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Orthogonal polynomials on Sobolev spaces: old and new directions

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Abstract

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During the last years, orthogonal polynomials on Sobolev spaces have attracted considerable attention. Algebraic properties, distribution of their zeros and Fourier expansions as well as their relevance in the analysis of spectral methods for partial differential equations provide a very large field to explore and to compare with the standard case. In this paper we present an introductory survey about the subject. The origin of the problems and their development show the interest and the promising future of this field.

Keywords: Orthogonal polynomials; Sobolev inner product; recurrence relations; zeros; differential equations.

1. Introduction

The interest in the study of orthogonal polynomials with respect to the inner product

$$(f, g)_w = \sum_{i=0}^{p} \int_a^b f^{(i)}(x) g^{(i)}(x) \, d\mu_i, \tag{1}$$

where $(\mu_i)_{i=0}^{p}$ are positive finite Borel measures whose support is contained in the interval (a, b), is justified by several reasons.

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(a) The comparison with the standard theory of orthogonal polynomials in L^2-spaces (see [8,14,35] as classical references in the subject).

(b) The spectral theory for ordinary differential equations (see [12,13]).

(c) The analysis of spectral methods in the numerical treatment of partial differential equations (see [7,31]).

(d) The search of algorithms for computing Fourier–Sobolev series as well as the approximation to both a function and its derivative in terms of Sobolev orthogonal polynomials. For instance, the standard Legendre projection produces poor approximation to the derivative, which might be pointwise worse by orders of magnitude than the underlying approximation to the function (see [16]).

(e) The extension of Gauss quadrature formulas. In [17] a framework for this question is provided, but apparently without connection with orthogonal polynomials of Sobolev-type. As we will analyze below, the recurrence relation [17, (2.4)] is intimately connected with such kind of orthogonal polynomials.

In this paper we consider several cases of the inner product (1) which appear in the literature. We intend to provide a general framework for the theory, taking into account some basic ideas concerning the representation of polynomials from different points of view. Moreover, some open questions are presented.

2. A motivation from data fitting

Let $\mu_0, \mu_1, \ldots, \mu_p$ be $p+1$ positive finite Borel measures supported on an interval $I=[a, b]$ in $\mathbb{R}$. Let f be a function in $\mathscr{C}^{p-1}(J)$ where J is an interval such that $J \supset I$. We suppose that the pth derivative $f^{(p)}$ exists μ_p-almost everywhere and belongs to $L^2_{\mu_p}(I)$. We will denote by $\mathscr{P}_n$ the linear space of polynomials with complex coefficients and degree less than or equal to n. Also, $\mathscr{P}$ means $\bigcup_{n\in\mathbb{N}}\mathscr{P}_n$.

The main problem in smooth data fitting is to find the best approximant to f in $\mathscr{P}_n$ with respect to the norm

$$\|h\|_w = \left(\sum_{i=0}^{p} \int_a^b |h^{(i)}(x)|^2 \, d\mu_i \right)^{1/2},$$

induced by the inner product (1). This problem was considered in [22], but nothing is said about the corresponding sequence of orthonormal polynomials (q_n) defined in $\mathscr{P}$ using the Gram–Schmidt process for the basis $(x^n)_{n=0}^{\infty}$. In terms of this sequence we can reformulate some results of [22] in the following way.

Proposition 2.1. *The best polynomial approximant $P_n(x; f)$ is given by*

$$P_n(x; f) = \sum_{i=0}^{p} \int_a^b f^{(i)}(y) S_n^{(0,i)}(x, y) \, d\mu_i(y), \tag{2}$$

where $S_n^{(0,i)}(x, y) = \sum_{j=0}^{n} q_j(x) q_j^{(i)}(y)$.

Proof. The best polynomial approximant to f in $\mathscr{P}_n$ is $P_n(x; f) = \sum_{j=0}^{n} \alpha_{nj} q_j(x)$ with $\alpha_{nj} = (f, q_j)_w$. Using (1), equation (2) follows immediately. □

It is possible to compare this best approximant with the mth Taylor polynomial $T_m(x; b; f)$ of f in b, where $p \leqslant m \leqslant n$. In fact, the following result is deduced in [22].

Proposition 2.2. *Let f be a function of class $\mathscr{C}^{m-1}$. We suppose also that $f^{(m)}$ exists almost everywhere, is of bounded variation, continuous at $x = b$ and $f^{(m-1)}$ is absolutely continuous. Then*

$$T_m(x; b; f) - P_n(x; f) = \frac{1}{m!} \int_a^b G_n^m(x, t) \, \mathrm{d}f^{(m)}(t) \tag{3}$$

holds for $x \in J$, where $G_n^m(x, t) \in \mathscr{P}_n$ is the best polynomial approximant to the function

$$g_m(x) = \begin{cases} (x-t)^m, & \text{if } x \leqslant t, \\ 0, & \text{otherwise}. \end{cases}$$

(In the case $m = p$, we assume that $f^{(p)}$ and μ_p have no points of discontinuity in common).

From (3), if G_n^m is uniformly bounded and $G_n^m \to g_m$ a.e. with respect to $\mathrm{d}f^{(m)}$, then $P_n(x; f) \to f(x)$ uniformly on J.

This result has obvious connections with some well-known results concerning the convergence of series of general orthogonal polynomials.

The paper [22] is the origin of several contributions [2,15,21], among others, in more specific situations.

Basically, their approach corresponds to some particular measures connected with classical orthogonal polynomials in the case $p = 1$. In this situation, more information for the sequence (q_n) is obtained.

3. Classical Sobolev orthogonal polynomials

The first example of a Sobolev inner product (1) is given by Althammer [2]. He considers $\mathrm{d}\mu_0 = \mathrm{d}x$, $\mathrm{d}\mu_1 = \lambda \, \mathrm{d}x$ and $I = [-1, 1]$. Then, a link with monic Legendre orthogonal polynomials (P_n) appears in a natural way. Let (Q_n) denote the sequence of monic orthogonal polynomials (SMOP) with respect to (1), and let $Z_n(x) = \int_{-1}^x Q_n(t) \, \mathrm{d}t$ be the primitive function $Q_n(x)$ such that $Z_n(-1) = 0$.

