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Symmetric Orthogonal Polynomials for Sobolev-Type Inner Products

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In this paper, families of symmetric orthogonal polynomials (Q_n) with respect to the Sobolev-type inner product, $\langle f, g\rangle = \int_I fg\, d\mu + \sum_{j=0}^{r} M_j f^{(j)}(0)\, g^{(j)}(0)$ where I is a symmetric interval and μ is a symmetric positive Borel measure with infinite support on I and whose moments are all finite, are considered. If $Q_{2n}(x) = U_n(x^2)$ and $Q_{2n+1}(x) = xV_n(x^2)$, we deduce that U_n and V_n are Sobolev-type orthogonal polynomials and, in several particular cases, standard orthogonal polynomials. We study the zeros of Q_n showing that, in some cases, Q_n has two complex conjugate zeros; moreover a partial result about separation of the zeros is given. We also discuss the symmetrization problem for this kind of inner products. Finally, some Sobolev-type inner products with two symmetric mass points are considered.

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1. Introduction

In the last years, topics such as algebraic and asymptotic properties, distribution of the zeros and differential equations about orthogonal polynomials with respect to different inner products involving derivatives (Sobolev-type inner products) have been studied (see [1, 2, 4, 5, 10–14, 16–19]).

In some of these papers the authors, looking for a differential equation satisfied by the orthogonal polynomials, found formulas similar to the ones appearing in the process of symmetrization studied by Chihara for the real line [6] and Marcellán and Sansigre for the unit circle [15].

More precisely, Marcellán and Ronveaux (see [14]) show that the sequence (Q_n) of monic orthogonal polynomials (SMOP) corresponding to the inner product

$$\langle f, g\rangle = \int_R fge^{-x^2}\,dx + Nf'(0)\,g'(0) \qquad (N \geqslant 0)$$

verifies the decomposition

$$Q_{2n}(x) = L_n^{-1/2}(x^2)$$
$$Q_{2n+1}(x) = xL_n^{1/2}(x^2) + x\alpha_n L_{n-1}^{3/2}(x^2),$$

where $L_n^\alpha(x)$ is the nth monic Laguerre polynomial. We point out that the monic Hermite polynomials are the symmetric orthogonal polynomials corresponding to $(L_n^{-1/2})$. $(L_n^{\alpha+1})$ is the sequence of kernel polynomials associated to (L_n^α).

Also, Alfaro and others (see [1]) have proved that the SMOP (Q_n) corresponding to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} fg(1-x^2)^{\lambda-1/2}\,dx + Mf(0)\,g(0) + Nf'(0)\,g'(0)$$

with $\lambda > -1/2$ and $M, N \geqslant 0$ verifies

$$Q_{2n}(x) = S_n(x^2) + M_n S_{n-1}^*(x^2)$$
$$Q_{2n+1}(x) = xS_n^*(x^2) + N_n xS_{n-1}^{**}(x^2),$$

where M_n and N_n depend on M and n, respectively, N and n, and if P_n^λ denotes the nth-monic Gegenbauer polynomial,

$$S_n(x^2) = P_{2n}^{(\lambda)}(x) \qquad xS_n^*(x^2) = P_{2n+1}^{(\lambda)}(x)$$

(R_n^* means the nth monic kernel polynomial corresponding to the sequence (R_n) with parameter $k=0$, see [6]).

From now on we will say the polynomials R_n are symmetric if they are even or odd depending on the parity of n. The above results seem to suggest a link between symmetric orthogonal polynomials Q_n with respect to a Sobolev-type inner product

$$\langle f, g\rangle = \int_{(-a,a)} fg\,d\mu + Mf(0)\,g(0) + Nf'(0)\,g'(0) \qquad (N>0)$$

and standard orthogonal polynomials P_n with respect to the symmetric positive Borel measure μ on $(-a, a)$.

Before showing our results we briefly mention some general properties about the symmetrization of sequences of standard orthogonal polynomials.

Let μ be a symmetric positive Borel measure on an interval $I=(-a, a)$, where $0<a\leqslant\infty$ (that is, $\mu(-A)=\mu(A)$ for every Borel set $A\subset I$). Let (P_n) denote the SMOP with respect to μ. Then

$$\begin{aligned} P_{2n}(x) &= S_n(x^2) \\ P_{2n+1}(x) &= xS_n^*(x^2), \end{aligned} \tag{1}$$

where $S_n^*(x)$ (see [6]) is the monic kernel associated with S_n, evaluated in the point $(x, 0)$, that is, S_n^* can be expressed in terms of S_n by

$$S_n^*(x)=\frac{\|S_n\|^2}{S_n(0)}\sum_{h=0}^{n}\frac{S_h(0)\,S_h(x)}{\|S_h\|^2}$$

($\|S_h\|$ denotes the L_μ^2-norm of S_h). Note that, for every n,

$$\begin{aligned} S_n(0) &= P_{2n}(0)\neq 0 \\ S_n^*(0) &= P'_{2n+1}(0)\neq 0. \end{aligned} \tag{2}$$

Chihara, see [6], points out that if (P_n) is a SMOP with respect to $w(x)\,dx$ on I, then (S_n) is a SMOP with respect to $x^{-1/2}w(x^{1/2})\,dx$ on $J=(0, a^2)$ and (S_n^*) is a SMOP with respect to $x^{1/2}w(x^{1/2})\,dx$ on J. This result is true not only for the weight functions but for any positive measure μ. More precisely:

(i) (S_n) is a SMOP with respect to the measure ν on J, where ν is the image of the measure μ under the mapping $\phi(x)=x^2$, i.e., $\nu(B)=\mu(\phi^{-1}(B))$ for every Borel set $B\subset J$. In the sequel, we will write $\nu=\phi(\mu)$.

