \deg p_2 - 2\} \geqslant 0$.

Proof. (i) If $\deg \mathcal{F} x^n < n$, for all $n \in \mathbb{N}$, we can consider the expansion

$$\mathcal{F} Q_n = \sum_{i=0}^{n-1} a_{n,i} Q_i.$$

Thus, using the symmetry of $\mathcal{F}$, we have

$$a_{n,i} = \frac{(\mathcal{F} Q_n, Q_i)_S}{(Q_i, Q_i)_S} = \frac{(Q_n, \mathcal{F} Q_i)_S}{(Q_i, Q_i)_S} = 0, \quad i = 0, 1, \ldots, n-1.$$

(ii) Let d_0, d_1, d_2 be the degree of the polynomials p_0, p_1, p_2, respectively and

$$p_0(x) = \sum_{i=0}^{d_0} a_i x^i, \qquad p_1(x) = \sum_{j=0}^{d_1} b_j x^j \quad \text{and} \quad p_2(x) = \sum_{k=0}^{d_2} c_k x^k.$$

Including, if necessary, some zero coefficients in the expansions of the polynomials p_0, p_1, p_2, we can write

$$\begin{aligned} \mathcal{F} x^n &= \sum_{i=0}^{r} a_i x^{i+n} + \sum_{j=0}^{r+1} n b_j x^{j+n-1} + \sum_{k=0}^{r+2} n(n-1) c_k x^{k+n-2} \\ &= (a_r + n b_{r+1} + n(n-1) c_{r+2}) x^{n+r} + \text{lower degree terms}, \end{aligned}$$

and the result follows. □

Remark. Observe that, as a consequence, if the Sobolev bilinear form (3) is regular, then $\mathcal{F}$ never reduces the degree of *all the polynomials*, and $r \geqslant 0$.

COROLLARY 2 (Difference–Differential Relation). *For every* $n \geqslant r$, *where* r *is as in Corollary* 1, *the following relation holds:*

$$\mathcal{F} Q_n = \sum_{i=n-r}^{n+r} \alpha_{n,i} Q_i,$$

where

$$\alpha_{n,n+r} = \frac{(\mathcal{F} Q_n, Q_{n+r})_S}{(Q_{n+r}, Q_{n+r})_S} \neq 0$$

except at most for two values of n.

Proof. Consider the Fourier expansion of the polynomial $\mathcal{F} Q_n$ in terms of Q_n, $\mathcal{F} Q_n = \sum_{i=0}^{n+r} \alpha_{n,i} Q_i$. Then, the result follows from the symmetry of the operator $\mathcal{F}$ and Corollary 1(ii). □

A linear functional u on $\mathbb{P}$ will be called a *Pearson* functional if there exist two polynomials ϕ and ψ, non simultaneously zero, such that

$$D(\phi u) = \psi u. \tag{8}$$

Let us recall that a linear functional u is semiclassical (see Hendriksen *et al.* (1985)), if it is regular, i.e., there exists a sequence of monic orthogonal polynomials with respect to the linear functional u, and it satisfies (8).

If a linear functional u is semiclassical (regular and Pearson) with polynomials ϕ and ψ, then $\deg \psi \geqslant 1$ and $\phi \not\equiv 0$.

Observe that there exist linear functionals which are Pearson and non regular, for instance, δ_a ($\langle \delta_a, f \rangle = f(a)$) and $(x-a)^{-1}u$, where u is a regular functional

$$\left(\langle (x-a)^{-1}u, f \rangle = \langle u, \frac{f(x) - f(a)}{x-a} \rangle \right).$$

Remark. By Theorem 1, if $\mathcal{F} = p_0 I + p_1 D + p_2 D^2$ is symmetric (non trivially symmetric, i.e., $\mathcal{F} \not\equiv cI$, for any constant c) with respect to (3) then the functional u_1 is Pearson and besides if p_0 is constant, u_0 is also Pearson.

3. The Case where u_0 and u_1 are Given by Positive Borel Measures

Let u be a linear functional given by a positive Borel measure μ on the real line $\mathbb{R}$, that is, for all $f \in \mathbb{P}$, we have

$$\langle u, f \rangle = \int_{\mathbb{R}} f \, \mathrm{d}\mu. \tag{9}$$

Assume that all the moments exist and are finite.

Recall that the *spectrum* of μ (see Chihara (1978), Chapter 2) is defined by

$$S(\mu) = \{x; \mu(x-\epsilon, x+\epsilon) > 0 \text{ for all } \epsilon > 0\}.$$

If $S(\mu)$ is an infinite set, the linear functional u defined by the relation (9) is positive definite.

