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Asymptotics of Sobolev orthogonal polynomials for Hermite coherent pairs

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Abstract

Let Q_n be the polynomials orthogonal with respect to the Sobolev inner product

$$(f,g)_S = \int fg \, \mathrm{d}\mu_0 + \int f'g' \, \mathrm{d}\mu_1,$$

being (μ_0, μ_1) a coherent pair where one of the measures is the Hermite measure. The outer relative asymptotics for Q_n with respect to Hermite polynomials are found. On the other hand, we consider the Sobolev scaled polynomials and we obtain the Plancherel–Rotach asymptotics for those as well as a consequence about their zeros. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let μ_0 and μ_1 be finite positive Borel measures on the real line $\mathbb{R}$ and $\lambda > 0$, and consider the Sobolev inner product

$$(f,g)_S = \int fg \, \mathrm{d}\mu_0 + \lambda \int f'g' \, \mathrm{d}\mu_1.$$

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In the study of the Sobolev polynomials Q_n, orthogonal with respect to $(.,.)_S$, an interesting situation occurs when the pair of measures (μ_0,μ_1) is a coherent pair (respectively, a symmetrically coherent pair), which means that there exist non-zero constants σ_n such that the monic polynomials T_n and P_n orthogonal with respect to μ_1 and μ_0, respectively, satisfy

$$T_n = \frac{P'_{n+1}}{n+1} + \sigma_n \frac{P'_n}{n} \quad (n \geqslant 1)$$

(respectively, $T_n = P'_{n+1}/(n+1) + \sigma_n P'_{n-1}/(n-1)$, $(n \geqslant 2)$).

Under this condition, it can be derived an algebraic relation between the polynomials Q_n and T_n, which has been the departure point to study some asymptotic properties of the polynomials Q_n; among them, the relative asymptotic behaviour of Q_n with respect to either P_n or T_n. Many of these results have been generalized for a more wide class of measures (see [4]).

Meijer has proved that if (μ_0,μ_1) is either a coherent or a symmetrically coherent pair, then at least μ_0 or μ_1 is a classical measure (that is, Jacobi, Laguerre or Hermite measure), and has completely classified all the coherent and the symmetrically coherent pairs (see [5]).

For coherent pairs where one of the measures is the Jacobi one, the strong asymptotics for the polynomials Q_n have been found in [1,3]. More recently, the analogous result for the Laguerre case has been obtained in [6]. Also, asymptotic properties have been studied in [2] for the nondiagonal case, which recovers a type of Laguerre coherent pairs as a particular case. However, nothing has been made when one of the measures is the Hermite one.

The aim of this paper is to fill this gap. Note that, according to Meijer's result, if (μ_0,μ_1) is a symmetrically coherent pair being one of the measures the Hermite measure, only the following cases can occur:

- *Case* I: $d\mu_0 = (x^2+\xi^2)e^{-x^2}\,dx$ with $\xi \in \mathbb{R}$ and $d\mu_1 = e^{-x^2}\,dx$,
- *Case* II: $d\mu_0 = e^{-x^2}\,dx$ and $d\mu_1 = \frac{e^{-x^2}}{x^2+\xi^2}\,dx$ with $\xi \in \mathbb{R}\backslash\{0\}$.

In both cases, we deduce the relative asymptotics and this is carried out in Section 2 (see Theorems 2.3 and 2.7). However, we obtain more information about the Sobolev polynomials in the unbounded case if we scale the polynomials Q_n. In this way, in Section 3, we obtain for scaled Q_n properties similar to those ones that verify the scaled Hermite polynomials (see [8]). First, we derive the corresponding scaled relative asymptotics (Theorem 3.1) and we deduce from here the Plancherel–Rotach asymptotics for $Q_n(\sqrt{n}x)$ in $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$. As an immediate consequence we get the location of scaled zeros.