From $(Q_n, 1)_w = \int_{-1}^1 Q_n(t) \, \mathrm{d}t = 0$, it follows that $Z_n(1) = 0$ for $n \geqslant 1$. But if $r \in \mathscr{P}_{n-1}$,

$$0 = (Q_n, r)_w = \int_{-1}^1 [Q_n(t) r(t) + \lambda Q_n'(t) r'(t)] \, \mathrm{d}t.$$

Integration by parts gives

$$0 = \int_{-1}^1 [-Z_n(t) + \lambda Z_n''(t)] r'(t) \, \mathrm{d}t,$$

and so $\lambda Z_n''(t) - Z_n(t)$ belongs to the orthogonal complement of the subspace $\mathscr{P}_{n-2}$ in $\mathscr{P}_{n+1}$ with respect to the standard inner product associated to the Lebesgue measure in $[-1, 1]$.

As the polynomials Q_n are even or odd, depending on the parity of n,

$$Z_n(-x) - Z_n(0) = (-1)^{n+1}[Z_n(x) - Z_n(0)],$$

and

$$\lambda Z_n''(t) - Z_n(t) = \alpha_n P_{n+1}(t) + \beta_n P_{n-1}(t)$$

follows, where the parameters α_n, β_n are given by

$$\alpha_n = -\frac{1}{n+1}, \qquad \beta_n = \frac{(Q_n, Q_n)_w}{n\int_{-1}^{1} P_{n-1}^2(t)\, \mathrm{d}t}. \tag{4}$$

Moreover,

$$\lambda Q_n''(t) - Q_n(t) = \alpha_n P_{n+1}'(t) + \beta_n P_{n-1}'(t), \tag{5}$$

and as an immediate consequence,

$$Q_n(t) = -\sum_{k=0}^{[n/2]} \lambda^k \left[\alpha_n P_{n+1}^{(2k+1)}(t) + \beta_n P_{n-1}^{(2k+1)}(t)\right].$$

But taking into account that

(a) $\displaystyle\int_{-1}^{1} P_n^2(t)\, \mathrm{d}t = \frac{n^2}{4n^2-1}\int_{-1}^{1} P_{n-1}^2(t)\, \mathrm{d}t,$

(b) $P_n^{(2k+2)}(x) = n(n-1)\cdots(n-2k-1)P_{n-2k-2}^{(2k+2,2k+2)}(x)$

($P_n^{(\alpha,\beta)}(x)$ denotes the nth monic Jacobi polynomial),

(c) $\displaystyle P_n^{(\alpha,\beta)}(1) = \frac{\binom{n+\alpha}{n}}{\binom{2n+\alpha+\beta}{n}} 2^n,$

(d) $\displaystyle (Q_n, Q_n)_w = (Q_n, P_n)_w = \int_{-1}^{1} P_n^2(t)\, \mathrm{d}t + 2\lambda P_n(1)Q_n'(1),$

(e) $\displaystyle Q_n'(1) = \frac{1}{n+1}\sum_{k=0}^{[(n-1)/2]} \lambda^k P_{n+1}^{(2k+2)}(1) - \beta_n \sum_{k=0}^{[(n-1)/2]-1} \lambda^k P_{n-1}^{(2k+2)}(1),$

β_n in (4) becomes

$$\beta_n = \frac{n}{4n^2-1} + \frac{2^n\lambda}{n\binom{2n}{n}} \frac{(2n-1)!!\,(2n-3)!!}{[(n-1)!]^2} Q_n'(1),$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$. By straightforward calculation,

$$\beta_n = \frac{n\sum_{k=0}^{[(n+1)/2]}\left(\frac{1}{4}\lambda\right)^k \frac{(n+2k+1)!}{(n-2k+1)!\,(2k)!}}{(4n^2-1)\sum_{k=0}^{[(n-1)/2]}\left(\frac{1}{4}\lambda\right)^k \frac{(n+2k-1)!}{(n-2k-1)!\,(2k)!}}.$$

Remark 3.1. From (5) a Rodrigues-type formula for the SMOP (Q_n) is obtained in the following way:

$$\begin{aligned}(\lambda D^2 - I)Q_n(t) &= D[\alpha_n P_{n+1}(t) + \beta_n P_{n-1}(t)]\\ &= D\left[A_{n+1}\alpha_n D^{n+1}(1-t^2)^{n+1} + A_{n-1}\beta_n D^{n-1}(1-t^2)^{n-1}\right]\\ &= D^n\left[(1-t^2)^{n-1}\left[A_{n-1}\beta_n - 2A_{n+1}((2n+1)t^2-1)\right]\right]\\ &= D^n\left[(1-t^2)^{n-1}(A_n t^2 + M_n)\right], \qquad (6)\end{aligned}$$

with

$$M_n = A_{n-1}\beta_n + 2A_{n+1}, \quad A_n = (-1)^n \frac{n!}{(2n)!}.$$

Proposition 3.2. *The following are valid.*
(a) $Q_n(1) > 0$ *for every* $n \in \mathbb{N}$;

(b) $$(n+2)P_{n+1}(x) = Q'_{n+2}(x) - \frac{Q_{n+2}(1)}{Q_n(1)} Q'_n(x). \qquad (7)$$

Proof. Let $R_{n+2}(x) = \int_{-1}^{x}(n+2)P_{n+1}(t)\,\mathrm{d}t$. As the polynomials R_n are even or odd, depending on the parity of n, we can write

$$R_{n+2}(x) = Q_{n+2}(x) + \sum_{j=0}^{n} \alpha_{n+2,j} Q_j(x). \qquad (8)$$

For $0 \leqslant j \leqslant n$,

$$\alpha_{n+2,j} = \frac{(R_{n+2}, Q_j)_w}{(Q_j, Q_j)_w} = \frac{\int_{-1}^{1} R_{n+2}(x)Q_j(x)\,\mathrm{d}x}{(Q_j, Q_j)_w} = \frac{-\int_{-1}^{1}(n+2)P_{n+1}(x)Z_j(x)\,\mathrm{d}x}{(Q_j, Q_j)_w}.$$

This last expression is zero for $0 \leqslant j \leqslant n-1$. Then, from (8),

$$R_{n+2}(x) = Q_{n+2}(x) + \alpha_{n+2,n}Q_n(x) \qquad (9)$$

and

$$\alpha_{n+2,n} = \frac{-\int_{-1}^{1} \frac{n+2}{n+1} P_{n+1}^2(x)\,\mathrm{d}x}{(Q_n, Q_n)_w} < 0.$$

But

$$R_{n+2}(1) = \int_{-1}^{1}(n+2)P_{n+1}(t)\,\mathrm{d}t = 0, \qquad (10)$$

and thus $Q_{n+2}(1) = -\alpha_{n+2,n}Q_n(1)$. Using the fact that $Q_0(1) = Q_1(1) = 1$, we deduce that $Q_n(1) > 0$ and (7) follows. □

