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Asymptotic Properties of Balanced Extremal Sobolev Polynomials: Coherent Case

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For each $n \in \mathbb{N}$ and $\lambda_n \geqslant 0$, Q_{n,λ_n} is the monic polynomial of degree n that minimizes the norm $\|q\|^2 = \int |q|^2 \, d\mu_0 + \lambda_n \int |q'|^2 \, d\mu_1$ in the class of all monic polynomials of degree n. Asymptotic properties of $\{Q_{n,\lambda_n}\}$ as $n \to \infty$ are studied under additional assumption that (μ_0, μ_1) is a coherent pair of measures on $[-1, 1]$ and the sequence $\{\lambda_n\}$ is regularly decreasing and satisfies $\lim_n n^2 \lambda_n = L \in [0, +\infty]$. The behavior of the norms and zeros of these polynomials is also studied. We show that in some cases the sequence $\{Q_{n,\lambda_n}\}$ asymptotically behaves as the monic orthogonal polynomials sequence corresponding to a new measure constructed as a combination of μ_0 and μ_1; we conjecture that this result is valid in a more general setting. © 1999 Academic Press

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1. INTRODUCTION

Assume that μ_0 and μ_1 are two finite Borel measures, compactly supported on $\mathbb{R}$; in what follows, $\mathrm{supp}(\mu_0) = [-1, 1]$. The study of orthogonal polynomials with respect to an inner product of the type

$$(p, q) = \int pq \, d\mu_0 + \int p'q' \, d\mu_1 \tag{1}$$

has a relatively short, although rich history, which we can trace back to the work of Lewis [1]. First asymptotic properties of these polynomials (as the degree goes to infinity) were established in the so-called "discrete" case, that is when μ_1 is a collection of a finite number of mass points [2, 3]. The "continuous" case is more subtle and needed different tools for its investigation. One of the first results was obtained in [5] (see also [9]) and can be stated as follows: if Q_n and T_n denote the monic polynomials of degree n, orthogonal with respect to (1) and μ_1, respectively, then

$$\lim_n \frac{Q_n(z)}{T_n(z)} = \frac{2}{\varphi'(z)}, \tag{2}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1, 1]$, where $\varphi(z) = z + \sqrt{z^2 - 1}$ with $\sqrt{z^2 - 1} > 0$ when $z > 1$.

In [5], the asymptotics (2) was established with the additional assumption of a link between μ_0 and μ_1, called coherence (see below). Later, (2) was proved under much milder conditions on μ_0 and μ_1 (see [4]), namely, when μ_0 and μ_1 are two arbitrary Borel measures supported on the same sufficiently smooth Jordan curve or arc, where they satisfy the well-known Szegő condition. This result was extended to Sobolev products with higher order derivatives in [6].

A closer look at the inner product (1) reveals that the measures μ_0 and μ_1 do not play an equivalent role: differentiation makes the leading coefficients of the polynomials involved in the second integral of (1) to be multiplied by their degrees. This effect is the more important the larger these degrees are, explaining the apparent independence of the limit (2) from the measure μ_0.

These considerations motivate to "balance" the role of both terms in (1) by considering only monic polynomials. In other words, we are interested in the monic polynomials Q_n of degree n, which minimize the norm

$$\|Q_n\|^2 = \int Q_n^2 \, d\mu_0 + \int \left(\frac{Q_n'}{n}\right)^2 d\mu_1$$

in the class of all monic polynomials of degree n. In a more general setting, we study orthogonality with respect to (1), where the second integral is multiplied by a parameter which depends on the degree of the polynomial.

Thus, we proceed with some notations. In what follows, $\mathbb{P}$ is the space of all polynomials with real coefficients. For μ_0 and μ_1 as above and $\lambda \geqslant 0$, denote by $(\cdot, \cdot; \lambda)$ the expression

$$(p, q; \lambda) = \int pq \, d\mu_0 + \lambda \int p'q' \, d\mu_1,$$

where $p, q \in \mathbb{P}$; for any fixed $\lambda \geqslant 0$ it defines an inner product in $\mathbb{P}$. Further, denote

$$\langle p, q \rangle_i = \int pq \, d\mu_i, \qquad i = 0, 1, \; p, q \in \mathbb{P}.$$

For $\lambda > 0$ and $n \in \mathbb{N}$ we can consider three monic orthogonal polynomial systems (MOPS); all the corresponding notation is gathered in the following table:

Inner product	MOPS	Square of the norm
$\langle \cdot, \cdot \rangle_0$	P_n	$\pi_n = \langle P_n, P_n \rangle_0$
$\langle \cdot, \cdot \rangle_1$	T_n	$\tau_n = \langle T_n, T_n \rangle_1$
$(\cdot, \cdot; \lambda)$	$Q_{n,\lambda}$	$\kappa_n(\lambda) = (Q_{n,\lambda}, Q_{n,\lambda}; \lambda)$

In particular, $Q_{n,0} = P_n$, for $n \in \mathbb{N}$.