Along this paper, we will use the following notations: H_n denotes the Hermite polynomial with the normalization $H_n(x) = 2^n x^n + \cdots$, and Q_n the Sobolev polynomial orthogonal with respect to $(.,.)_S$ normalized by $Q_n(x) = 2^n x^n + \cdots$. Also, φ is the conformal mapping of $\mathbb{C}\backslash[-1,1]$ onto the exterior of the closed unit disk, that is, $\varphi(x) = x + \sqrt{x^2-1}$ with $\sqrt{x^2-1} > 0$ when $x > 1$.

2. Relative asymptotics

Through this section, we denote $k_n = \int_{\mathbb{R}} H_n^2(x)e^{-x^2}\,dx = 2^n n!\sqrt{\pi}$.

2.1. *Case* I: ($d\mu_0 = (x^2+\xi^2)e^{-x^2}\,dx$, $\xi \in \mathbb{R}$, and $d\mu_1 = e^{-x^2}\,dx$).

Lemma 2.1. *The polynomials H_n and Q_n satisfy the relation*

$$H_{n+2} = Q_{n+2} + a_n Q_n, \quad (n \geqslant 0) \tag{2.1}$$

being $H_1 = Q_1$, $H_0 = Q_0$ and

$$a_n = \frac{k_{n+2}}{4(Q_n, Q_n)_S}, \quad (n \geqslant 0). \tag{2.2}$$

Moreover, a_n are constants depending on ξ and λ satisfying the nonlinear recurrence formula

$$a_n = \frac{4(n+1)(n+2)}{2[2(2\lambda+1)n + 1 + 2\xi^2] - a_{n-2}}, \quad (n \geqslant 2). \tag{2.3}$$

Proof. Let us consider the Fourier expansion $H_{n+2} = Q_{n+2} + \sum_{i=0}^{n+1} a_i Q_i$. A suitable use of the orthogonality of Q_n and H_n shows that $a_i = 0$ for $i = 0, \ldots, n-1$, $a_n = (H_{n+2}, Q_n)_S/(Q_n, Q_n)_S$ and, from the symmetrical character of H_n, we get $a_{n+1} = 0$.

Using adequately (2.1) and the orthogonality of Q_n,

$$\begin{aligned}(Q_n, Q_n)_S = (H_n, Q_n)_S &= (H_n, H_n - a_{n-2} Q_{n-2})_S \\ &= (H_n, H_n)_S - a_{n-2}(H_n, H_{n-2} - a_{n-4} Q_{n-4})_S \\ &= (H_n, H_n)_S - a_{n-2}(H_n, H_{n-2})_S\end{aligned}$$

and $(H_{n+2}, Q_n)_S = (H_{n+2}, H_n)_S$. Hence,

$$a_n = \frac{(H_{n+2}, H_n)_S}{(H_n, H_n)_S - a_{n-2}(H_n, H_{n-2})_S}, \quad (n \geqslant 2). \tag{2.4}$$

Moreover,

$$(H_{n+2}, H_n)_S = \int_{\mathbb{R}} H_{n+2} H_n \, \mathrm{d}\mu_0 = \frac{1}{4} k_{n+2} = 2^n (n+2)! \sqrt{\pi}$$

and

$$\begin{aligned}(H_n, H_n)_S &= \int_{\mathbb{R}} H_n^2 \, \mathrm{d}\mu_0 + \lambda 4 n^2 k_{n-1} \\ &= \frac{1}{4} \int_{\mathbb{R}} (2x H_n)^2 \mathrm{e}^{-x^2} \mathrm{d}x + (\xi^2 + 2\lambda n) k_n = \sqrt{\pi} 2^n n! \left(2\lambda n + \xi^2 + n + \frac{1}{2}\right),\end{aligned}$$

where we have used the well known properties of the classical Hermite polynomials (see [7]).