Remark 3.3. Let us consider the Fourier expansion of R_n in terms of the polynomials P_n:

$$R_{n+2}(x) = P_{n+2}(x) + \sum_{j=0}^{n} \beta_{n+2,j} P_j(x).$$

As

$$\beta_{n+2,j} = \frac{\int_{-1}^{1} R_{n+2}(x) P_j(x)\,\mathrm{d}x}{\int_{-1}^{1} P_j^2(x)\,\mathrm{d}x} = \frac{-\int_{-1}^{1} \frac{n+2}{j+1} P_{n+1}(x) R_{j+1}(x)\,\mathrm{d}x}{\int_{-1}^{1} P_j^2(x)\,\mathrm{d}x} = 0,$$

for $0 \leqslant j \leqslant n-1$, we obtain

$$R_{n+2}(x) = P_{n+2}(x) + \beta_{n+2,n} P_n(x).$$

By using (10), $\beta_{n+2,n} = -P_{n+2}(1)/P_n(1)$ and therefore

$$R_{n+2}(x) = P_{n+2}(x) - \frac{P_{n+2}(1)}{P_n(1)} P_n(x) = (x^2-1) P_n^{(1,1)}(x).$$

Finally, from (9), the next formula is satisfied:

$$(x^2-1) P_n^{(1,1)}(x) = Q_{n+2}(x) - \frac{Q_{n+2}(1)}{Q_n(1)} Q_n(x).$$

Thus, we can deduce the following corollary.

Corollary 3.4. *The formulas*

(a) $$\frac{Q_{n+2}(x)}{Q_{n+2}(1)} - \frac{Q_n(x)}{Q_n(1)} = \frac{P_{n+2}(1)}{Q_{n+2}(1)} \left(\frac{P_{n+2}(x)}{P_{n+2}(1)} - \frac{P_n(x)}{P_n(1)} \right),$$

(b) $$\frac{Q_{n+2}(x)}{Q_{n+2}(1)} = \frac{P_{n+2}(x)}{Q_{n+2}(1)} + \sum_{j=0}^{[n/2]} \gamma_{nj} \frac{P_{n-2j}(x)}{P_{n-2j}(1)},$$

with

$$\gamma_{nj} = \frac{P_{n-2j}(1)}{Q_{n-2j}(1)} - \frac{P_{n-2j+2}(1)}{Q_{n-2j+2}(1)}, \quad 0 \leqslant j \leqslant \left[\tfrac{1}{2}n\right],$$

are true.

Concerning the zeros of the SMOP (Q_n), several authors (see [2,9,33]) have obtained quite a few results. Some of them are given in what follows.

Proposition 3.5. (a) *The zeros of the polynomials* $Q_n(x)$ *are real, simple and belong to* $(-1, 1)$.
(b) *The zeros of the polynomials* $Q_n(x)$ *are interlaced with the zeros of* $P_{n-1}(x)$ *whenever* $\lambda \geqslant 2/n$.

Proof. See [2,33] for (a) and [9] for (b). □

However, some questions remain open.

Problem 3.6. Do the zeros of the polynomial $Q_n(x)$ interlace with the zeros of $Q_{n+1}(x)$?

Problem 3.7. Are the positive zeros of the polynomial $Q_n(x)$ monotone functions of λ?

From another set of ideas, concerning differential properties, it is well known that classical orthogonal polynomials (Jacobi, Laguerre, Hermite) are eigenfunctions for certain second-order linear differential operators.

In particular, Legendre polynomials are eigenfunctions of the differential operator $(1-t^2)D^2 - 2tD$. This property can be easily deduced from the so-called structure relation (see [26,35])

$$(x^2-1)P'_{n+1}(x) = (n+1)P_{n+2}(x) + \xi_n P_n(x).$$

In our case, from (5),

$$(I - \lambda D^2)Q_n(x) = P_n^{(1,1)}(x) - (n-1)\beta_n P_{n-2}^{(1,1)}(x).$$

Then, using Corollary 3.4, we get

$$(x^2-1)(I-\lambda D^2)Q_n(x)$$
$$= Q_{n+2}(x) - \frac{Q_{n+2}(1)}{Q_n(1)}Q_n(x) - (n-1)\beta_n\left(Q_n(x) - \frac{Q_n(1)}{Q_{n-2}(1)}Q_{n-2}(x)\right),$$

that is,

$$(x^2-1)(I-\lambda D^2)Q_n(x) = Q_{n+2}(x) + a_n Q_n(x) + b_n Q_{n-2}(x).$$

This expression leads to an analog of the structure relation and in a natural way we can formulate the following problem.

Problem 3.8. Are Legendre–Sobolev orthogonal polynomials (Q_n) the eigenfunctions of some linear differential operator?

Finally, in the framework of asymptotics for sequences of standard orthogonal polynomials it seems natural to compare (Q_n) and (P_n) when n tends to infinity. In fact, the following result is valid.

Proposition 3.9. (see [33]). *Let* q_n *denote the orthonormal polynomial associated to* Q_n. *The formula*

$$q'_n(x) = \frac{n}{k_n}P_{n-1}(x) + O(n^{-3/2}),$$

where $k_n^2 = (Q_n, Q_n)_w$, *holds for* $x \in (-1, 1)$.

Problem 3.10. What about $\lim_{n\to\infty} Q_n(x)/P_n(x)$ for $x \in \mathbb{C}\setminus[-1, 1]$ if such a limit exists (relative asymptotics)?

Problem 3.11. What about $\lim_{n\to\infty} Q_n(x)/Q_{n-1}(x)$ for $x \in \mathbb{C}\setminus[-1, 1]$ if such a limit exists (ratio asymptotics)?

Problem 3.12. Extend the above questions to the case $d\mu_0 = (1-x)^\alpha(1+x)^\beta\, dx$ and $d\mu_1 = \lambda(1-x)^\alpha(1+x)^\beta\, dx$, $\alpha, \beta > -1$, i.e., consider Jacobi–Sobolev orthogonal polynomials.