(ii) (S_n^*) is a SMOP with respect to the measure $x\,d\nu(x)$ on J.

(iii) Besides,

$$\begin{aligned} \|P_{2n}\|_{L^2(\mu, I)} &= \|S_n\|_{L^2(\nu, J)} \\ \|P_{2n+1}\|_{L^2(\mu, I)} &= \|S_n^*\|_{L^2(x\,d\nu, J)}. \end{aligned} \tag{3}$$

(The above results can be directly obtained from the definition of the image of a measure.)

Let $K_n(x, y)$, $\kappa_n(x, y)$, and $\kappa_n^*(x, y)$ be the kernels associated to the sequences (P_n), (S_n), and (S_n^*), respectively; that is, $K_n(x, y) = \sum_{h=0}^{n} (P_h(x)\, P_h(y)/\|P_h\|^2)$, $\kappa_n(x, y) = \sum_{h=0}^{n} (S_h(x)\, S_h(y)/\|S_h\|^2)$, and $\kappa_n^*(x, y) = \sum_{h=0}^{n} (S_h^*(x)\, S_h^*(y)/\|S_h^*\|^2)$. Note that, because of (2), $\kappa_n(x, 0)$ and $\kappa_n^*(x, 0)$ are polynomials of degree n.

We denote, as usual, $K_n^{(j,k)}(x, y) = \sum_{h=0}^{n} (P_h^{(j)}(x)\, P_h^{(k)}(y)/\|P_h\|^2)$. An analogous notation will be used for the derivatives of the kernels κ_n and κ_n^*.

LEMMA 1. *The formulas*

$$K_{2n-1}^{(0,2j)}(x, 0) = (j+1)_j\, \kappa_{n-1}^{(0,j)}(x^2, 0) \tag{4}$$

$$K_{2n}^{(0,2j+1)}(x, 0) = (j+1)_{j+1}\, x\kappa_{n-1}^{*(0,j)}(x^2, 0), \tag{5}$$

where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$, $n \in N$, *and* $(\alpha)_0 = 1$, *hold.*

Proof. Let us prove the formula (4). Taking into account (1) and using Taylor's formula, we obtain

$$P_{2n}^{(2j)}(0) = (j+1)_j\, S_n^{(j)}(0), \qquad j \geqslant 1. \tag{6}$$

Now, it suffices to consider that $K_{2n-1}^{(0,2j)}(x, 0) = \sum_{h=0}^{n-1} (P_{2h}(x)\, P_{2h}^{(2j)}(0)/\|P_{2h}\|^2)$ and use (1), (3), and (6). ∎

Remark. More general relations for the kernels K_n, κ_n, and κ_n^* and their derivatives can be deduced, by doing similar calculations to the previous one. For instance,

$$\begin{aligned} K_{2n-1}^{(2j,2j)}(0, 0) &= [(j+1)_j]^2\, \kappa_{n-1}^{(j,j)}(0, 0), \\ K_{2n}^{(2j+1,2j+1)}(0, 0) &= [(j+1)_{j+1}]^2\, \kappa_{n-1}^{*(j,j)}(0, 0), \\ K_{2n}(x, y) &= \kappa_n(x^2, y^2) + xy\kappa_{n-1}^*(x^2, y^2), \\ K_{2n+1}(x, y) &= \kappa_n(x^2, y^2) + xy\kappa_n^*(x^2, y^2). \end{aligned}$$

In this paper we consider families of symmetric orthogonal polynomials (Q_n) associated with some Sobolev-type inner products. For a more convenient presentation, the paper is structured in three sections. In the first two, the inner product has only one mass point and in the last section, it has two mass points symmetrically located.

In Section 2, we obtain that the corresponding odd and even associated polynomials to Q_n (V_n and U_n, respectively) are orthogonal in the Sobolev-type sense. We point out that in several particular cases U_n and V_n are standard orthogonal polynomials; this enables us to obtain some information about separation properties for the zeros of Q_n. On the other hand we study the symmetrization problem, that is, the construction of a sequence of symmetric orthogonal polynomials from a given SMOP. In the standard case it is well known (see [6]) that, given a SMOP (S_n) with respect to a measure ν on $J=(0, a^2)$, there exist infinitely many sequences of monic orthogonal polynomials (P_n) such that $P_{2n}(x)=S_n(x^2)$, but only one of them is a sequence of symmetric polynomials. This unique sequence is orthogonal with respect to the measure μ on $I=(-a, a)$ where $\phi(\mu)=\nu$ and satisfies $P_{2n+1}(x)=xS_n^*(x^2)$. We give the solution of this problem for Sobolev-type orthogonal polynomials. Before doing this, we generalize an analog of the Christoffel–Darboux formula.

By using the results obtained in the above section, we discuss the position of the zeros of Q_n in Section 3. In some particular cases, we show that Q_n has two complex conjugate zeros whenever $r \geqslant 2$ (r is the highest order of derivatives in the inner product). Moreover, an improvement of the separation of the zeros is obtained.

In Section 4 we consider symmetric Sobolev-type inner products with two mass points and $r=1$. A seven-term recurrence relation and other algebraic properties are given.