Now (2.2) is immediate and substituting the above values in (2.4), formula (2.3) follows. □

Lemma 2.2. *The sequence $(b_n)_{n \geqslant 1} = (a_n/2(n+2))_{n \geqslant 1}$ is bounded by $1/(1+2\lambda)$ and converges to $b = 1/\varphi(2\lambda+1)$.*

Proof. From (2.2) we write $b_n = k_{n+2}/[8(n+2)(Q_n, Q_n)_S]$.

Let (P_n) be the sequence of polynomials orthogonal with respect μ_0, with $P_n(x) = 2^n x^n + \cdots$. Using the extremal property of the norm of the orthogonal polynomials, we have

$$(Q_n, Q_n)_S = \int_{\mathbb{R}} Q_n^2 \, d\mu_0 + \lambda \int_{\mathbb{R}} (Q_n')^2 \, d\mu_1 \geqslant \int_{\mathbb{R}} P_n^2 \, d\mu_0 + 4\lambda n^2 k_{n-1}$$
$$= \int_{\mathbb{R}} ((x+\xi)P_n)^2 \, d\mu_1 + 4\lambda n^2 k_{n-1} \geqslant \frac{1}{4} k_{n+1} + 4\lambda n^2 k_{n-1}$$

and thus $b_n \leqslant 1/(1+2\lambda) < 1$, for all $n \geqslant 1$.

Besides, from (2.3)

$$b_n = \frac{n+1}{2(2\lambda+1)n + 1 + 2\xi^2 - nb_{n-2}}, \quad n \geqslant 2. \tag{2.5}$$

Suppose that (b_n) converges to b, then from (2.5), $b^2 - 2(2\lambda+1)b + 1 = 0$ and since $b < 1$, one has $b = 1/\varphi(2\lambda+1)$. But

$$|b_n - b| = \left| \frac{n + 1 - b[2(2\lambda+1)n + 1 + 2\xi^2] + nb[b_{n-2} - b] + nb^2}{2(2\lambda+1)n + 1 + 2\xi^2 - nb_{n-2}} \right|$$
$$\leqslant \frac{|n[b^2 - 2(2\lambda+1)b + 1] + 1 - b(1+2\xi^2)|}{(4\lambda+1)n} + \frac{b}{4\lambda+1} |b_{n-2} - b|,$$

which implies

$$\limsup_n |b_n - b| \leqslant \frac{b}{4\lambda+1} \limsup_n |b_{n-2} - b|$$

and the result follows. □

As a consequence of Perron's formula and the relation between Hermite and Laguerre polynomials (see [7, p. 199]), we have

$$\lim_n \frac{nH_n(x)}{H_{n+2}(x)} = -\frac{1}{2} \tag{2.6}$$

uniformly on compact sets of $\mathbb{C} \backslash \mathbb{R}$.

Now, we can derive the following result.

Theorem 2.3. *The Sobolev orthogonal polynomials* Q_n *satisfy*

$$\lim_n \frac{Q_n(x)}{H_n(x)} = \frac{\varphi(2\lambda+1)}{\varphi(2\lambda+1) - 1} \tag{2.7}$$

uniformly on compact sets of $\mathbb{C} \backslash \mathbb{R}$.

Proof. With the notation

$$f_n(x) = \frac{Q_n(x)}{H_n(x)}, \quad c_n(x) = -a_n \frac{H_n(x)}{H_{n+2}(x)}$$

formula (2.1) reads as

$$f_{n+2}(x) = 1 + c_n(x) f_n(x), \tag{2.8}$$

where f_n and c_n are analytic functions in $\mathbb{C}\backslash\mathbb{R}$ and Lemma 2.2 and (2.6) give us $\lim_n c_n(x) = 1/\varphi(2\lambda+1) = b$ uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$.