In the same order of ideas, a study of Laguerre–Sobolev orthogonal polynomials was developed by Brenner in 1969 (see [6]). His approach is very different from the Althammer one. He uses a characterization of such kind of orthogonal polynomials in terms of the minimization problem

$$\Omega_n = \inf \int_0^\infty e^{-t}\left[y_n^2(t) + \lambda y_n'^2(t)\right] dt, \tag{11}$$

with the constraint $y_n^{(n)}(t) = n!$. Using the method of Lagrange multipliers, Brenner reduces the problem to the minimization of the expression

$$\int_0^\infty e^{-t}\left[y^2(t) + \lambda y'^2(t)\right] + 2\alpha(t)\left[y^{(n)}(t) - n!\right] dt.$$

From Euler's condition

$$e^{-x}\left[\lambda y''(x) - \lambda y'(x) - y(x)\right] = (-1)^n \alpha^{(n)}(x) \tag{12}$$

and boundary values

$$\begin{aligned}
&\alpha(0) = \alpha'(0) = \cdots = \alpha^{(n-2)}(0) = 0,\\
&\lim_{x\to\infty} \alpha(x) = \cdots = \lim_{x\to\infty} \alpha^{(n-2)}(x) = 0,\\
&\lambda y'(0) + (-1)^{n-1}\alpha^{(n-1)}(0) = 0,\\
&\lim_{x\to\infty}\left[\lambda e^{-x} y'(x) + (-1)^{n-1}\alpha^{(n-1)}(x)\right] = 0,
\end{aligned}$$

we can deduce that

$$\alpha(x) = -e^{-x}\left(x^n + a_n x^{n-1}\right).$$

Then,

$$\left(I + \lambda D - \lambda D^2\right)^{-1}\left[(-1)^{n-1} e^x D^n \alpha(x)\right]$$

gives a Rodrigues-type representation for the Laguerre–Sobolev orthogonal polynomials in the same sense as (6) for the Legendre–Sobolev orthogonal polynomials.

Proposition 3.13. (see [6]). *Let (L_n) be the sequence of monic Laguerre polynomials. Then the following results are true.*

(a) $Q_n(0) \neq 0$;

(b) $(n+2)L_{n+1}(x) = Q'_{n+2}(x) - \frac{Q_{n+2}(0)}{Q_{n+1}(0)} Q'_{n+1}(x);$

(c) $\lambda[Q''_n(x) - Q'_n(x)] - Q_n(x) = -\frac{1}{(n+1)} L'_{n+1}(x) + \eta_n L'_n(x).$

With respect to the zeros of Q_n, they are real, simple and positive. But nothing more is known about them as interlacing properties and monotone character in terms of λ.

Problem 3.14. Extend the above questions to the case $d\mu_0 = x^\alpha e^{-x}\, dx$, $d\mu_1 = \lambda x^\alpha e^{-x}\, dx$, $\alpha > -1$, i.e., consider Sonin–Laguerre–Sobolev orthogonal polynomials.

Finally, a more general study for Sobolev orthogonal polynomials with respect to (1) with $d\mu_i = \lambda_i\, dx$ is given in [21]. The basic tool is the variational problem

$$\inf_{y^{(n)}(t)=n!} \sum_{k=0}^{p} \int_a^b [y^{(k)}(x)]^2 \lambda_k\, dx.$$

A Rodrigues-type formula is deduced, but the constraint linked with the integration by parts in the linear functional does not allow to consider the general situation for positive measures.

As a final remark it seems very difficult to consider inner products (1) when the measures are not classical measures, at least for the study of algebraic properties and zeros. The approach presented in this section is conditioned by the "good" behaviour of classical orthogonal polynomials with respect to the derivative operator. In fact, the character of preserving orthogonality for such an operator plays a very important role in the above study.

4. Coherent pairs of measures

A more general attempt concerning orthogonal polynomials with respect to an inner product as (1) with $p = 1$ is investigated in [16].

They consider two positive finite Borel measures μ_0, μ_1 and introduce a real parameter λ such that (1) becomes

$$(f, g)_w = \int_{\mathbb{R}} f(x)g(x)\, d\mu_0 + \lambda \int_{\mathbb{R}} f'(x)g'(x)\, d\mu_1. \tag{13}$$

The parameter λ can be absorbed into $d\mu_1$ but the analysis of the dependence of the SMOP (Q_n) with respect to (13) in terms of λ is the key idea in the investigation by the authors.

Another important remark is the fact that the supports of μ_0 and μ_1 are infinite sets. Then there exist two sequences of monic orthogonal polynomials (P_n) and (R_n) associated with μ_0 and μ_1 respectively.

Since the sequence (P_n) spans $\mathscr{P}$, there exist functions $r_{n0}(\lambda), r_{n1}(\lambda), \ldots, r_{nn}(\lambda)$ such that

$$Q_n(x; \lambda) = \sum_{k=0}^{n} r_{nk}(\lambda) P_k(x). \tag{14}$$

Remark that

$$r_{nn}(\lambda) = 1 \quad \text{and} \quad r_{n0}(\lambda) = \frac{(Q_n, P_0)_{\mu_0}}{(P_0, P_0)_{\mu_0}} = \frac{(Q_n, P_0)_w}{(P_0, P_0)_w} = 0, \qquad \text{for } n \geq 1.$$

Then, (14) becomes

$$Q_n(x;\lambda)=P_n(x)+\sum_{k=1}^{n-1} r_{nk}(\lambda)P_k(x). \tag{15}$$

Since $(Q_n, P_m)_w=0$, for $1\leqslant m\leqslant n-1$, we have

$$0=r_{nm}(\lambda)\|P_m\|^2_{\mu_0}+\lambda\left(\sum_{k=1}^{n-1} r_{nk}(\lambda)(P_k', P_m')_{\mu_1}+(P_n', P_m')_{\mu_1}\right).$$

If we write $c_{ij}=(P_i', P_j')_{\mu_1}$ and $d_i=\|P_i\|^2_{\mu_0}$, then $(r_{nk})_{k=1}^{n-1}$ are the solutions of the linear system

$$\begin{pmatrix} d_1+\lambda c_{11} & \lambda c_{21} & \cdots & \lambda c_{n-1,1}\\ \lambda c_{12} & d_2+\lambda c_{22} & \cdots & \lambda c_{n-1,2}\\ \vdots & \vdots & & \vdots\\ \lambda c_{1,n-1} & \lambda c_{2,n-1} & \cdots & d_{n-1}+\lambda c_{n-1,n-1}\end{pmatrix}\begin{pmatrix} r_{n1}\\ r_{n2}\\ \vdots\\ r_{n,n-1}\end{pmatrix}=-\lambda\begin{pmatrix}(P_n', P_1')_{\mu_1}\\ (P_n', P_2')_{\mu 1}\\ \vdots\\ (P_n', P_{n-1}')_{\mu_1}\end{pmatrix},$$

that is,

$$(D_{n-1}+\lambda C_{n-1})r^{(n)}=-\lambda c^{(n)}, \tag{16}$$

where $D_{n-1}=\operatorname{diag}\{d_1, d_2,\ldots, d_{n-1}\}$, C_{n-1} is the symmetric matrix $(c_{ij})_{i,j=1}^{n-1}$, $c^{(n)}=(c_{n1}, c_{n2},\ldots, c_{n,n-1})^{\mathrm T}$ and $r^{(n)}=(r_{n1}, r_{n2},\ldots, r_{n,n-1})^{\mathrm T}$.