2. Symmetric Sobolev-Type Inner Products with One Mass Point

Now let us consider the inner product

$$\langle f, g\rangle_{r,\mu}=\int_I fg\,d\mu+\sum_{j=0}^{r} M_j f^{(j)}(0)\,g^{(j)}(0),$$

where $r \geqslant 1$, $M_r>0$, and $M_j \geqslant 0$ for every $j=0, ..., r-1$. Let (Q_n) denote the SMOP with respect to the inner product $\langle\ ,\ \rangle_{r,\mu}$. If we consider the representation of Q_n in terms of P_k,

$$Q_n(x)=P_n(x)+\sum_{k=0}^{n-1} \alpha_{nk} P_k(x)$$

from the orthogonality of P_k,

$$\alpha_{nk}=\|P_k\|^{-2}\int_I Q_n P_k\,d\mu=-\|P_k\|^{-2}\sum_{j=0}^{r} M_j Q_n^{(j)}(0)\,P_k^{(j)}(0),$$

and then

$$Q_n(x) = P_n(x) - \sum_{j=0}^{r} M_j Q_n^{(j)}(0)\, K_{n-1}^{(0,j)}(x, 0). \tag{7}$$

The polynomials Q_n are symmetric and thus we can write

$$\begin{aligned} Q_{2n}(x) &= U_n(x^2) \\ Q_{2n+1}(x) &= xV_n(x^2). \end{aligned} \tag{8}$$

Using Taylor's formula, we have

$$\begin{aligned} Q_{2n}^{(2j)}(0) &= (j+1)_j\, U_n^{(j)}(0) \\ Q_{2n+1}^{(2j+1)}(0) &= (j+1)_{j+1}\, V_n^{(j)}(0) \end{aligned} \tag{9}$$

for every $j = 1, 2, \ldots$.

THEOREM 2. *In the above conditions, Q_n is a SMOP w.r.t. $\langle\, , \rangle_{r,\mu}$ if and only if (U_n) is a SMOP with respect to the inner product*

$$\int_J fg\, dv + \sum_{j=0}^{[r/2]} \bar{M}_{2j} f^{(j)}(0)\, g^{(j)}(0)$$

and (V_n) is a SMOP with respect to the inner product

$$\int_J fgx\, dv + \sum_{j=0}^{[(r-1)/2]} \bar{M}_{2j+1} f^{(j)}(0)\, g^{(j)}(0),$$

where $v = \phi(\mu)$ and

$$\begin{aligned} \bar{M}_0 &= M_0 \\ \bar{M}_{2j} &= [(j+1)_j]^2\, M_{2j} \\ \bar{M}_{2j+1} &= [(j+1)_{j+1}]^2\, M_{2j+1}. \end{aligned}$$

Moreover, the polynomials U_n and V_n satisfy the formulas

$$\begin{aligned} U_n(x) &= S_n(x) - \sum_{j=0}^{[r/2]} \bar{M}_{2j} U_n^{(j)}(0)\, \kappa_{n-1}^{(0,j)}(x, 0) \\ V_n(x) &= S_n^* - \sum_{j=0}^{[(r-1)/2]} \bar{M}_{2j+1} V_n^{(j)}(0)\, \kappa_{n-1}^{*(0,j)}(x, 0), \end{aligned}$$

where S_n and S_n^* are defined by (1).

Proof. If we compute the inner products $\langle Q_{2n}, Q_{2m}\rangle$, $\langle Q_{2n+1}, Q_{2m+1}\rangle$, and $\langle Q_{2n+1}, Q_{2m}\rangle$ taking into account (8) and (9), we obtain

$$\langle Q_{2n}, Q_{2m}\rangle_{r,\mu} = \int_I Q_{2n}, Q_{2m}\, d\mu + \sum_{j=0}^{r} M_j Q_{2n}^{(j)}(0)\, Q_{2m}^{(j)}(0)$$
$$= \int_J U_n U_m\, dv + \sum_{j=0}^{[r/2]} [(j+1)_j]^2 M_{2j} U_n^{(j)}(0)\, U_m^{(j)}(0)$$

and

$$\langle Q_{2n+1}, Q_{2m+1}\rangle_{r,\mu}$$
$$= \int_I Q_{2n+1} Q_{2m+1}\, d\mu + \sum_{j=0}^{r} M_j Q_{2n+1}^{(j)}(0)\, Q_{2m+1}^{(j)}(0)$$
$$= \int_J V_n V_m x\, dv + \sum_{j=0}^{[(r-1)/2]} [(j+1)_{j+1}]^2 M_{2j+1} V_n^{(j)}(0)\, V_m^{(j)}(0)$$

and

$$\langle Q_{2n+1}, Q_{2m}\rangle_{r,\mu} = \int_I x V_n(x^2)\, U_m(x^2)\, d\mu$$
$$+ \sum_{j=0}^{r} M_j Q_{2n+1}^{(j)}(0)\, Q_{2m}^{(j)}(0) = 0.$$

The expressions of U_n and V_n can be deduced as an easy consequence of (1), (7), (9), and Lemma 1. ∎

Remark. If $r=1$, (U_n) and (V_n) are standard SMOP. More precisely (U_n) is orthogonal w.r.t. $v + M_0\delta_0$ and (V_n) is orthogonal w.r.t. $x\, dv + M_1\delta_0$.