Let $\{\lambda_n\}$ be a decreasing sequence of real positive numbers such that

$$\lim_n n^2 \lambda_n = L \in [0, +\infty]. \tag{3}$$

We consider only *regularly* decreasing sequences, which means that we assume additionally that

$$\lim_n n^2 (\lambda_{n-1} - \lambda_n) = \lim_n \left(\frac{\lambda_{n-1}}{\lambda_n} - 1 \right) = 0. \tag{4}$$

Notice that when $0 < L < \infty$, (4) follows from (3). Thus, one (and only one) of these limits imposes a restriction on $\{\lambda_n\}$ only in the extremal cases $L = 0$ and $L = +\infty$.

We are interested in the asymptotic behavior of the sequence $\{Q_{n,\lambda_n}\}$ as $n \to \infty$.

This study will be carried out under an additional assumption that (μ_0, μ_1) is a coherent pair of measures. We recall the definition (see, e.g., [5]):

DEFINITION 1. (μ_0, μ_1) is a coherent pair of measures if there exist non-zero constants $\sigma_1, \sigma_2, ...,$ such that

$$T_n(x) = \frac{P'_{n+1}(x)}{n+1} - \sigma_n \frac{P'_n(x)}{n}, \qquad n \geqslant 1. \tag{5}$$

We say that (μ_0, μ_1) is a *coherent pair on* $[-1, 1]$, if $\operatorname{supp} \mu_0 = [-1, 1]$.

Recently, Meijer [7] classified all coherent pairs of measures (see below). From his work it follows that whenever (μ_0, μ_1) is a coherent pair on $[-1, 1]$, the limit

$$\Psi(z) \stackrel{\text{def}}{=} \lim_n \frac{T_n(z)}{P_n(z)} \tag{6}$$

exists and holds locally uniformly in $\overline{\mathbb{C}} \setminus [-1, 1]$ (thus, Ψ is analytic in this domain).

The goal of this paper is to prove the following

THEOREM 1. *Let (μ_0, μ_1) be a coherent pair of measures on $[-1, 1]$, and the sequence $\{\lambda_n\}$ satisfies* (3)–(4). *Then, with the notation introduced above,*

(i) *There exists the limit*

$$\lim_n \frac{\pi_n}{\kappa_n(\lambda_n)} = k(L) \in [0, 1]; \tag{7}$$

(ii) *Uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1, 1]$,*

$$\lim_n \frac{Q_{n,\lambda_n}(z)}{P_n(z)} = \frac{\Psi(z)}{k(L)\, \Psi(z) + (1 - k(L))\, \varphi'(z)/2}, \tag{8}$$

where $\varphi(z) = z + \sqrt{z^2 - 1}$ with $\sqrt{z^2 - 1} > 0$ when $z > 1$, and Ψ is defined in (6).

Notice that for $\lambda_n \equiv \text{const} > 0$, limit L in (3) is infinity; we will show below (see (22)) that $k(\infty) = 0$, and (2) is a particular case of (8).

Since the right hand side of (6) is a non-vanishing analytic function outside of the set of accumulation points of zeros of T_n, using Hurwitz' theorem the following corollary is immediate:

COROLLARY 1. *The sets of accumulation points of zeros of Q_{n,λ_n} and T_n coincide.*

Using the same techniques; similar results can be obtained for symmetrically coherent pairs with compact support.

The structure of the paper is as follows. In the next section we introduce some preliminary and auxiliary results, necessary for establishing (in Section 3) the asymptotics of the Sobolev norms and an explicit expression for $k(L)$ (see Proposition 2); the proof of Theorem 1 is concluded in Section 4. Finally, we show that in some cases the sequence $\{Q_{n,\lambda_n}\}$ asymptotically behaves as the manic orthogonal polynomials sequence corresponding to a new measure constructed as a combination of μ_0 and μ_1; we conjecture that this result is valid in a more general setting.

2. PRELIMINARY RESULTS

Using the result of Meijer [7], we can classify all coherent pairs of measures on $[-1, 1]$ as follows. Let w_0, w_1 be two non-negative weights on $(-1, 1)$ related by

$$\frac{w_1(x)}{w_0(x)} = \frac{1-x^2}{|x-\xi|}, \qquad \xi \in \mathbb{R} \setminus (-1, 1), \tag{9}$$

and v_0, v_1 be the corresponding absolutely continuous measures on $[-1, 1]$:

$$dv_i(x) = w_i(x)\, dx, \qquad i = 0, 1. \tag{10}$$

Furthermore, denote $\rho^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}$.