Operating in (16) and taking into account (15), we get

$$Q_n(x;\lambda)=P_n(x)-(P_1(x),\ldots, P_{n-1}(x))(D_{n-1}+\lambda C_{n-1})^{-1}\lambda c^{(n)}$$

$$=\frac{1}{\det(D_{n-1}+\lambda C_{n-1})}\begin{vmatrix} & & & \lambda c_{n1}\\ & D_{n-1}+\lambda C_{n-1} & & \vdots\\ & & & \lambda c_{n,n-1}\\ P_1(x) & \cdots & P_{n-1}(x) & P_n(x)\end{vmatrix}. \tag{17}$$

As such, this formula is not very useful for arbitrary Borel measures, but if we can write $c_{ij}=a_{\min\{i,j\}}/A_iA_j$ for all $i, j=1, 2,\ldots$ and we denote $d_i'=A_i^2d_i$, then (17) can be reduced to

$$(A_1A_2\cdots A_{n-1})^2A_n\det(D_{n-1}+\lambda C_{n-1})Q_n(x;\lambda)$$

$$=\begin{vmatrix} d_1'+\lambda a_1 & \lambda a_1 & \lambda a_1 & \cdots & \lambda a_1\\ \lambda a_1 & d_2'+\lambda a_2 & \lambda a_2 & \cdots & \lambda a_2\\ \lambda a_1 & \lambda a_2 & d_3'+\lambda a_3 & \cdots & \lambda a_3\\ \vdots & \vdots & \vdots & & \vdots\\ \lambda a_1 & \lambda a_2 & \lambda a_3 & \cdots & \lambda a_{n-1}\\ A_1P_1(x) & A_2P_2(x) & A_3P_3(x) & \cdots & A_nP_n(x)\end{vmatrix}$$

$$=\begin{vmatrix} d_1' & 0 & \cdots & 0 & \lambda a_1\\ \lambda(a_1-a_2) & d_2' & \cdots & 0 & \lambda a_2\\ \vdots & \vdots & & \vdots & \vdots\\ \lambda(a_1-a_{n-1}) & \lambda(a_2-a_{n-1}) & \cdots & d_{n-1}' & \lambda a_{n-1}\\ \tilde p_1(x) & \tilde p_2(x) & \cdots & \tilde p_{n-1}(x) & A_nP_n(x)\end{vmatrix},$$

where, for the sake of simplicity, in the last determinant we have written $\tilde{p}_j(x)$ instead of $A_jP_j(x)-A_nP_n(x)$.

To evaluate this determinant we expand it from the nth row and thus we obtain

$$(A_1A_2\cdots A_{n-1})^2A_n\det(D_{n-1}+\lambda C_{n-1})Q_n(x;\lambda)$$
$$=A_n\left[d_1'\cdots d_{n-1}'-\sum_{j=1}^{n-1}\alpha_j(\lambda)\right]P_n(x)+\sum_{j=1}^{n-1}A_j\alpha_j(\lambda)P_j(x),$$

where $\alpha_j(\lambda)$ are polynomials in λ of degree j and independent of n, provided that $a_j\neq a_{j+1}$ for every j.

The leading coefficient of this polynomial is a polynomial in λ of degree $n-1$; we will denote it by $\beta_{n-1}(\lambda)$, that is, $\beta_{n-1}(\lambda)=(A_1A_2\cdots A_{n-1})^2A_n\det(D_{n-1}+\lambda C_{n-1})$.

Comparing the leading coefficients in the above formula, we conclude that $\beta_{n-1}(\lambda)=A_n[d_1'\cdots d_{n-1}'-\Sigma_{j=1}^{n-1}\alpha_j(\lambda)]$.

So, the final representation for $Q_n(x;\lambda)$ is

$$\beta_{n-1}(\lambda)Q_n(x;\lambda)=\beta_{n-1}(\lambda)P_n(x)+\sum_{j=1}^{n-1}A_j\alpha_j(\lambda)P_j(x). \tag{18}$$

The above arguments justify the following definition.

Definition 4.1. We say that the pair $\{\mu_0,\mu_1\}$ is a *coherent pair* if there exist constants A_n, a_n with $A_n\neq 0$ and $a_n\neq a_{n+1}$ for every n such that $c_{ij}=a_{\min\{i,j\}}/A_iA_j$ for all $i, j=1,2,\ldots,$ where $c_{ij}=(P_i',P_j')_{\mu_1}$.

From (18), we can state the following.

Corollary 4.2. *If $\{\mu_0,\mu_1\}$ is a coherent pair, the relation*

$$\beta_n(\lambda)Q_{n+1}(x)-\beta_{n-1}(\lambda)Q_n(x)=\beta_n(\lambda)P_{n+1}(x)+[A_n\alpha_n(\lambda)-\beta_{n-1}(\lambda)]P_n(x)$$

holds.

The condition of coherence may seem very difficult to verify. However, it is possible to give a condition far less technical.

Proposition 4.3 (see [16]). *Let (P_n) and (R_n) be, respectively, the SMOP with respect to μ_0 and μ_1. The pair $\{\mu_0,\mu_1\}$ is coherent if and only if there exist nonzero constants A_n such that*

$$R_n(x)=\frac{P_{n+1}'(x)}{n+1}-\frac{A_n}{A_{n+1}}\frac{P_n'(x)}{n}$$

holds.

Problem 4.4 Given μ_0, describe all positive finite Borel measures μ_1 such that $\{\mu_0,\mu_1\}$ is a coherent pair.

If μ_0 is a classical measure (Jacobi, Laguerre, Hermite), then $(P'_{n+1}/(n+1))$ is a SMOP. Thus, according to our last proposition, (R_n) is a SMOP which is quasi orthogonal of order 1 with respect to $P'_{n+1}/(n+1)$. The framework of this problem is the theory of orthogonal polynomials with respect to a regular (or quasi definite) inner product.