Let (P_n^c) denote the SMOP w.r.t. the measure $d\mu_1 = x^2\, d\mu$. We next consider the relation between U_n, V_n, and P_n^c. As the polynomials P_n^c are symmetric, we get

$$P_{2n}^c(x) = R_n(x^2)$$
$$P_{2n+1}^c(x) = xR_n^*(x^2) \tag{10}$$

then,

PROPOSITION 3. *Suppose $r=1$. The formulas*

$$U_n(x) = R_n(x) + a_{2n} R_{n-1}(x)$$
$$V_n(x) = R_n^*(x) + a_{2n+1} R_{n-1}^*(x),$$

where $a_n = \|P^c_{n-2}\|^{-2}_{\mu_1} \langle Q_n, Q_n \rangle$, *hold. Hence, SMOP* (U_n) *and* (V_n) *are strictly quasi-orthogonal of order* 1 *w.r.t. the measures* $v_1 = \phi(\mu_1)$ *and* $x\, dv_1$, *respectively.*

Proof. Since $\int_I Q_n P^c_j x^2\, d\mu = \langle Q_n, x^2 P^c_j \rangle = 0$ for $j < n-2$, we can write $Q_n(x) = P^c_n(x) + \sum_{j=n-2}^{n-1} \alpha_{nj} P^c_j(x)$. Because of the symmetry of the polynomials Q_n and P^c_j it follows that $\alpha_{n,n-1} = 0$ and, therefore, $Q_n(x) = P^c_n(x) + a_n P^c_{n-2}(x)$. Moreover $a_n = \|P^c_{n-2}\|^{-2}_{\mu_1} \int_I Q_n P^c_{n-2} x^2\, d\mu = \|P^c_{n-2}\|^{-2}_{\mu_1} \langle Q_n, x^2 P^c_{n-2} \rangle = \|P^c_{n-2}\|^{-2}_{\mu_1} \langle Q_n, Q_n \rangle$. Now, it suffices to use formulas (8) and (10). ∎

Remark. In a similar way, we can obtain that the sequences (U_n) and (V_n) are strictly quasi-orthogonal of order s w.r.t. the measures $v_s = \phi(\mu_s)$ and $x\, dv_s$, respectively, where $d\mu_s = x^{2s}\, d\mu$.

Now, we are going to study the symmetrization problem.

Let (T_n) be the SMOP with respect to the inner product

$$[f, g] = \int_{(0, a^2)} fg\, dv + \sum_{j=0}^{r} N_j f^{(j)}(0)\, g^{(j)}(0) \tag{11}$$

with $N_r > 0$ and $r \geqslant 1$ such that $T_n(0) \neq 0$ for every $n \in N$. (There are polynomials satisfying this condition, see [17].)

Let $L_n(x, y)$ be the nth-kernel associated with the SMOP (T_n), that is, $L_n(x, y) = \sum_{j=0}^{n} (T_j(x)\, T_j(y) / [T_j, T_j])$. It is known that $L_n(x, y)$ satisfies the reproducing property: $[L_n(x, y), R(x)] = R(y)$ for every polynomial R with degree less than or equal to n.

To study the symmetrization problem for this inner product we will give the corresponding version of the Christoffel–Darboux formula satisfied by T_n. This formula appears in [1] for $r = 1$ and in [9] for $r \geqslant 1$ whenever the measure v is the Laguerre weight function.

PROPOSITION 4. *Let* γ_{hj} *be the coefficients in the Fourier expansion*

$$x^{r+1} T_h(x) = \sum_{j=0}^{h+r+1} \gamma_{hj} T_j(x),$$

where $\gamma_{h,h+r+1} = 1$. *Then the formula*

$$[x^{r+1} - y^{r+1}]\, L_n(x, y) = \sum_{h=n-r}^{n} \left\{ \sum_{j=n+1}^{h+r+1} \frac{\gamma_{hj}}{[T_h, T_h]} (T_j(x)\, T_h(y) - T_h(x)\, T_j(y)) \right\} \tag{12}$$

holds.

Proof. First at all, let us note that $\gamma_{hj}=0$ for $j=0, ..., h-r-2$ by the orthogonality of T_n. On the other hand, as $[x^{r+1}P, Q]=[P, x^{r+1}Q]$ for all polynomials P, Q, we have

$$\gamma_{hj}[T_j, T_j]=\gamma_{jh}[T_h, T_h]$$

for $j=h-r-1, ..., h+r+1$.

Now it suffices to handle this in the usual way (for instance [6, p. 23]) and the formula follows. ∎

Next we obtain some symmetrization results. More precisely, we are going to analyze a sequence (Q_n) if we choose its corresponding even (respectively odd) associated polynomials to be orthogonal w.r.t. some Sobolev-type inner product.

THEOREM 5. *Let (T_n) be the SMOP with respect to the inner product* (11) *with $T_n(0)\neq 0$ and let (Q_n) be a sequence of symmetric orthogonal polynomials with respect to an inner product*

$$\langle f, g\rangle_{p,\mu}=\int_{(-a,a)} fg\,d\mu+\sum_{j=0}^{p} M_j f^{(j)}(0)\,g^{(j)}(0), \tag{13}$$

where $\phi(\mu)=v$ and $M_p>0$. Then $Q_{2n}(x)=T_n(x^2)$ holds for every $n\in N$ if and only if $[p/2]\geqslant r$, $M_{2j}=[(j+1)_j]^{-2}N_j$ for $j=0, ..., r$, and $M_{2j}=0$ for $j>r$.

Proof. Given (T_n), if the sequence (Q_n) verifies $Q_{2n}(x)=T_n(x^2)$ for every $n\in N$, then from Theorem 2, (T_n) must be orthogonal w.r.t. $[\cdot,\cdot]_{[p/2],v}$ with $\phi(\mu)=v$, so we get $[p/2]\geqslant r$, $M_{2j}=[(j+1)_j]^{-2}N_j$ for $j=0, ..., r$, and $M_{2j}=0$ for $j>r$.

The converse is an easy consequence of Theorem 2. ∎

Remark. Note that if $p>2r$ then p is odd and $M_{2r}\neq 0$.