PROPOSITION 1. *Let μ_0, μ_1 be two measures, and the support* $\operatorname{supp}(\mu_0) = [-1, 1]$. *Then, (μ_0, μ_1) form a coherent pair of measures if and only if one of the following cases holds*:

Case 1 (*absolutely continuous μ_1*).

$$\mu_0 = v_0 + M\delta_\xi, \qquad \mu_1 = v_1, \qquad M \geqslant 0,$$

where either $w_0(x) = \rho^{(\alpha,\beta)}(x)$ or $w_1(x) = \rho^{(\alpha,\beta)}(x)$.

Moreover, $M \neq 0$ if and only if

$$w_1(x) = \rho^{(0,\beta)}(x) \qquad \text{and} \qquad \xi = 1,$$

or

$$w_1(x) = \rho^{(\alpha,0)}(x) \qquad \text{and} \qquad \xi = -1.$$

Case 2 (*mass point in μ_1*).

$$\mu_0 = v_0, \qquad w_0(x) = \rho^{(\alpha,\beta)}(x),$$

and

$$\mu_1 = v_1 + M\delta_\xi, \qquad M > 0.$$

In both cases v_0 *and* v_1 *are related by* (9), (10) *and* $\alpha, \beta \in \mathbb{R}$ *can take any admissible value* (*i.e., such that* $w_0, w_1 \in L_1[-1, 1]$).

Thus, in the absolutely continuous case the asymptotic behavior of the sequence $\{T_n/P_n\}$ and of the norms π_n and τ_n is determined by the Szegő function of the ratio w_1/w_0 given in (9). In Case 2, with a mass point outside of $[-1, 1]$, analogous results can be obtained applying standard techniques (see, e.g., [10, Sect. 7]). Furthermore, using this information and formula (5), in [5] the asymptotics of the sequence $\{\sigma_n\}$ was computed. We gather all these results in the following Lemma, which we state without proof (see [5, 9] for details):

LEMMA 1. *Under assumptions of Proposition* 1, *the following limits exist*:

$$\sigma \stackrel{\text{def}}{=} \lim_n \sigma_n = \begin{cases} \dfrac{1}{2\varphi(\xi)}, & \text{in Case } 1, \\ \dfrac{\varphi(\xi)}{2} & \text{in Case } 2, \end{cases} \tag{11}$$

$$\lim_n \frac{\pi_{n+1}}{\pi_n} = \lim_n \frac{\tau_{n+1}}{\tau_n} = \frac{1}{4}, \qquad \lim_n \frac{\tau_n}{\pi_n} = |\sigma|, \tag{12}$$

$$\Psi(z) = \lim_n \frac{T_n(z)}{P_n(z)} = \begin{cases} \dfrac{\varphi'(z)}{2}\left(1 - \dfrac{1}{\varphi(\xi)\,\varphi(z)}\right), & \text{in Case } 1, \\ \dfrac{\varphi'(z)}{2}(1 - \varphi(\xi)/\varphi(z)), & \text{in Case } 2, \end{cases} \tag{13}$$

this last limit, locally uniformly in $\overline{\mathbb{C}} \setminus [-1, 1]$. *Here we take by continuity* $\varphi(\pm 1) = \pm 1$.

Coherence of measures μ_0 and μ_1 has a very important consequence: the structure of the sequence of Sobolev polynomials $\{Q_{n,\lambda}\}$ can be described by means of the following relation ([5], see also [11, Proposition 5.4.3]):

LEMMA 2. *For any* $\lambda > 0$, $n \in \mathbb{N}$,

$$P_{n+1}(x) - \sigma_n \frac{n+1}{n} P_n(x) = Q_{n+1,\lambda}(x) - \alpha_n(\lambda)\, Q_{n,\lambda}(x), \tag{14}$$

where σ_n *are the coherence parameters introduced in* (5), *and*

$$\alpha_n(\lambda) = \sigma_n \frac{n+1}{n} \frac{\pi_n}{\kappa_n(\lambda)}. \tag{15}$$

The identity (14) is the key to the study of the sequence $\{Q_{n,\lambda_n}\}$. Nevertheless, in order to compute the limit of the parameters α_n, we need to find the asymptotic behavior of the norms κ_n first.