Let u be a classical linear functional and let (P_n) denote the corresponding sequence of classical monic orthogonal polynomials. It is well known that $(P'_{n+1}/(n+1))$ is a SMOP with respect to the functional $\phi(x)u$ where $\phi(x)$ is a polynomial of degree at most 2 such that $D[\phi(x)u] = kP_1(x)u$ (see [27]).

Using a result of [26], there exists a polynomial of degree 1, $h(x) = N(x-c)$, such that $h(x)v = \phi(x)u$ where v is the linear functional whose associated SMOP is (R_n). Then,

$$v = M\delta(x-c) + [N(x-c)]^{-1}\phi(x)u$$

is the general solution. Among these functionals, all the positive definite functionals are the solutions to the coherent pairs.

Remarks 4.5. (1) In [16], the concept of symmetric coherent pairs is introduced as follows. For symmetric measures μ_0, μ_1, the pair $\{\mu_0, \mu_1\}$ is called *symmetrically coherent* (from now on s-coherent) if $c_{ij} = a_{\min\{i,j\}}/A_i A_j$, when $i+j$ is an even number, and $c_{ij} = 0$, otherwise. A characterization of the s-coherent pairs is also obtained in [16]. The pair $\{\mu_0, \mu_1\}$ is s-coherent if and only if there exist nonzero constants A_n, such that

$$R_n(x) = \frac{P'_{n+1}(x)}{n+1} - \frac{A_{n-1}}{A_{n+1}} \frac{P'_{n-1}(x)}{n-1}$$

holds.

In the Hermite case, that is, $d\mu_0 = e^{-x^2}\, dx$, the set of symmetric measures μ_1 such that $\{\mu_0, \mu_1\}$ is s-coherent is contained in the set of linear functionals v such that there exists a polynomial $h(x)$ of degree 2 satisfying the functional equation $h(x)v = u$, where $\langle u, p(x)\rangle = \int_{\mathbb{R}} p(x)e^{-x^2}\, dx$. Then the solution is completely determined.

Also, if $d\mu_0 = (1-x^2)^\nu\, dx$ (the Gegenbauer case), the solutions are included in the set of linear functionals v such that there exists a polynomial $h(x)$ of degree 2 such that $h(x)v = (x^2 - 1)u$ where $\langle u, p(x)\rangle = \int_{-1}^{1} p(x)(1-x^2)^\nu\, dx$.

(2) Another interesting aspect of coherent pairs is the efficient calculation of Sobolev–Fourier coefficients for functions $f \in W_2^1(\mathbb{R}; d\mu_0, d\mu_1)$ (see [16]).

5. Sobolev-type orthogonal polynomials

If we consider an inner product as (1) with $p = 1$ and μ_1 an atomic measure, some recent contributions have shown a more realistic approach to the problems considered in Section 3.

Bavinck and Meijer [3] studied the inner product

$$(f, g)_w = \int_{-1}^{1} f(x)g(x)(1-x^2)^\alpha\, dx + M[f(1)g(1) + f(-1)g(-1)] + N[f'(1)g'(1) + f'(-1)g'(-1)], \quad M, N \geqslant 0,\ \alpha > -1.$$

The connection with Gegenbauer polynomials is obtained in terms of differential operators as follows.

Proposition 5.1. *Let* (Q_n) *be the SMOP with respect to the above inner product. There exist constants* a_n, b_n, c_n, d_n, e_n *such that the formula*

$$Q_n(x) = (a_n x^2 D^2 + b_n xD + c_n) P_n^{(\alpha,\alpha)}(x)$$
$$= \left(a_n(1-x^2)^2 D^4 + d_n(1-x^2)D^2 + e_n\right) P_n^{(\alpha,\alpha)}(x)$$

holds.

Moreover, a representation in terms of hypergeometric functions ${}_4F_3$ is given. For such a family of orthogonal polynomials, a recurrence relation appears and represents some performance with respect to the classical Sobolev orthogonal polynomials. Using the fact that the multiplication operator by $(x^2-1)^2$ in $\mathscr{P}$ is self-adjoint,

$$(x^2-1)^2 Q_n(x) = \sum_{k=n-4}^{n+4} \alpha_{nk} Q_k(x),$$

with $\alpha_{n,n+4} = 1$ and $\alpha_{n,n-4} \neq 0$, holds. But this recurrence relation is not minimal in the sense of the number of elements involved. Indeed, Bavinck and Meijer show that the recurrence relation

$$(x^3 - 3x)Q_n(x) = \sum_{k=n-3}^{n+3} \beta_{nk} Q_k(x)$$

also holds.

Recently, in [11] is shown that this last one is minimal.

With respect to the zeros of Q_n, they are real and simple. If $N \neq 0$, Q_n has exactly two real zeros $\pm x_0$ outside $(-1, 1)$ for n sufficiently large, $n \gg 1$.

Problem 5.2. Consider the above questions for symmetric measures and masses located in the ends of the interval.

A particular case of this problem was analyzed in [5].

In a natural way, and taking into account Laguerre polynomials as a limit situation for Jacobi polynomials, Koekoek [18] has considered

$$(f, g)_w = \int_0^\infty f(x)g(x)x^\alpha e^{-x}\, dx + \sum_{k=0}^{p} M_k f^{(k)}(0) g^{(k)}(0),$$

with $M_k \geqslant 0$ and $\alpha > -1$. In this situation a connection between Q_n and Laguerre polynomials via a differential operator is given as

$$Q_n(x) = \left(\sum_{k=0}^{r} B_{nk} D^k\right) L_n^\alpha(x),$$

where $r = \{\min(n, p+1)\}$.

In this situation a representative of Q_n as a ${}_{p+2}F_{p+2}$ hypergeometric function is obtained, as well as the fact that (Q_n) satisfies the second-order linear differential equation

$$A(x;\, n)Q_n''(x) + B(x;\, n)Q_n'(x) + C(x;\, n)Q_n(x) = 0,$$

where A, B, C are polynomials of degrees independent of n.

Concerning the zeros, whenever $p = 1$, the main result is the following proposition.