If we put

$$g_n(x) = f_n(x) - \frac{1}{1-b}, \tag{2.9}$$

then we can write (2.8) as

$$g_{n+2}(x) = bg_n(x) + \varepsilon_n(x), \tag{2.10}$$

with

$$\varepsilon_n(x) = (c_n(x) - b)\left(g_n(x) + \frac{1}{1-b}\right).$$

Since $0 < b < 1$ from (2.8), it is straightforward to deduce that (f_n) and, consequently, (g_n) are uniformly bounded on compact subsets of $\mathbb{C}\backslash\mathbb{R}$. Therefore, $\lim_n \varepsilon_n(x) = 0$ uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$ and from (2.10), we deduce the same behaviour for g_n and the result follows. □

2.2. *Case* II: ($\mathrm{d}\mu_0 = \mathrm{e}^{-x^2}\,\mathrm{d}x$ and $\mathrm{d}\mu_1 = \frac{\mathrm{e}^{-x^2}}{x^2+\xi^2}\,\mathrm{d}x$, $\xi \in \mathbb{R}\backslash\{0\}$).

In what follows, (T_n) stands for the sequence of polynomials orthogonal with respect to μ_1, with $T_n(x) = 2^n x^n + \cdots$, and we denote $k'_n = \int_{\mathbb{R}} T_n^2\,\mathrm{d}\mu_1$.

From the coherence condition and the fact that $H'_n = 2nH_{n-1}$, we have

$$T_n = \frac{H'_{n+1}}{2(n+1)} + \sigma_{n-1}\frac{H'_{n-1}}{2(n-1)} = H_n + \sigma_{n-1}H_{n-2}, \quad (n\geqslant 2) \tag{2.11}$$

and then

$$\sigma_{n-1} = \frac{k'_n}{4k_{n-2}}, \quad (n\geqslant 2). \tag{2.12}$$

Lemma 2.4. *The sequence* $(\alpha_n)_{n\geqslant 1} = (\sigma_n/2n)_{n\geqslant 1}$ *converges to* 1.

Proof. We will distinguish according to n be either even or odd.

In the first case, by a process of symmetrization, (2.11) becomes the coherence relation for one of the Laguerre coherent pairs, being σ_{2n} the corresponding coherence parameters, and so, as it has been proved in [6], $\sigma_{2n}/4n$ tends to 1.

When n is odd, integrating (2.11) with respect to μ_1, $\sigma_{2n-1} = (-\int_{\mathbb{R}} H_{2n}\,\mathrm{d}\mu_1)/(\int_{\mathbb{R}} H_{2n-2}\,\mathrm{d}\mu_1)$ and it suffices to estimate $\int_{\mathbb{R}} H_{2n}\,\mathrm{d}\mu_1$. For that, using the relation between Hermite and Laguerre polynomials (see [7, p. 106]), the Rodrigues formula for the Laguerre ones (see [7, p. 101]), after integration by parts n times, we obtain

$$\begin{aligned}\int_{\mathbb{R}} H_{2n}\,\mathrm{d}\mu_1 &= (-1)^n 2^{2n} n! \int_0^{+\infty} \frac{L_n^{(-1/2)}(x)}{x+\xi^2}\mathrm{e}^{-x}x^{-1/2}\,\mathrm{d}x \\ &= (-1)^n 2^{2n} \int_0^{+\infty} \frac{1}{x+\xi^2} D^n(x^{n-1/2}\mathrm{e}^{-x})\,\mathrm{d}x \\ &= (-1)^n 2^{2n} n! \int_0^{+\infty} \frac{x^{n-1/2}\mathrm{e}^{-x}}{(x+\xi^2)^{n+1}}\,\mathrm{d}x = \frac{(-1)^n 2^{2n} n!}{|\xi|} \int_0^{+\infty} \frac{x^{n-1/2}\mathrm{e}^{-\xi^2 x}}{(1+x)^{n+1}}\,\mathrm{d}x.\end{aligned}$$

Using the Laplace's method as in Lemma 4.3 in [6], we find that

$$\int_0^{+\infty} \frac{x^{n-1/2}e^{-\xi^2 x}}{(1+x)^{n+1}}\,dx \sim e^{-\xi^2 M_n} M_n^{-1/2} \left(\frac{M_n}{1+M_n}\right)^n \sqrt{\frac{M_n}{n}\pi},$$

if $n \to +\infty$, where $M_n = -\frac{1}{2} + \sqrt{\frac{1}{4} + (n+1)/\xi^2}$.