In the standard case, if (P_n) is a SMOP with respect to a measure ψ on $(0, a^2)$ ($P_n(0)\neq 0$ for all n), the kernels $K_n(x, 0)$ are orthogonal with respect to $x\,d\psi$. In our case, an analogous result is not true. However, we obtain the following:

LEMMA 6. *Let (T_n) be the SMOP with respect to the inner product* (11) *and let $L_n(x, y)$ be the nth-kernel associated with the SMOP (T_n). Let $[f, g]_2$ denote the inner product $[f, g]_{s, x\,dv}$, that is,*

$$[f, g]_2=\int_{(0, a^2)} fgx\,dv+\sum_{i=0}^{s} m_i f^{(i)}(0)\,g^{(i)}(0).$$

Then for every $n \in N$ there exist $B_0^{(n)}, B_1^{(n)}, \ldots, B_{r+s}^{(n)}$ independent of x such that the polynomials $V_n(x) = \sum_{j=0}^{r+s} B_j^{(n)} L_n^{(0,j)}(x, 0)$ are orthogonal with respect to $[\cdot, \cdot]_2$.

Proof. First we remark that the polynomials $L_n^{(0,j)}(x, 0)$, $j = 0, 1, \ldots, \min\{r+s, n\}$, are linearly independent. Indeed suppose

$$\sum_{j=0}^{\min\{r+s,n\}} A_j L_n^{(0,j)}(x, 0) \equiv 0,$$

then

$$\sum_{j=0}^{\min\{r+s,n\}} A_j T_h^{(j)}(0) = 0$$

for all $h \in \{0, 1, \ldots, n\}$. Now $h = 0$ gives $A_0 = 0$ and proceeding in this way all A_j, $j = 0, \ldots, \min\{r+s, n\}$, are zero. This implies that the lemma is trivial if $n \leqslant r+s$.

Suppose now $n \geqslant r+s+1$. Since every polynomial of degree $\leqslant n-1$ can be represented by

$$R(x) = \sum_{i=0}^{r+s-1} c_i x^i + \sum_{k=0}^{n-r-s-1} d_k x^{r+s} T_k(x)$$

it suffices to show that there exist $B_0^{(n)}, \ldots, B_{r+s}^{(n)}$ such that

$$[V_n, x^i]_2 = 0 \qquad \text{for} \quad i = 1, \ldots, r+s-1 \tag{14}$$

$$[V_n, x^{r+s} T_k]_2 = 0 \qquad \text{for} \quad k = 0, \ldots, n-r-s-1. \tag{15}$$

The Christoffel–Darboux formula (Proposition 4) implies

$$x^{r+1} L_n^{(0,j)}(x, 0) = c_j L_n^{(0,j-r-1)}(x, 0) + \sum_{i=n-r}^{n+r+1} \beta_{ji} T_i(x)$$

for $j = 0, \ldots, r+s$, where $c_j = 0$ if $j \leqslant r$ and $c_j = j!/(j-r-1)!$ if $r+1 \leqslant j \leqslant r+s$.

Since, for all polynomials P of degree $\leqslant n$,

$$[L_n^{(0,j)}(x, 0), P(x)] = P^{(j)}(0) \tag{16}$$

then

$$[V_n, x^{r+s} T_k]_2 = \int_{(0,a^2)} V_n x^{r+s+1} T_k \, dv = [x^{r+1} V_n, x^s T_k] = 0$$

for $k = 0, \ldots, n-r-s-1$ and the condition (15) is always true.

Assertion (14) can be rewritten as

$$0=\int_{(0,a^2)} V_n x^{i+1}\,dv+i!\,m_i V_n^{(i)}(0)$$
$$=[V_n, x^{i+1}]-(i+1)!\,N_{i+1} V_n^{(i+1)}(0)+i!\,m_i V_n^{(i)}(0).$$

(Here we have to take $N_{i+1}=0$ if $i+1>r$ and $m_i=0$ if $i>s$.)
By using (16), we have

$$[V_n, x^{i+1}]=\sum_{j=0}^{r+s} B_j^{(n)}[L_n^{(0,j)}(x,0), x^{i+1}]=(i+1)!\,B_{i+1}^{(n)}.$$

Then (14) is equivalent to

$$(i+1)\,B_{i+1}^{(n)}=(i+1)\,N_{i+1}\sum_{j=0}^{r+s} B_j^{(n)} L_n^{(i+1,j)}(0,0)$$
$$-m_i\sum_{j=0}^{r+s} B_j^{(n)} L_n^{(i,j)}(0,0), \qquad i=0,1,...,r+s-1. \quad (17)$$

The system (17) is a system of $r+s$ homogeneous linear equations in $r+s+1$ unknown. So it has a non-trivial solution $B_0^{(n)}, ..., B_{r+s}^{(n)}$. Since the $L_n^{(0,j)}(x,0)$, $j=0,1,...,r+s$, are linearly independent the corresponding polynomial V_n is not identically zero. Obviously it is possible to choose the $B_0^{(n)}, ..., B_{r+s}^{(n)}$ in such a way that the leading coefficient of V_n is equal to 1. ∎

In a similar way as in Theorem 5, we get

THEOREM 7. *Let (T_n) be the SMOP with respect to the inner product* (11), *let (Q_n) be a sequence of O.P. with respect to the inner product* (13), *and let (V_n) be the SMOP introduced in Lemma* 6. *Then $Q_{2n+1}(x)=xV_n(x^2)$ holds for every $n\in N$ if and only if $[(p-1)/2]\geqslant s$, $M_{2j+1}=[(j+1)_{j+1}]^{-2} m_j$ for $j=0,...,s$ and $M_{2j+1}=0$ for $j>s$. Besides if $p>2s+1$, p is even and $M_{2s+1}\neq 0$.*

Remark. In the case $r=s$, we have $Q_{2n}(x)=T_n(x^2)$ and $Q_{2n+1}(x)=xV_n(x^2)$ for every $n\in N$ if and only if $p=2r+1$, $M_{2j}=[(j+1)_j]^{-2} N_j$, and $M_{2j+1}=[(j+1)_{j+1}]^{-2} m_j$ for $j=0,...,r$. This result should be compared to Theorem 8.1 in [6].