3. ASYMPTOTICS OF THE SOBOLEV NORMS

We begin with the following elementary

LEMMA 3. *With our assumptions on* $\{\lambda_n\}$,

$$\frac{\lambda_n}{\lambda_{n-1}}\kappa_n(\lambda_{n-1})\leqslant\kappa_n(\lambda_n)\leqslant\kappa_n(\lambda_{n-1}),$$
$$\kappa_n(\lambda_{n+1})\leqslant\kappa_n(\lambda_n)\leqslant\frac{\lambda_n}{\lambda_{n+1}}\kappa_n(\lambda_{n+1}). \tag{16}$$

In particular,

$$\lim_n\frac{\kappa_n(\lambda_n)}{\kappa_n(\lambda_{n-1})}=\lim_n\frac{\kappa_n(\lambda_n)}{\kappa_n(\lambda_{n+1})}=1. \tag{17}$$

Proof. Using the extremal property of the norms of the monic orthogonal polynomials and the fact that $0<\lambda_n\leqslant\lambda_{n-1}$, we have

$$\begin{aligned}\kappa_n(\lambda_n)&=\langle Q_{n,\lambda_n},Q_{n,\lambda_n}\rangle_0+\lambda_n\langle Q'_{n,\lambda_n},Q'_{n,\lambda_n}\rangle_1\\&\leqslant\langle Q_{n,\lambda_{n-1}},Q_{n,\lambda_{n-1}}\rangle_0+\lambda_n\langle Q'_{n,\lambda_{n-1}},Q'_{n,\lambda_{n-1}}\rangle_1\leqslant\kappa_n(\lambda_{n-1}).\end{aligned}$$

On the other hand, analogous arguments lead us to

$$\begin{aligned}\kappa_n(\lambda_{n-1})&\leqslant\frac{\lambda_{n-1}}{\lambda_n}\left\{\frac{\lambda_n}{\lambda_{n-1}}\langle Q_{n,\lambda_n},Q_{n,\lambda_n}\rangle_0+\lambda_n\langle Q'_{n,\lambda_n},Q'_{n,\lambda_n}\rangle_1\right\}\\&\leqslant\frac{\lambda_{n-1}}{\lambda_n}\kappa_n(\lambda_n).\end{aligned}$$

The second inequality in (16) follows form the first one by a simple shift $\lambda_n\mapsto\lambda_{n+1}$. Now, (17) is a straightforward consequence of (4). ∎

LEMMA 4. *For a fixed* $\lambda>0$, *the sequence* $\{\kappa_n(\lambda)\}$ *satisfies*

$$\kappa_n(\lambda)=\pi_n\left(B_n(\lambda)-A_n\frac{\pi_{n-1}}{\kappa_{n-1}(\lambda)}\right),\qquad\kappa_1(\lambda)=\pi_1+\lambda\tau_0, \tag{18}$$

where

$$A_n = \sigma_{n-1}^2 \left(\frac{n}{n-1}\right)^2 \frac{\pi_{n-1}}{\pi_n}, \qquad B_n(\lambda) = 1 + \lambda n^2 \frac{\tau_{n-1}}{\pi_n} + A_n. \tag{19}$$

Proof. Using (14), we have that

$$\begin{aligned}\kappa_n(\lambda) &= (Q_{n,\lambda}, Q_{n,\lambda}; \lambda) \\ &= \Big(P_n - \frac{n\sigma_{n-1}}{n-1} P_{n-1} + \alpha_{n-1}(\lambda)\, Q_{n-1,\lambda}, P_n \\ &\qquad - \frac{n\sigma_{n-1}}{n-1} P_{n-1} + \alpha_{n-1}(\lambda)\, Q_{n-1,\lambda}; \lambda\Big).\end{aligned}$$

Now, we have

$$\begin{aligned}&\Big\langle P_n - \frac{n\sigma_{n-1}}{n-1} P_{n-1} + \alpha_{n-1}(\lambda)\, Q_{n-1,\lambda}, P_n \\ &\qquad - \frac{n\sigma_{n-1}}{n-1} P_{n-1} + \alpha_{n-1}(\lambda)\, Q_{n-1,\lambda}\Big\rangle_0 \\ &= \pi_n + \left(\frac{n\sigma_{n-1}}{n-1}\right)^2 \pi_{n-1} + \alpha_{n-1}^2(\lambda) \langle Q_{n-1,\lambda}, Q_{n-1,\lambda}\rangle_0 \\ &\qquad - 2\alpha_{n-1}(\lambda) \frac{n\sigma_{n-1}}{n-1} \pi_{n-1}.\end{aligned} \tag{20}$$