Conversely, if u is a positive definite linear functional, then there exists a positive Borel measure μ with infinite spectrum such that u can be represented as (9) (Chihara (1978), p. 56).

If $S(\mu)$ is finite, that is, $S(\mu) = \{x_1, x_2, \ldots, x_N\}$, then $\mu = \sum_{j=1}^{N} \alpha_j \delta_{x_j}$, where $\alpha_j > 0$, for $j = 1, 2, \ldots, N$, and δ_{x_j} denotes the Dirac mass measure supported on $\{x_j\}$.

In this section, we consider the case when u_0 and u_1 are given by positive Borel measures μ_0 and μ_1 on $\mathbb{R}$, and the Sobolev bilinear form (3) is an inner product. Thus, the spectrum of at least one of the measures is an infinite set. In this situation, we can write (3) as

$$(f, g)_S = \int_{\mathbb{R}} fg \, \mathrm{d}\mu_0 + \int_{\mathbb{R}} f'g' \, \mathrm{d}\mu_1, \tag{10}$$

for all polynomials f and g.

According to $S(\mu_0)$ and $S(\mu_1)$ being either finite or infinite, we obtain the corresponding operator $\mathcal{F}$ associated with (10). In every case, we recover the results about $\mathcal{F}$ already known in the literature.

3.1. $S(\mu_0)$ FINITE

Let $S(\mu_0)$ be finite and nonempty, then $\mu_0 = \sum_{i=1}^{N} \alpha_i \delta_{x_i}$, where $N \geqslant 1$, x_i are distinct real points and $\alpha_i > 0$ for $i = 1, 2, \ldots, N$. Since (10) is an inner product, $S(\mu_1)$ is infinite and u_1 is positive definite.

Using Theorem 1, if u_0 and u_1 satisfy (4) and (5), u_1 is semiclassical and the corresponding $\mathcal{F}$ is symmetric with respect to (10).

Define $q(x) = \prod_{i=1}^{N}(x - x_i)$. In this way, $qu_0 = 0$, and $q'(x_i) \neq 0$, $i = 1, 2, \ldots, N$.

Multiplying (4) by q, we get $-p_2 q' u_0 + p_0' q u_1 = 0$, which leads to $p_0' q^2 u_1 = 0$. Therefore $p_0' = 0$ and by (4), u_0 is Pearson with equation $D(p_2 u_0) = p_1 u_0$. The polynomials p_2 and p_1 contain the factor q. Indeed, from last equation we have $qD(p_2 u_0) = 0$, that is, $p_2 q' u_0 = 0$, and $q'(x_i) p_2(x_i) = 0$, $i = 1, 2, \ldots, N$, and q divides p_2. From $p_2 u_0 = 0$ and (4), q also divides p_1.

The linear operator $\mathcal{F}$ can be expressed as

$$\mathcal{F} = p_0 I + q\big[\tilde{p}_1 D + \tilde{p}_2 D^2\big],$$

where p_0 is a constant, and $p_i = q\tilde{p}_i$, $i = 1, 2$.

Observe that this result includes as a particular case, the previously one obtained in Alfaro *et al.* (1999).

3.2. $S(\mu_1)$ FINITE

Orthogonal polynomials associated with (10) when μ_1 is a finite spectrum positive Borel measure (the so-called Sobolev-type) have been studied exhaustively by several authors.

In 1995, Evans *et al.* (1995) gave a characterization of the Sobolev orthogonal polynomials satisfying a recurrence relation. In particular, they obtain that there exists a nonconstant polynomial h such that $(hf, g)_S = (f, hg)_S$ if and only if μ_1 has finite spectrum. Moreover, $\deg h \geqslant 2$ and h is not necessarily of minimal degree.

In our case, taking $\mathcal{F} = p_0 I$, with p_0 a nonconstant polynomial, Theorem 1 says that $\mathcal{F}$ is symmetric with respect to the Sobolev inner product (10) if and only if $p_0' u_1 = 0$. Therefore $\mu_1 = \sum_{j=1}^{N} \alpha_j \delta_{x_j}$, for some integer N, where $\alpha_j \geqslant 0$ (at least one must be nonzero), and $\{x_j\}_1^N$ are the distinct real roots of p_0'. In this case, $\deg p_0 \geqslant 2$.

Finally, we can observe that p_0 could be not of minimal degree, in the sense that we can consider only the nonzero α's, and we will obtain another polynomial in the same conditions.
Moreover, we point out that the *difference-differential relation* (Corollary 2) for the orthogonal polynomials associated with (10) is a recurrence relation with at least five terms.