Proposition 5.3 (see [19]). (a) *The polynomial Q_n has n real zeros which are simple. At least $n - 1$ of them belong to $(0, +\infty)$.*

(b) *If $M_1 > 0$ and $n \gg 1$, then Q_n has a zero x_{n1} in $(-\infty, 0]$. Moreover, if $M_0 > 0$, it is possible to give bounds for such a zero as $-x_{n2} < x_{n1} \leqslant 0$ and $-\frac{1}{2}\sqrt{M_1/M_0} \leqslant x_{n1} \leqslant 0$. For all $M_0 \geqslant 0$, we have $\lim_{n\to\infty} x_{n1} = 0$.*

The interesting work [18] has been extended in two ways. The first one corresponds to a generalization for measures whose support is in $(0, +\infty)$. Meijer [29] considers the following inner product:

$$(f,\, g)_w = \int_0^\infty f(x)g(x)\, \mathrm{d}\mu_0 + \sum_{k=0}^{1} M_k f^{(k)}(0)g^{(k)}(0).$$

The technique for the analysis of the sequence (Q_n) is very different from [18]. The constraint about the hypergeometric character of the initial sequence with respect to μ_0 disappears and then a very elegant algebraic approach is provided.

Proposition 5.4 (see [29]). *The polynomials $Q_n(x)$ can be expressed as*

$$Q_n(x) = \alpha_n P_n(x;\, \mathrm{d}\mu) + \beta_n x P_{n-1}(x;\, x^2\, \mathrm{d}\mu) + \gamma_n x^2 P_{n-2}(x;\, x^4\, \mathrm{d}\mu).$$

The last representation plays a very important role in the study of the distribution of the zeros.

In fact, the following proposition holds.

Proposition 5.5 (see [29]). *Let us suppose $M_1 > 0$.*

(a) *The polynomial Q_n has n real and simple zeros; at most one of them is outside $(0, +\infty)$.*

(b) *If $Q_{n_0}(x)$ has a zero in $(-\infty, 0]$, then $Q_n(x)$ has a zero in $(-\infty, 0)$ for all n with $n > n_0$.*

(c) *Let $y_{n-1,1} < y_{n-1,2} < \cdots < y_{n-1,n-1}$ denote the zeros of $P_{n-1}(x;\, x^2\, \mathrm{d}\mu)$ being ordered by increasing size. If we suppose that Q_n has a zero x_{n1} in $(-\infty, 0]$, then Q_n has a zero in $(0, y_{n-1,1})$ and a zero in every interval $(y_{n-1,i}, y_{n-1,i+1})$, for $i = 1, 2, \ldots, n-2$. Moreover, $-y_{n-1,1} < x_{n1} \leqslant 0$.*

The second direction has been explored in [1]. There the inner product

$$(f,\, g)_w = \int_I f(x)g(x)\, \mathrm{d}\mu_0 + M_0 f(c)g(c) + M_1 f'(c)g'(c),$$

where $M_i \geqslant 0$, $i = 0, 1$, and $c \in \mathbb{R}$, is studied.

Algebraic properties as, for instance, recurrence relations or Christoffel–Darboux formulas are independent of the location of the point c with respect to the support of μ_0. As in [18], the main tool is the self-adjointness of multiplication by $(x-c)^2$. Then the recurrence relation is

$$(x-c)^2 Q_n(x) = \sum_{k=n-2}^{n+2} \alpha_{nk} Q_k(x),$$

with $\alpha_{n,n-2} \neq 0$. Such kind of relations appears in [17].

With respect to the distribution of the zeros, the following result is deduced.

Proposition 5.6 (see [1]). (a) *If* $M_1 > 0$ *and* $n \geqslant 3$, Q_n *has a least* $n-2$ *different zeros with odd multiplicity in* $\mathring{I}$ ($\mathring{I}$ *denotes the interior of the interval* I).

(b) *Moreover, if* $c = \inf I$ *or* $c = \sup I$, *then the zeros of* Q_n *are real and simple. At least* $n-1$ *of them are located in* $\mathring{I}$.

(c) *If* $c = \sup I$ *and there exists a zero* x_{nn} *outside* I *and* x_{n1} *denotes the smallest zero of* Q_n, *then*

$$c < x_{nn} < c + \frac{c - x_{n1}}{n-1}$$

and $|x_{nn} - c| < |x_{n,n-1} - c|$. *If* $M_0 \neq 0$, *then* $x_{nn} - c < \frac{1}{2}\sqrt{M_1/M_0}$ *holds.*

If $c \in I$, some results have been obtained in [30], in terms of the so-called tangent property.

Now, we consider the inner product

$$(f, g)_w = \int_{-1}^{1} f(x) g(x) \, \mathrm{d}\mu_0(x) + \lambda f'(c) g'(c),$$

where $c \in \mathbb{R}$, $\lambda > 0$ and μ_0 belongs to Nevai's class $M(0, 1)$. Recall, if $\mu_0 \in M(0, 1)$, then $\operatorname{supp} \mu_0 = [-1, 1] \cup E$ where E is a bounded and countable set with $E' \subset \{-1, 1\}$.

In this situation, some results are obtained in [25] about the asymptotic behaviour of the sequence (q_n) with respect to the sequence (p_n), where (p_n), (q_n) are the corresponding orthonormal polynomials associated with μ_0 and the inner product $(\cdot, \cdot)_w$ respectively.

Proposition 5.7 (see [25]). (a) *If* $c \in \mathbb{R} \setminus \operatorname{supp} \mu_0$,

$$\lim_{n\to\infty} \frac{q_n(x)}{p_n(x)} = \frac{1}{|c + \sqrt{c^2-1}|} \left(1 - \frac{\sqrt{c^2-1}}{x + \sqrt{x^2-1}} \, \frac{\left(x + \sqrt{x^2-1}\right) - \left(c + \sqrt{c^2-1}\right)}{x - c} \right),$$

uniformly on compact sets in $\mathbb{C} \setminus [\operatorname{supp} \mu_0 \cup \{c\}]$. *Moreover,*

$$\lim_{n\to\infty} \frac{q_n(c)}{p_n(c)} = 0 \quad \textit{and} \quad \lim_{n\to\infty} q_n'(c) p_n'(c) = 2\lambda^{-1} \left| \sqrt{c^2-1} \right|.$$

(b) *If* $c \in \operatorname{supp} \mu_0$, $\lim_{n\to\infty} q_n(x)/p_n(x) = 1$ *uniformly on compact sets in* $\mathbb{C} \setminus \operatorname{supp} \mu_0$. *Moreover,*

$$\lim_{n\to\infty} q_n'(c) p_n'(c) = 0.$$

Nevai's class $M(0, 1)$ can be characterized in terms of the asymptotic behaviour of the parameters which appear in the three-term recurrence relation satisfied by the orthonormal polynomials (see [32]). If

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x),$$

then $\lim_{n\to\infty} a_n = \frac{1}{2}$ and $\lim_{n\to\infty} b_n = 0$.