On the other hand, by (5),

$$\begin{aligned}&\Big\langle P'_n - \frac{n\sigma_{n-1}}{n-1} P'_{n-1} + \alpha_{n-1}(\lambda)\, Q'_{n-1,\lambda}, P'_n \\ &\qquad - \frac{n\sigma_{n-1}}{n-1} P'_{n-1} + \alpha_{n-1}(\lambda)\, Q'_{n-1,\lambda}\Big\rangle_1 \\ &= \langle nT_{n-1} + \alpha_{n-1}(\lambda)\, Q'_{n-1,\lambda}, nT_{n-1} + \alpha_{n-1}(\lambda)\, Q'_{n-1,\lambda}\rangle_1 \\ &= n^2\tau_{n-1} + \alpha_{n-1}^2(\lambda) \langle Q'_{n-1,\lambda}, Q'_{n-1,\lambda}\rangle_1.\end{aligned}$$

Thus, taking into account (20), we have that

$$\begin{aligned}\kappa_n(\lambda) &= \pi_n + \left(\frac{n\sigma_{n-1}}{n-1}\right)^2 \pi_{n-1} \\ &\qquad - 2\alpha_{n-1}(\lambda) \frac{n\sigma_{n-1}}{n-1} \pi_{n-1} + \lambda n^2 \tau_{n-1} + \alpha_{n-1}^2(\lambda)\, \kappa_{n-1}(\lambda),\end{aligned}$$

and it remains to substitute the value of $\alpha_{n-1}(\lambda)$ from (15) into the last identity to obtain (18) and (19). ∎

COROLLARY 2. $\lim_n 4^n[\kappa_n(\lambda_{n-1})-\kappa_n(\lambda_n)]=0$.

Proof. By (16),

$$0\leqslant\kappa_n(\lambda_{n-1})-\kappa_n(\lambda_n)\leqslant\left(1-\frac{\lambda_n}{\lambda_{n-1}}\right)\kappa_n(\lambda_{n-1}).$$

Furthermore, taking into account that $A_n>0$, from (18) we have that

$$\kappa_n(\lambda_{n-1})\leqslant\pi_n+\sigma_{n-1}^2\left(\frac{n}{n-1}\right)^2\pi_{n-1}+\lambda_{n-1}n^2\tau_{n-1}. \tag{21}$$

Thus, using (3)–(4), (21) and the well-known fact (see, e.g., [12, formula (12.7.2); 10, Lemma 16, p. 132, and Lemma 2, p. 39]) that both $4^n\pi_n$ and $4^n\tau_n$ converge, we can conclude the proof. ∎

Observe that we have showed additionally that for $L<+\infty$, the sequence $\{4^n\kappa_n(\lambda_n)\}$ is bounded; in fact, it converges. This follows from the first assertion of Theorem 1, which we proceed to prove now.

PROPOSITION 2. *Under assumptions* (3)–(4),

$$k(L)=\lim_n\frac{\pi_n}{\kappa_n(\lambda_n)}=\frac{1}{2\,|\sigma|\,\varphi(L+\Theta)}, \tag{22}$$

where σ was defined in (11) *and*

$$\Theta=|\sigma|+\frac{1}{4\,|\sigma|}\geqslant 1. \tag{23}$$

Proof. For $L=\infty$ this is a trivial consequence of the inequality

$$\kappa_n(\lambda_n)\geqslant\pi_n+\lambda_n n^2\tau_{n-1}$$

and (12). Assume now $L<\infty$.

Denote $s_n=\kappa_n(\lambda_n)/\pi_n$; then (18) can be rewritten as

$$s_n=B_n(\lambda_n)-\frac{A_n^*}{s_{n-1}},\qquad n\geqslant 2, \tag{24}$$

where $A_n^*=A_n\kappa_{n-1}(\lambda_{n-1})/\kappa_{n-1}(\lambda_n)$. Define a new sequence $\{q_n\}$ by $q_{n+1}=s_nq_n$, $q_1=1$. Then $\{q_n\}$ satisfies the three-term recurrence relation

$$q_{n+1}-B_n(\lambda_n)\,q_n+A_n^*q_{n-1}=0, \tag{25}$$

with $q_1=1$, $q_2=\kappa_1(\lambda_1)/\pi_1$. By (12) and (17), its coefficients, given by (19), converge,

$$\lim_n A_n^* = \lim_n A_n = 4\sigma^2, \qquad \lim_n B_n(\lambda_n) = 1 + 4\,|\sigma|\,L + 4\sigma^2.$$

For $L>0$ or for $L=0$ and $\xi \neq \pm 1$ ($\sigma \neq \pm 1/2$, see (11)), it is straightforward to check that the roots of the characteristic equation

$$q^2 - (1 + 4\,|\sigma|\,L + 4\sigma^2)\,q + 4\sigma^2 = 0 \tag{26}$$

are real, simple and have different absolute values. Thus, by Poincaré's Theorem (see, e.g., [8]), $s_n = q_{n+1}/q_n$ converges to one of these roots. We can choose to which one noticing that $\kappa_n(\lambda_n) \geqslant \pi_n$, so that $k(L) \leqslant 1$.