3.3. $S(\mu_0)$ AND $S(\mu_1)$ INFINITE

In this case, the linear functionals u_0 and u_1 are positive definite, and from Equation (5), u_1 is semiclassical with $p_2 \not\equiv 0$, and $\deg(p_1 + p_2') \geqslant 1$.

Here, we will analyze two cases from the literature: The so-called *semiclassical case*, introduced and studied in Marcellán *et al.* (1995), and the *coherent pairs*, concept introduced by Iserles *et al.* (1991), and developed by several authors.

Finally, a study when u_0 and u_1 are defined from weight functions is given.

3.3.1. *The Semiclassical Case*

The existence of a linear operator $\mathcal{F}$ symmetric with respect to (10) when u_0 and u_1 are positive definite functionals satisfying

$$Au_0 = Bu_1, \tag{11}$$

where A and B are nonzero polynomials, and

$$D(\phi_1 u_1) = \psi_1 u_1, \tag{12}$$

is shown in Marcellán *et al.* (1995). Since u_1 is regular, it is semiclassical, $\phi_1 \not\equiv 0$ and $\deg \psi_1 \geqslant 1$. Moreover, u_0 is semiclassical because of (11), in fact u_0 is semiclassical if and only if u_1 is semiclassical. In Marcellán *et al.* (1995), the authors proved that the linear operator

$$\mathcal{F} = B\phi_1 I - A(\psi_1 - \phi_1')D - A\phi_1 D^2 \tag{13}$$

is symmetric with respect to the Sobolev inner product (10).

The hypothesis (11) and (12), are a particular case of the hypothesis (4)–(5) of Theorem 1.

In fact, multiplying (11) by ϕ_1, and taking derivatives, we obtain

$$A\phi_1 Du_0 + [(A\phi_1)' - A(\psi_1 - \phi_1')]u_0 - (B\phi_1)'u_1 = 0. \tag{14}$$

The above expression is equal to (4), where

$$p_0 = B\phi_1, \quad p_1 = -A(\psi_1 - \phi_1'), \quad p_2 = -A\phi_1,$$

and we recover (13).

Observe that (11) is a particular case of (6), since $Au_0 - Bu_1 = 0$ is always a solution of (5)–(12).

Now, we can ask for the reciprocal. Let us asssume that we have a linear operator $\mathcal{F}$, symmetric with respect to (10), whose explicit expression is given by (13). Applying Theorem 1, we obtain a wider class of linear functionals that includes, as a particular case, the originals u_0 and u_1.

Remark that the case $u_0 = u$, $u_1 = \lambda u$, $\lambda > 0$, considered in Marcellán *et al.* (1994b, 1996a), where u is either the classical Gegenbauer or the classical Laguerre functional, respectively, is a particular case of the semiclassical one, with $A = \lambda$, and $B = 1$.

3.3.2. *Coherent Pairs*

Coherent pairs have been the subject of a great number of papers during the last few years. This concept for positive definite functionals u_0 and u_1, was introduced in Iserles *et al.* (1991).

In Marcellán *et al.* (1995), it has been proved that coherent pairs are a particular case of 3.3.1. In fact, u_1 is semiclassical with $D(\phi_1 u_1) = \psi_1 u_1$ and $\phi_1 u_0 = Bu_1$ where $\deg \phi_1 \leqslant 3$, $\deg \psi_1 \leqslant 2$ and $\deg B = 2$.

Using Theorem 1, in this case, we deduce that

$$\mathcal{F} = B\phi_1 I - \phi_1(\psi_1 - \phi_1')D - \phi_1^2 D^2,$$

is symmetric with respect to (10).

Recently, Meijer (1997) has shown that if $\{u_0, u_1\}$ is a coherent pair, then at least either u_0 or u_1 is a Laguerre or Jacobi functional.

3.3.3. *Weight Functions*

To conclude this section, we study the case when the positive linear functionals u_0 and u_1, are defined by means of weight functions w_0 and w_1, respectively, i.e., for all polynomials $f \in \mathbb{P}$, we have

$$\langle u_i, f\rangle = \int_{\mathbb{R}} f w_i \, \mathrm{d}x, \quad i = 0, 1.$$

In this situation, Equations (4) and (5) can be written as

$$p_2 w_0' + (p_2' - p_1) w_0 + p_0' w_1 = 0, \tag{15}$$
$$p_2 w_1' - p_1 w_1 = 0. \tag{16}$$

Remark that $p_2 \not\equiv 0$, since u_1 is positive definite. Solving these differential equations, directely, we obtain

$$w_1 = k \exp \int \frac{p_1}{p_2}, \qquad w_0 = \frac{-p_0 + c}{p_2} w_1,$$

where $k > 0$, and $c \in \mathbb{R}$. This is a particular case of the semiclassical one.