From this result, we conclude. □

Lemma 2.5. *The polynomials H_n and Q_n satisfy the relation*

$$H_{n+2} + \sigma_n \frac{n+2}{n} H_n = Q_{n+2} + a_n Q_n, \quad (n \geqslant 1), \tag{2.13}$$

where the cofficients a_n are given by

$$a_n = \sigma_n \frac{n+2}{n} \frac{k_n}{(Q_n, Q_n)_S} \tag{2.14}$$

and satisfy the recurrence relation

$$a_n = \frac{(n+2)k_n\sigma_n/n}{k_n + n^2 k_{n-2}(\sigma_{n-2}/(n-2))^2 + 16\lambda n^2 k_{n-3}\sigma_{n-2} - nk_{n-2}(\sigma_{n-2}/(n-2))a_{n-2}}, \tag{2.15}$$

for all $n > 2$.

Proof. Let R_n be the polynomials defined by $R_{n+2}=H_{n+2}+\sigma_n((n+2)/n)H_n$. Observe that the coherence relation, implies that $R_n' = 2nT_{n-1}$. The expansion of R_n in terms of Q_n leads to $R_{n+2} = Q_{n+2} + a_n Q_n$. Handling as in Lemma 2.1, it follows that

$$a_n = \frac{(R_{n+2}, R_n)_S}{(R_n, R_n)_S - a_{n-2}(R_n, R_{n-2})_S}, \quad (n \geqslant 2). \tag{2.16}$$

Now using the definition of the polynomials R_n, the relation of their derivatives with T_n and formula (2.12), we can derive (2.14) and (2.15). □

Lemma 2.6. *The sequence $(b_n)_{n\geqslant 1} = (a_n/2(n+2))_{n\geqslant 1}$ is bounded by $(1+\xi^2)/(1+\xi^2+2\lambda)$ and converges to $b = 1/\varphi(2\lambda+1)$.*

Proof. From (2.14) and (2.12),

$$b_n = \frac{k_{n+1}'}{8nk_{n-1}} \frac{k_n}{(Q_n, Q_n)_S}. \tag{2.17}$$

Notice that, in (2.17), the first factor is bounded by 1 since

$$k_n = \int_{\mathbb{R}} H_n^2\, d\mu_0 = \int_{\mathbb{R}} (x^2+\xi^2) H_n^2\, d\mu_1 = \int_{\mathbb{R}} [(x+\xi)H_n]^2\, d\mu_1 \geqslant \frac{k_{n+1}'}{4}.$$

On the other hand, using again the extremal property of the norm of orthogonal polynomials,

$$(Q_n, Q_n)_S \geqslant k_n + 4\lambda n^2 k_{n-1}' \tag{2.18}$$

and then

$$b_n \leqslant \frac{k_n}{(Q_n, Q_n)_S} \leqslant \frac{1}{1+2\lambda n k'_{n-1}/k_{n-1}}.$$

Moreover, if we consider the Fourier expansion of H_n in terms of T_n, we have

$$4(x^2+\xi^2)H_n = T_{n+2} + 4\frac{k_n}{k'_n}T_n.$$

Multiplying by H_n both sides of the above formula, integrating with respect to μ_0 and using the recurrence relation for H_n we can obtain

$$\frac{k'_{n+2}}{4k_n} + \frac{4k_n}{k'_n} = 4(n+\xi^2)+2. \tag{2.19}$$

Finally,

$$b_n \leqslant \frac{1}{1+2\lambda n/(n+\xi^2-1/2)} \leqslant \frac{1}{1+2\lambda n/(n+\xi^2)} \leqslant \frac{1}{1+2\lambda/(1+\xi^2)} < 1, \quad \text{for all } n \geqslant 1.$$

In order to prove the convergence of the sequence (b_n), observe that, from (2.15) and the expression of σ_n in terms of α_n,

$$b_n = \frac{\alpha_n}{1+[n/(n-1)]\alpha_{n-2}^2 + 4\lambda[n/(n-1)]\alpha_{n-2} - [n/(n-1)]\alpha_{n-2}b_{n-2}} \tag{2.20}$$

holds.