It remains to consider the case $L=0$, $\sigma^2 = 1/4$ (when Poincaré's Theorem is no longer applicable); then,

$$\lim_n A_n^* = 1, \qquad \lim_n B_n(\lambda_n) = 2,$$

and we can choose $n_0 \in \mathbb{N}$ large enough such that for $n \geqslant n_0$, $A_n^* > 0$ and $B_n(\lambda_n) > 1$. Since $s_n \geqslant 1$, by (24),

$$s_n \leqslant B_n(\lambda_n), \qquad n \geqslant n_0,$$

so that

$$s_{n+1} \leqslant B_{n+1}(\lambda_{n+1}) - \frac{A_{n+1}^*}{B_n(\lambda_n)}.$$

Repeating this reasoning we obtain that for any fixed $j \in \mathbb{N}$,

$$1 \leqslant s_{n+j} \leqslant B_{n+j}(\lambda_{n+j}) - \cfrac{A_{n+j}^*}{B_{n+j-1}(\lambda_{n+j-1}) - \cfrac{A_{n+j-1}^*}{\ddots - \cfrac{A_{n+1}^*}{B_n(\lambda_n)}}}$$

It is easy to check that when $n \to \infty$, the right hand side of this inequality tends to $(j+2)/(j+1)$. Thus,

$$1 \leqslant \liminf_n s_n \leqslant \limsup_n s_n \leqslant \frac{j+2}{j+1},$$

and since $j \in \mathbb{N}$ is arbitrary, we obtain that $\lim_n s_n = 1$. The assertion of the proposition is established. ∎

4. ASYMPTOTICS OF SOBOLEV POLYNOMIALS

For the time being we have obtained the asymptotics of $\kappa_n(\lambda_n)$; in this way, we already know the limits of the coefficients in (14). The last preparatory step is the following

LEMMA 5. *Under assumptions* (3)–(4),

$$\lim_n \frac{Q_{n+1,\lambda_n}(z) - Q_{n+1,\lambda_{n+1}}(z)}{P_{n+1}(z)} = 0,$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1, 1]$.

Proof. By Proposition 1, μ_0 satisfies the Szegő's condition on $[-1, 1]$; therefore,

$$\lim_n \frac{2^n P_n(z)}{\varphi^n(z)}$$

exists and holds locally uniformly in $\overline{\mathbb{C}} \setminus [-1, 1]$, and defines a nonzero analytic function there. Thus, it is sufficient to prove that

$$\lim_n 2^{n+1} \left[\frac{Q_{n+1,\lambda_n}(z)}{\varphi^{n+1}(z)} - \frac{Q_{n+1,\lambda_{n+1}}(z)}{\varphi^{n+1}(z)} \right] = 0,$$

locally uniformly in this domain.

In order to simplify notation, put

$$U_n(z) = Q_{n,\lambda_{n-1}}(z), \qquad V_n(z) = Q_{n,\lambda_n}(z).$$

Then, by orthogonality of U_n and V_n,

$$\begin{aligned} \langle U_n - V_n, U_n - V_n \rangle_0 &\leqslant (U_n - V_n, U_n - V_n; \lambda_n) = (U_n, U_n - V_n; \lambda_n) \\ &= (U_n, U_n; \lambda_n) - (U_n, V_n; \lambda_n) \\ &= (U_n, U_n; \lambda_{n-1}) - (V_n, V_n; \lambda_n) + (\lambda_n - \lambda_{n-1}) \langle U_n', U_n' \rangle_1 \\ &\leqslant \kappa_n(\lambda_{n-1}) - \kappa_n(\lambda_n). \end{aligned} \tag{27}$$

Furthermore, since $|\varphi(x)| = 1$ for $x \in [-1, 1]$, from (27) we obtain that

$$\left\langle \frac{2^n (U_n - V_n)}{\varphi^n}, \frac{2^n (U_n - V_n)}{\varphi^n} \right\rangle_0 \leqslant 4^n [\kappa_n(\lambda_{n-1}) - \kappa_n(\lambda_n)],$$

and by Corollary 2, the left hand side tends to zero. But for each n, $2^n(U_n - V_n)/\varphi^n$ is in the Hardy class H_{2,μ_0} in $\overline{\mathbb{C}}\setminus[-1,1]$. Thus, standard arguments (see, e.g., [13, Corollary 7.4]) allow us to conclude that

$$\lim_n 2^n \frac{U_n(z) - V_n(z)}{\varphi^n(z)} = 0,$$

locally uniformly in $\overline{\mathbb{C}}\setminus[-1,1]$. ∎

Now we are ready to prove the second assertion of Theorem 1. With the notation

$$f_n(z) = \frac{Q_{n,\lambda_n}(z)}{P_n(z)}, \qquad a_n(z) = \alpha_n(\lambda_n)\,\frac{P_n(z)}{P_{n+1}(z)},$$