4. Degree-Preserving Operators

Corollary 1 shows how the operator $\mathcal{F}$ increases the degree of the polynomials when the Sobolev bilinear form (3) is regular. In this section we study when $\mathcal{F}$ is a degree preserving operator, that is, when $\deg \mathcal{F} f = \deg f$ for all $f \in \mathbb{P}$, or equivalently, $\deg \mathcal{F} x^n = n$ for every nonnegative integer n.

This situation is very interesting, since if $\mathcal{F}$ preserves the degree of the polynomials, as a consequence of the difference-differential relation (Corollary 2), the corresponding Sobolev orthogonal polynomials are the eigenfunctions of the differential operator $\mathcal{F}$, i.e.,

$$\mathcal{F} Q_n = \lambda_n Q_n, \quad n \geqslant 0,$$

that is, they satisfy a second-order differential equation

$$p_2 y'' + p_1 y' + p_0 y = \lambda_n y.$$

First, notice that it is easy to express this fact in terms of the polynomial coefficients of the operator $\mathcal{F}$:

LEMMA 1. *Set $\mathcal{F} = p_0 I + p_1 D + p_2 D^2$, then $\mathcal{F}$ is a degree preserving operator if and only if* $\deg[x^2 p_0 + n x p_1 + n(n-1) p_2] = 2$ *holds for every* $n \geqslant 0$.

This characterization implies that p_0 is a nonzero constant, $\deg p_1 \leqslant 1$, and $\deg p_2 \leqslant 2$.

Now, we will consider (3) as a regular bilinear form and we will deduce necessary and sufficient conditions about u_0 and u_1 in order to obtain a symmetric linear second-order differential operator $\mathcal{F}$ with polynomial coefficients preserving the degree. Observe that, whenever p_0 is constant, Equations (4) and (5) can be written as

$$D(p_2 u_0) = p_1 u_0, \tag{17}$$
$$D(p_2 u_1) = (p_1 + p_2') u_1. \tag{18}$$

THEOREM 2. *Let u_0 and u_1 be two regular linear functionals. Then, $\mathcal{F} = p_0 I + p_1 D + p_2 D^2$ is a degree preserving operator symmetric with respect to* (3) *if and only if u_0 is classical with distributional equation* (17)*, and u_1 is also classical satisfying $u_1 = p_2 u_0$.*

Proof. If $\mathcal{F}$ is symmetric and preserves the degree, then Theorem 1, Lemma 1 and the regularity of u_0 yield $\deg p_1 = 1$ and $\deg p_2 \leqslant 2$. So, u_0 is classical. In a similar way, we deduce that u_1 is also classical. Relation $u_1 = p_2 u_0$ follows from the canonical representations for classical functionals (see Marcellán *et al.* (1994a)).

Conversely, if u_0 is classical, then it satisfies (17) with $\deg p_2 \leqslant 2$ and $\deg p_1 = 1$. So, we have (4) with $p_0' = 0$. Using this fact and $u_1 = p_2 u_0$, we get (5). We can choose p_0 satisfiying Lemma 1 and therefore $\mathcal{F}$ preserves the degree. Applying Theorem 1, the result follows. □

It is well known that the only classical functionals are those associated with Hermite, Laguerre, Jacobi and Bessel polynomials. (The Pearson equation for these functionals can be seen, for instance, in Marcellán *et al.* (1994a).) So, we have

COROLLARY 3. *The only regular functionals u_0 and u_1 with a degree preserving operator $\mathcal{F}$ symmetric with respect to* (3) *are the following:*

(a) *u_0 and u_1 Hermite functionals.*
(b) *$u_0 = u^{(\alpha)}$ and $u_1 = u^{(\alpha+1)}$, where $u^{(\alpha)}$ is the Laguerre functional, with α not a negative integer.*
(c) *$u_0 = u^{(\alpha,\beta)}$ and $u_1 = u^{(\alpha+1,\beta+1)}$, where $u^{(\alpha,\beta)}$ is the Jacobi functional, with α, β and $\alpha+\beta+1$ not a negative integer.*
(d) *$u_0 = u^{(\alpha)}$ and $u_1 = u^{(\alpha+2)}$, where $u^{(\alpha)}$ is the Bessel functional, with $\alpha + 1$ not a negative integer.*

Moreover, $\mathcal{F} = p_0 I + \psi D + \phi D^2$, where p_0 is some constant and $D(\phi u_0) = \psi u_0$ is the Pearson equation satisfied by u_0.

Remark. The reader is referred to the contribution (Kwon *et al.*, 1998, Theorem 3.5), where the statements of Theorem 2 and Corollary 3 have been obtained, using a different method.

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