3. Zeros

Consider the special Sobolev-type inner product

$$\langle f, g\rangle_{r,\mu} = \int_I fg\, d\mu + M_r f^{(r)}(0)\, g^{(r)}(0), \tag{18}$$

where $I = (-a, a)$, μ is symmetric, $M_r > 0$, and $r \geqslant 2$. We will discuss the position of the zeros of the corresponding SMOP (Q_n).

Note that if $r = 1$, as a consequence of the remark after Theorem 2, we have that the positive zeros of Q_{2n} and Q_{2n+1} are the square root of the zeros of U_n and V_n, respectively. Moreover, the positive zeros of Q_{2n} and Q_{2n+2} mutually separate each other $(n \geqslant 1)$; the same property is verified by the positive zeros of Q_{2n-1} and Q_{2n+1} $(n \geqslant 1)$.

As before the (P_n) denote the standard SMOP with respect to (18) with $M_r = 0$.

If $n \leqslant r$ and $j \in \{0, 1, ..., n-1\}$, then $\langle x^j, P_n\rangle_{r,\mu} = 0$ thus $Q_n = P_n$ and the zeros of Q_n are just the zeros of P_n.

Suppose now $n \geqslant r + 1$.

Meijer [17] studied the discrete Sobolev inner product

$$[f, g] = \int_{(0, a^2)} fg\, d\psi + N f^{(k)}(0)\, g^{(k)}(0), \qquad \text{where} \quad N \geqslant 0. \tag{19}$$

Let (T_n) denote the SMOP with respect to (19) and (K_n) the standard SMOP with respect to (19) with $N = 0$.

It is proved in [17] that T_n has n simple real zeros. Moreover, for $N \neq 0$, and $n \geqslant k + 1$ these zeros and the zeros of K_n mutually separate each other. The smallest zero of T_n is less than the smallest zero of K_n. If N is sufficiently large, then the smallest zero of T_n is negative. More precisely, it is proved in [17] that there exists an increasing sequence $(\alpha_n)_{n=k+1}^{\infty}$ of positive numbers, depending on n, ψ, and k, but independent of N such that

(i) If $\alpha_n N < 1$, then T_n has n positive, simple zeros.

(ii) If $\alpha_n N = 1$, then $T_n(0) = 0$.

(iii) If $\alpha_n N > 1$, then T_n has a negative zero.

We apply the results of [17] to the SMOP (Q_n).

Theorem 8. *Let (Q_n) denote the SMOP with respect to the inner product (18) where $M_r > 0$, $r \geqslant 2$. Let (P_n) denote the corresponding standard SMOP. Then:*

(a) *If* $n \leqslant r$ *or* $n = r + 2m + 1$, $m \in \{0, 1, 2, ...\}$, *then* $Q_n = P_n$.

(b) *If* $n = r + 2m$, $m \in \{1, 2, ...\}$, *then between two consecutive positive zeros of* P_n *there is exactly one zero of* Q_n. *Moreover there exists a positive number* σ_n *depending on* n, μ, *and* r *but independent of* M_r *such that*

(i) *if* $\sigma_n M_r < 1$, *then* Q_n *has a zero in* $(0, x_1)$, *where* x_1 *denotes the smallest positive zero of* P_n,

(ii) *if* $\sigma_n M_r = 1$, *then* 0 *is a zero of* Q_n *of multiplicity* 2 (r *even*) *or* 3 (r *odd*),

(iii) *if* $\sigma_n M_r > 1$, *then* Q_n *has* 2 *complex conjugated zeros.*

Proof. If r is even, by Theorem 2, (V_n) is the SMOP with respect to the standard inner product $\int_I fgx\,dv$. Hence $V_n = S_n^*$ and $Q_{2n+1} = P_{2n+1}$. The polynomials U_n, however, are orthogonal with respect to the discrete Sobolev inner product

$$\int_{(0, a^2)} fg\,dv + \bar{M}_r f^{(\rho)}(0)\, g^{(\rho)}(0), \qquad \text{where} \quad \rho = \frac{r}{2} \geqslant 1 \tag{20}$$

which is of type (19).

Let as before (S_n) denote the standard SMOP with respect to (20) with $\bar{M}_r = 0$. Then for $n \geqslant \rho + 1$ the zeros of U_n interlace with those of S_n and the smallest zero of U_n is negative if $\bar{M}_r$ is sufficiently large. Let $x_1 < x_2 < \cdots < x_n$ denote the positive zeros of $P_{2n}(x) = S_n(x^2)$. If $2n \geqslant r + 2$ then $Q_{2n}(x) = U_n(x^2)$ has one zero in every interval (x_i, x_{i+1}), $i = 1, 2, ..., n - 1$.

Moreover there exists a positive number σ_{2n} $(2n \geqslant r + 2)$ such that if $\sigma_{2n} M_r < 1$, then Q_{2n} has a zero in $(0, x_1)$. If $\sigma_{2n} M_r > 1$, then U_n has a negative zero and Q_{2n} has two complex conjugated zeros. If $\sigma_{2n} M_r = 1$, then Q_{2n} has a zero of multiplicity 2 in $x = 0$.