$$b_n(z) = 1 - \sigma_n\,\frac{n+1}{n}\,\frac{P_n(z)}{P_{n+1}(z)} - \frac{Q_{n+1,\lambda_n}(z) - Q_{n+1,\lambda_{n+1}}(z)}{P_{n+1}(z)},$$

formula (14) reads as

$$f_{n+1}(z) = a_n(z)\, f_n(z) + b_n(z). \tag{28}$$

Observe that f_n, a_n, and b_n are analytic functions in $\overline{\mathbb{C}}\setminus[-1,1]$. Moreover, Lemmas 1 and 5, Proposition 2, and (15) give us the limits

$$a(z) \stackrel{\text{def}}{=} \lim_n a_n(z) = \frac{2\sigma k(L)}{\varphi(z)}, \qquad b(z) \stackrel{\text{def}}{=} \lim_n b_n(z) = 1 - \frac{2\sigma}{\varphi(z)},$$

which hold uniformly on compact subsets of $\overline{\mathbb{C}}\setminus[-1,1]$.

If we put

$$g_n(z) = f_n(z) - \frac{b(z)}{1 - a(z)},$$

then we can rewrite (28) as

$$g_{n+1}(z) = a(z)\, g_n(z) + \varepsilon(z), \tag{29}$$

with

$$\varepsilon_n(z) = [a_n(z) - a(z)]\, g_n(z) + b_n(z) - b(z)\,\frac{1 - a_n(z)}{1 - a(z)}.$$

Notice that $|\varphi(z)| > 1$ for $z \notin [-1, 1]$, and with account of (22), $|a(z)| < 1$ in this domain. In particular, for a fixed compact set $K \subset \overline{\mathbb{C}} \setminus [-1, 1]$ there exist constants $0 < r < 1$, $R > 0$, and $n_0 \in \mathbb{N}$ such that

$$|a_n(z)| \leqslant r, \qquad |b_n(z)| \leqslant R, \qquad \text{for} \quad n \geqslant n_0, z \in K.$$

Thus,

$$|f_{n+1}(z)| \leqslant r\,|f_n(z)| + R, \qquad n \geqslant n_0, \qquad z \in K,$$

and it is straightforward that $\{f_n\}$ and $\{g_n\}$ are uniformly bounded on compact subsets of $\overline{\mathbb{C}} \setminus [-1, 1]$. Consequently,

$$\lim_n \varepsilon_n(z) = 0,$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus [-1, 1]$. Using (29) it is easy to establish the same behavior for $g_n(z)$. In other words, we have proved that

$$\lim_n f_n(z) = \frac{b(z)}{1 - a(z)}, \tag{30}$$

also locally uniformly in $\overline{\mathbb{C}} \setminus [-1, 1]$. It remains to rewrite (30), in order to obtain (8). Indeed, by (5) and (6),

$$\Psi(z) = \frac{\varphi'(z)}{2} \left(1 - \frac{2\sigma}{\varphi(z)} \right),$$

and thus,

$$b(z) = \frac{2\Psi(z)}{\varphi'(z)}.$$

Analogously,

$$a(z) = k(L) \left(1 - \frac{2\Psi(z)}{\varphi'(z)} \right).$$

Substituting these expressions into (30) we arrive at the expression in the right bared side of (8). ∎

Since the zeros of T_n accumulate at the support of the measure μ_1, we can sharpen the statement of Corollary 1:

COROLLARY 3. *Under assumptions of Theorem* 1,

$$\bigcap_{n \geqslant 1} \overline{\bigcup_{k=n}^{\infty} \{z \colon Q_{k,\lambda_k}(z)=0\}} = \operatorname{supp} \mu_1 .$$

The assertions of Theorem 1 can be formulated in terms suitable for a general conjecture on asymptotics of the balanced Sobolev polynomials. We will restrict ourselves to the absolutely continuous case (Case 1 of Proposition 1).