Since the sequence (q_n) satisfies a five-term recurrence relation as

$$(x-c)^2 q_n(x) = \alpha_{n+2}q_{n+2}(x) + \beta_{n+1}q_{n+1}(x) + \gamma_n q_n(x) + \beta_n q_{n-1}(x) + \alpha_n q_{n-2}(x),$$

the following proposition holds.

Proposition 5.8 (see [25]). *For every* $c \in \mathbb{R}$, *we have*

$$\lim_{n\to\infty} \alpha_n = \tfrac{1}{4}, \qquad \lim_{n\to\infty} \beta_n = -c, \qquad \lim_{n\to\infty} \gamma_n = \tfrac{1}{2}(1+2c^2).$$

Problem 5.9. Discuss the asymptotic behaviour of the sequence (q_n) for other kind of measures, for example, when the support is unbounded and μ_0 is a Freud's weight function.

Finally, some authors have considered the inner product

$$(f, g)_w = \int_{\mathbb{R}} f(x)g(x)\, d\mu_0 + \lambda f^{(r)}(c) g^{(r)}(c),$$

where $\lambda > 0$ and $r \geqslant 1$ (see [24]).

An explicit expression for Q_n in terms of the polynomials P_n is obtained using its Fourier expansion with respect to this SMOP:

$$Q_n(x) = P_n(x) - \frac{\lambda P_n^{(r)}(c)}{1 + \lambda K_{n-1}^{(r,r)}(c, c)} K_{n-1}^{(0,r)}(x, c),$$

where

$$K_{n-1}^{(i,j)}(x, y) = \sum_{l=0}^{n-1} \frac{P_l^{(i)}(x) P_l^{(j)}(y)}{\| P_l \|_\mu^2}.$$

From the Christoffel–Darboux formula

$$(x-c)^{r+1} Q_n(x) = M_{r+1}(x; n) P_n(x) + N_r(x; n) P_{n-1}(x), \tag{19}$$

with

$$M_{r+1}(x; n) = (x-c)^{r+1} - \alpha_n(c; \lambda) T_r(x; c; P_{n-1}),$$
$$N_r(x; n) = \alpha_n(c; \lambda) T_r(x; c; P_n),$$

where

$$\alpha_n(c; \lambda) = r! \, \| P_{n-1} \|_{\mu_0}^{-2} \frac{\lambda P_n^{(r)}(c)}{1 + \lambda K_{n-1}^{(r,r)}(c, c)},$$

and $T_r(x; c; P_n)$ has been introduced in Section 2.

Moreover, from the self-adjointness of multiplication by $(x-c)^{r+1}$, a $(2r+3)$-term recurrence relation for the sequence (Q_n) can be deduced:

$$(x-c)^{r+1}Q_n(x) = \sum_{j=n-r-1}^{n+r+1} \gamma_{nj}Q_j(x), \tag{20}$$

with $\gamma_{n,n-r-1} \neq 0$.

The parameters γ_{nj} are easily computed using (19). They are solutions of an upper triangular system of linear equations.

If the support of μ_0 is contained in the interval $[0, a)$, where a may be finite or infinite, then some results concerning to the location of zeros of Q_n can be given.

Proposition 5.10 (see [28]). *Let us suppose* $\lambda > 0$ *and* $c = 0$.

(a) *The zeros of* Q_n *are real and simple. At least* $n-1$ *of them are in* $(0, a)$. *If the polynomial* Q_n *has a root* x_{n1} *outside* $(0, a)$, *then* $x_{n1} \leqslant 0$ *and* $n \geqslant r+1$. *Moreover, if* $x_{n1} < 0$, *then*

$$0 < -x_{n1} < rx_{n,r+1},$$

where $x_{n,r+1}$ *denotes the rth positive zero of* Q_n. *Besides, if a is finite, then* $-x_{n1} < ar/(n-r)$.

(b) *Suppose* $n \geqslant r+1$; *if* (y_{ni}) *denotes the set of zeros of* P_n, *then* $x_{n1} < y_{n1}$ *and* $y_{ni} < x_{n,i+1} < y_{n,i+1}$, *for* $i = 1, 2, \ldots, n-1$.

Problem 5.11 (see [28]). If $(x_{ni}(\lambda))_{i=1}^{n}$ is the set of zeros of $Q_n(x; \lambda)$, analyze its behaviour as a function of λ.

The case corresponding to $r = 1$ has been studied in [23]. The following result has been obtained.

Proposition 5.12 (see [23]). *Let us consider c outside the interval of orthogonality.*

(a) *For every* i, $x_{ni}(\lambda)$ *is an increasing function of* λ.

(b) *Let*

$$R_n(x) = P_n(x) - \frac{P_n'(c)}{K_{n-1}^{(1,1)}(c, c)} K_{n-1}^{(0,1)}(x, c)$$

be the limit polynomial when $\lambda \to +\infty$; *then it has real and simple roots* z_{ni} *such that* $y_{ni} < x_{ni}(\lambda) < z_{ni}$ *for* $i = 1, \ldots, n$.

On the other hand, from (20) it seems natural to consider an analog of Favard's theorem. It is well known for standard polynomials (P_n) satisfying a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

with $\gamma_n \neq 0$ for every $n \in \mathbb{N}$, that a linear functional u on $\mathscr{P}$ exists such that

$$\langle u, P_n P_j \rangle = M_n \delta_{nj}.$$

For $c = 0$ we search symmetric bilinear forms B on $\mathscr{P} \times \mathscr{P}$ such that the corresponding sequences of monic orthogonal polynomials satisfy (20). In fact the following characterization is obtained in [10].

Proposition 5.13. *The following assertions are equivalent.*

(a) *There exists a real function* μ_0 *and constant real numbers* $(M_k)_{k=1}^r$ *such that*

$$B(f, g)=\int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}\mu_0+\sum_{k=1}^{r} M_k f^{(k)}(0)g^{(k)}(0).$$

(b) *The operator* $H:\mathscr{P}\to\mathscr{P}$ *such that* $H(p)=x^{r+1}p(x)$ *is self-adjoint with respect to* B, $B(x^{r+1}f, xg)=B(xf, x^{r+1}g)$ *and the Gram matrix of* B *with respect to* (x^k) *is structured as follows*: $B(x^k, x^m)=B(1, x^{k+m})$ *when* $1\leqslant k, m\leqslant r$ *and* $k\neq m$.

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