The case r odd is treated in a similar way. Now (U_n) is orthogonal with respect to a standard inner product and $Q_{2n} = P_{2n}$. The polynomials V_n are orthogonal with respect to the discrete Sobolev inner product

$$\int_{(0, a^2)} fgx\,dv + \bar{M}_r f^{(\rho)}(0)\, g^{(\rho)}(0),$$

where $\rho = (r-1)/2 \geqslant 1$. If $V_n(0) = 0$, then $x = 0$ is a zero of multiplicity 3 of $Q_{2n+1}(x) = xV_n(x^2)$. ∎

It is well known that the zeros of two consecutive standard orthogonal polynomials mutually separate each other. Until now, nothing was known about an analogous property for the zeros of Sobolev-type orthogonal polynomials. At the beginning of this section, we pointed out a partial result in this way for $r = 1$. However, this property is not true, in general, as we will show below.

From the symmetric character of the polynomials P_n^c and the separation property for the zeros of standard orthogonal polynomials we can assure, see [18],

LEMMA 9. *Between two consecutive zeros of P_n^c there is exactly one positive zero of P_{n-2}^c.*

Now, let us consider the inner product (18) with $r=1$. As a consequence of the above lemma we can derive some relations among the zeros of Q_n, P_n^c, and P_{n-2}^c:

PROPOSITION 10. *Between two consecutive positive zeros of P_n^c (or P_{n-2}^c) there is exactly one zero of Q_n. Moreover, the largest positive zero of Q_n is less than the largest positive zero of P_n^c and greater than the largest positive zero of P_{n-2}^c.*

Proof. In Proposition 3 we had

$$Q_n(x)=P_n^c(x)+a_nP_{n-2}^c(x) \tag{21}$$

with $a_n>0$. Since P_{n-2}^c has the opposite sign in two consecutive zeros of P_n^c, the same is true for Q_n. Thus, if x_{nj}^c, $j=1, ..., [n/2]$, denotes the positive zeros of P_n^c, then Q_n has at least one, hence exactly one, zero on each interval $(x_{nj}^c, x_{n,j+1}^c)$, $j=1, ..., [n/2]-1$.

(The result for P_{n-2}^c can be deduced in a smilar way.)

Moreover, whenever $x\geqslant x_{n,[n/2]}^c$, it follows that $Q_n(x)>0$.

On the other hand, since $P_n^c(x_{n-2,[(n-2)/2]}^c)<0$ we have $Q_n(x_{n-2,[(n-2)/2]}^c)<0$. ∎

THEOREM 11. *Suppose $r=1$. The polynomials Q_n and Q_{n+1} either have two symmetric common zeros or they have no common zeros.*

Proof. Suppose that the recurrence relation satisfied by the sequence (P_n^c) is $xP_n^c(x)=P_{n+1}^c(x)+B_{n+1}^cP_{n-1}^c(x)$ where $B_{n+1}^c>0$. From (21) we have

$$\begin{aligned} Q_{n+1}(x)&=xP_n^c(x)+(a_{n+1}-B_{n+1}^c)\,P_{n-1}^c(x)\\ Q_n(x)&=\left(1-\frac{a_n}{B_n^c}\right)P_n^c(x)+\frac{a_n}{B_n^c}\,xP_{n-1}^c(x), \end{aligned} \tag{22}$$

where $a_n\neq B_n^c$ for every n. Indeed, if there exists an $n\in N$ such that $a_n=B_n^c$, then from (22) we get $Q_n(x)=xP_{n-1}^c(x)$. This formula is obviously false when n is even. If n is odd, by using (7), $Q_n(x)=P_n(x)-(M_1P_n'(0)/(1+M_1K_{n-1}^{(1,1)}(0,0)))\,K_{n-1}^{(0,1)}(x,0)$, then $Q_n(x)\neq P_n(x)$ but, for every odd n, $P_n(x)=xP_{n-1}^c(x)$ so we have a contradiction.

Eliminating $P^c_{n-1}(x)$ in (22), we obtain

$$r_2(x, n)\, P^c_n(x) = r_1(x, n)\, Q_{n+1}(x) + r_0(x, n)\, Q_n(x) \tag{23}$$

with

$$r_2(x, n) = a_n x^2 - (a_{n+1} - B^c_{n+1})(B^c_n - a_n)$$
$$r_1(x, n) = a_n x$$
$$r_0(x, n) = B^c_n(B^c_{n+1} - a_{n+1}).$$

Note that $r_1(x, n) > 0$ for $x \in (0, a)$ and $r_2(x, n)$ is a symmetric polynomial.

Suppose Q_n and Q_{n+1} have a common positive zero ξ. From (23), we have $r_2(\xi, n)\, P^c_n(\xi) = 0$. But, from (22), Q_{n+1} and P^c_n have no common zeros; hence ξ must be the only positive zero of $r_2(x, n)$. By the symmetry of Q_n, the result follows. ▮

Remark. Let μ be a symmetric positive Borel measure on $(-a, a)$ and let m_n denote the nth-moment of μ. Note that, since μ is positive, $m_0 m_4 > m_2^2$. It is not difficult to prove that, choosing $M_1 = m_0 m_4 / m_2 - m_2$, the polynomials Q_2 and Q_3 have a common positive zero.

The question remains open if for any consecutive polynomials Q_n and Q_{n+1}, a similar result is true for a suitable choice of M_1.

4. Symmetric Sobolev-Type Inner Products with Two Mass Points

Next, we consider the symmetric Sobolev-type inner product

$$\langle f, g \rangle = \int_I fg\, d\mu + M[f(c)\, g(c) + f(-c)\, g(-c)]$$
$$+ N[f'(c)\, g'(c) + f'(-c)\, g'(-c)], \tag{24}$$

where μ is a symmetric positive Borel measure on $I = (-a, a)$, $M, N \geqslant 0$, and $0 < c < +\infty$.