Then, to conclude it suffices to proceed as in Lemma 2.2 taking in mind that

$$|b_n - b| \leqslant \frac{|\alpha_n - b[1+[n/(n-1)]\alpha_{n-2}^2 + 4\lambda[n/(n-1)]\alpha_{n-1}] + [n/(n-1)]\alpha_{n-2}b^2|}{1+[n/(n-1)]\alpha_{n-2}^2 + (4\lambda-1)[n/(n-1)]\alpha_{n-2}} + \frac{b}{1+4\lambda}|b_{n-2}-b|$$

and Lemma 2.4. □

Theorem 2.7. *The Sobolev orthogonal polynomials Q_n satisfy*

$$\lim_n \frac{Q_n(x)}{H_n(x)} = 0$$

uniformly on compact sets of $\mathbb{C}\backslash\mathbb{R}$.

Proof. Working as in the Theorem 2.3, from (2.13), taking into account (2.6) and Lemmas 2.4 and 2.6, we can write

$$f_{n+2}(x) = c_n(x) f_n(x) + d_n, \quad f_n(x) = \frac{Q_n(x)}{H_n(x)},$$

where $d_n \to 0$ and $\lim_n c_n(x) = 1/\varphi(2\lambda+1)$ uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$. Then, the result follows. □

3. Scaled relative asymptotics

Additional information about the polynomials Q_n can be obtained from the relative asymptotics for the scaled Sobolev polynomials with respect to the scaled Hermite polynomials.

A result we will need later is the ratio asymptotics for the scaled Hermite polynomials: for every nonnegative integer i,

$$\lim_n \frac{\sqrt{2n}\,H_{n-1}(\sqrt{n-i}\,x)}{H_n(\sqrt{n-i}\,x)} = \frac{1}{\varphi(x/\sqrt{2})} \tag{3.1}$$

holds uniformly on compact subsets of $\mathbb{C}\setminus[-\sqrt{2},\sqrt{2}]$, (see [8, p. 126]).

Theorem 3.1.

(i) *In Case I*,

$$\lim_n \frac{Q_n(\sqrt{n}\,x)}{H_n(\sqrt{n}\,x)} = \frac{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})}{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})+1}$$

holds uniformly on compact sets of $\mathbb{C}\setminus[-\sqrt{2},\sqrt{2}]$.

(ii) *In Case* II,

$$\lim_n \frac{Q_n(\sqrt{n}\,x)}{H_n(\sqrt{n}\,x)} = \frac{(\varphi^2(x/\sqrt{2})+1)\varphi(2\lambda+1)}{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})+1}$$

holds uniformly on compact sets of $\mathbb{C}\setminus[-\sqrt{2},\sqrt{2}]$.

Proof. (i) From (2.1),

$$Q_n = H_n - a_{n-2}Q_{n-2} = H_n - a_{n-2}(H_{n-2} - a_{n-4}Q_{n-4}).$$

Iteration yields

$$Q_n = H_n + \sum_{i=1}^{[n/2]}(-1)^i\left(\prod_{j=1}^{i} a_{n-2j}\right)H_{n-2i},$$