For $0 \leqslant L < \infty$ we introduce the measure μ^* on $[-1, 1]$,

$$d\mu^*(x) = \{\mu_0'(x) + L\,|\varphi'(x)|^2\,\mu_1'(x)\}\,dx, \qquad x \in [-1, 1]. \tag{31}$$

Let $R_n(x) = x^n + \cdots$ be the sequence of monic polynomials, orthogonal on $[-1, 1]$ with respect to μ^* and

$$\varrho_n(L) = \|R_n\|^2_{L^2(\mu^*)} = \int_{-1}^{1} |R_n(x)|^2\,d\mu^*(x).$$

Then, the statement of Theorem 1 corresponding to the absolutely continuous case is equivalent to the following

COROLLARY 4. *Let* (μ_0, μ_1) *be a coherent pair of measures satisfying the Szegő condition on* $[-1, 1]$ (*cf. Case* 1 *of Proposition* 1), *and the sequence* $\{\lambda_n\}$ *as in* (3)–(4). *Then,*

$$\lim_n \frac{\varrho_n(L)}{\kappa_n(\lambda_n)} = 1, \tag{32}$$

and

$$\lim_n \frac{Q_{n,\lambda_n}(z)}{R_n(z)} = 1, \tag{33}$$

locally uniformly in $\overline{\mathbb{C}} \setminus [-1, 1]$.

In other words, the sequence $\{Q_{n,\lambda_n}\}$ asymptotically behaves as the monic orthogonal polynomials sequence corresponding to the measure (31).

Proof. Due to relation (9) and the definition of φ, in our case

$$d\mu^*(x) = \frac{x-\eta}{x-\xi}\,w_0(x)\,dx, \tag{34}$$

where

$$\eta = \xi + L \operatorname{sgn} \xi.$$

Thus, the problem is reduced to the asymptotic behavior of polynomials corresponding to a rational modification of the weight w_0. This situation has been thoroughly studied; an obliged reference is the monograph [10]. In particular, from Lemma 10 of [10, Sect. 6.1], it is easy to obtain that

$$\lim_n \frac{\pi_n}{\varrho_n(L)} = \frac{\varphi(\xi)}{\varphi(\eta)}. \tag{35}$$

Thus, in order to prove (32) it is sufficient to show that the right hand side of (35) coincides with the value of $k(L)$ given in (22). This is straightforward if we notice that Eq. (26) for $k(L)$ can be rewritten in this case as

$$[\varphi(\xi)]^2\, q^2 - 2\eta\varphi(\xi)\, q + 1 = 0.$$

Now we turn to formula (33); again, it is sufficient to prove that the function in the right hand side of (8) describes the ratio asymptotics of R_n/P_n, which is reduced to the computation of some simple Szegő functions. In fact, for $a \in \overline{\mathbb{C}} \setminus [-1, 1]$ let

$$\mathscr{F}(z;\, a) = \frac{|\varphi(a)|^2}{2\varphi(a)}\, \frac{\varphi(z) - \varphi(a)}{z - a}\, \frac{\varphi(z)}{\varphi(z)\, \varphi(a) - 1}. \tag{36}$$

Notice that $\mathscr{F}(z; a)$ is analytic single-valued and non-vanishing in $\overline{\mathbb{C}} \setminus [-1, 1]$, $\mathscr{F}(\infty; a) = 1$, and for $x \in (-1, 1)$,

$$\lim_{y \to 0} |\mathscr{F}(x + iy;\, a)| = \left| \frac{\varphi(a)}{2(x - a)} \right|.$$

Thus, Szegő's theory (see, e.g., [12, Theorem 12.1.2, 13, Theorem 9.1]) along with (34) yield that

$$\lim_n \frac{R_n}{P_n}(z) = \left(\frac{\mathscr{F}(z;\, \eta)}{\mathscr{F}(z;\, \xi)} \right)^{1/2},$$

locally uniformly in $\overline{\mathbb{C}} \setminus [-1, 1]$, where the branch of the root is fixed by the value 1 at infinity. By (6) and (8), it is sufficient to establish that

$$\frac{\mathscr{F}(z;\, \eta)}{\mathscr{F}(z;\, \eta)} = \left(\frac{\Psi(z)}{k(L)\, \Psi(z) + (1 - k(L))\, \varphi'(z)/2} \right)^2. \tag{37}$$

This can be done by direct computation if we use the explicit expressions (cf. (13) and (35))

$$\Psi(z) = \frac{\varphi'(z)}{2} \left(1 - \frac{1}{\varphi(\xi)\, \varphi(z)} \right), \qquad k(L) = \frac{\varphi(\xi)}{\varphi(\eta)},$$

the fact that $\xi, \eta \in \mathbb{R}\backslash(-1, 1)$ and the identity

$$\frac{\varphi(z)-\varphi(a)}{2\varphi(a)(z-a)}=\frac{\varphi(z)}{\varphi(z)\,\varphi(a)-1}, \qquad a \in \mathbb{R}\backslash(-1, 1).$$

The corollary is proved. ∎

Finally, we pose the following

Conjecture 1. The assertions of Corollary 4 hold when both μ_0 and μ_1 satisfy the Szegő condition on $[-1, 1]$ and μ^* is given by (31).

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