where $[n]$ means the biggest integer less than or equal to n. So,

$$\frac{Q_n(\sqrt{n}\,x)}{H_n(\sqrt{n}\,x)} = 1 + \sum_{i=1}^{[n/2]}(-1)^i\left(\prod_{j=1}^{i}\frac{a_{n-2j}}{2n}\right)\frac{(2n)^i H_{n-2i}(\sqrt{n}\,x)}{H_n(\sqrt{n}\,x)} = \sum_{i=0}^{[n/2]} f_{n,i}(\sqrt{n}\,x),$$

where

$$f_{n,i}(x) = (-1)^i\left(\prod_{j=1}^{i} a_{n-2j}\right)\frac{H_{n-2i}(x)}{H_n(x)} \quad \text{for } 0 < i \leqslant [\tfrac{n}{2}] \quad \text{and} \quad f_{n,0} \equiv 1.$$

As a consequence of Lemma 2.2 and (3.1), for every fixed nonnegative integer i,

$$\lim_n f_{n,i}(\sqrt{n}\,x) = f_i(x) = \left(\frac{-1}{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})}\right)^i \tag{3.2}$$

holds uniformly on compact subsets of $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$ and the sequence $(f_{n,i}(\sqrt{n}x))_{n=0}^{+\infty}$ is uniformly bounded on compact subsets of $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$. More precisely, for $i=0,1,\ldots,[n/2]$, given a compact set $K\subset\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$

$$|f_{n,i}(\sqrt{n}x)|\leqslant Mc^i \tag{3.3}$$

holds for $x\in K$, where $c=1/(1+2\lambda)$ and the constant M only depends on K.

Notice that, from (3.2) and (3.3), we find that

$$\lim_n \sum_{i=0}^{[n/2]} f_{n,i}(\sqrt{n}x)=\sum_{i=0}^{+\infty} f_i(x)$$

uniformly on compact subsets of $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$ and the result follows.

(ii) As above, from the relation of polynomials R_n in terms of Q_n (see Lemma 2.5), we get

$$Q_n=R_n+\sum_{i=1}^{[n/2]}(-1)^i\left(\prod_{j=1}^{i} a_{n-2j}\right)R_{n-2i}.$$

Recall that, $R_n=H_n+\sigma_{n-2}[n/(n-2)]H_{n-2}$. Using Lemma 2.4 and (3.1) one easily shows that

$$\lim_n \frac{R_n(\sqrt{n}x)}{H_n(\sqrt{n}x)}=1+\frac{1}{\varphi^2(x/\sqrt{2})}$$

holds uniformly on compact subsets of $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$.

Now, it suffices to use the same techniques as in (i) and, taking in mind Lemma 2.6, we conclude. □

Next, we give some consequences of the Theorem 3.1.

Corollary 3.2. *The following analogues of Plancherel–Rotach asymptotics hold*:

(i) *Case* I

$$\lim_n \frac{Q_n(\sqrt{n}x)}{(2^n n!\sqrt{\pi})^{1/2}\prod_{k=1}^n \varphi(\sqrt{n/2k}x)}=\left(\frac{x^2-2}{x^2}\right)^{-1/4}\frac{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})}{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})+1}$$

(ii) *Case* II

$$\lim_n \frac{Q_n(\sqrt{n}x)}{(2^n n!\sqrt{\pi})^{1/2}\prod_{k=1}^n \varphi(\sqrt{n/(2k)}x)}=\left(\frac{x^2-2}{x^2}\right)^{-1/4}\frac{(\varphi^2(x/\sqrt{2})+1)\varphi(2\lambda+1)}{\varphi(2\lambda+1)\varphi^2(x/\sqrt{2})+1}$$

uniformly on compact subsets of $\mathbb{C}\backslash[-\sqrt{2},\sqrt{2}]$.

Proof. Is an inmediate consequence of the Theorem 3.1 and the Plancherel–Rotach asymptotics for Hermite polynomials, see [8]. □

Moreover, from Hurwitz's theorem we get:

Corollary 3.3. *In both cases, zeros of the polynomials* $Q_n(\sqrt{n}x)$ *accumulate on* $[-\sqrt{2},\sqrt{2}]$.

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