The study of such inner products was started by Bavinck and Meijer for Gegenbauer weight functions and $c = 1$ in [2, 3]. The hypergeometric character constitutes a key element to analyze the algebraic properties and the representation of the new polynomials. Also, the location of the mass points plays an important role in the computation of several parameters which appear in the Fourier expansion of orthogonal polynomials of Sobolev type in terms of the first ones. More recently, the same authors have generalized their results for even weight functions in $[-1, 1]$ with two mass points located in the ends of the interval, see [4]. We study some analogous problems without constraints about the interval and the location of the mass points.

Let (Q_n) denote the sequence of MOP with respect to the inner product (24). Since the polynomials Q_n are symmetric, they verify formula (8).

If we compute the inner products $\langle Q_{2n}, Q_{2m}\rangle$ and $\langle Q_{2n+1}, Q_{2m+1}\rangle$ $(n, m \geqslant 0)$ and we substitute Q_{2n} and Q_{2n+1} by U_n and V_n, respectively, we obtain that (U_n) is the SMOP with respect to the inner product

$$\langle f, g\rangle = \int_J fg\, dv + 2Mf(c^2)\, g(c^2) + 8Nc^2 f'(c^2)\, g'(c^2)$$

and (V_n) is the SMOP with respect to

$$\langle f, g\rangle = \int_J fgx\, dv + (f(c^2), f'(c^2)) \begin{pmatrix} 2(c^2M+N) & 4c^2N \\ 4c^2N & 8c^4N \end{pmatrix} \begin{pmatrix} g(c^2) \\ g'(c^2) \end{pmatrix},$$

where $v = \phi(\mu)$ and $J = (0, a^2)$.

We want to point out that letting $c \to 0$ we recover the previous results.

The above formulas suggest the interest to study the inner product

$$B(f, g) = \int_I fg\, d\mu + (f(c), f'(c))\, H \begin{pmatrix} g(c) \\ g'(c) \end{pmatrix},$$

where $H = \left(\begin{smallmatrix} M & A \\ A & N \end{smallmatrix}\right)$ $(M, N, A \in R)$ is a positive semidefinite matrix. Note that if H is diagonal we recover the Sobolev-type inner product.

There are some other reasons to study $B(f, g)$. For instance, it provides an example of an inner product which it is not of Sobolev-type in a strict sense (because the terms $Af(c)\, g'(c)$ and $Af'(c)\, g(c)$ appear) and whose corresponding orthogonal polynomials satisfy a five-term recurrence relation.

Because the operator multiplication by $(x^2 - c^2)^2$ is self-adjoint with respect to the inner product (24), the polynomials Q_n verify the formula

$$(x^2 - c^2)^2\, Q_n(x) = \sum_{j=n-4}^{n+4} \xi_{nj} Q_j(x).$$

We note that this relation is not minimal since the operator multiplication by $x^3 - 3c^2x$ is also self-adjoint w.r.t. (24) and thus

$$(x^3 - 3c^2x)\, Q_n(x) = \sum_{j=n-3}^{n+3} \gamma_{nj} Q_j(x).$$

This last relation is minimal. This fact had already appeared in the case studied by Bavinck and Meijer in [3] and it also appears in a more general situation in a paper by Evans and others (see [8]).

PROPOSITION 12. *The SMOP (Q_n) satisfy the seven-term recurrence relation*

$$(x^3 - 3c^2x)\,Q_n(x) = Q_{n+3}(x) + \gamma_n Q_{n+1}(x) + \gamma_{n-1}\lambda_n Q_{n-1}(x) + \lambda_n\lambda_{n-1}\lambda_{n-2}Q_{n-3}(x), \qquad (25)$$

where

$$\lambda_n = \frac{\langle Q_n, Q_n\rangle}{\langle Q_{n-1}, Q_{n-1}\rangle}$$

and

$$\gamma_n = \frac{\langle (x^3 - 3c^2x)\,Q_n, Q_{n+1}\rangle}{\langle Q_{n+1}, Q_{n+1}\rangle}.$$

Proof. From

$$(x^3 - 3c^2x)\,Q_n(x) = Q_{n+3}(x) + \sum_{j=0}^{n+2} \gamma_{nj} Q_j(x)$$

it follows that

$$\gamma_{nj} = \frac{\langle (x^3 - 3c^2x)\,Q_n, Q_j\rangle}{\langle Q_j, Q_j\rangle}.$$

By using the self-adjointness of operator multiplication by $(x^3 - 3c^2x)$ we get $\gamma_{nj} = 0$ for $j \leqslant n-4$.

Since the polynomials Q_n are symmetric we have $\gamma_{n,n+2} = \gamma_{n,n} = \gamma_{n,n-2} = 0$. Finally,

$$\gamma_{n,n+1} = \frac{\langle (x^3 - 3c^2x)\,Q_n, Q_{n+1}\rangle}{\langle Q_{n+1}, Q_{n+1}\rangle} = \gamma_n$$

$$\gamma_{n,n-1} = \frac{\langle (x^3 - 3c^2x)\,Q_{n-1}, Q_n\rangle}{\langle Q_{n-1}, Q_{n-1}\rangle} = \gamma_{n-1}\lambda_n$$

$$\gamma_{n,n-3} = \frac{\langle Q_n, Q_n\rangle}{\langle Q_{n-3}, Q_{n-3}\rangle} = \lambda_n\lambda_{n-1}\lambda_{n-2}$$

and the result follows. ∎

Remark. The initial conditions are given by $Q_{-3}(x) = Q_{-2}(x) = Q_{-1}(x) = 0$, $Q_0(x) = 1$, $Q_1(x) = x$, and $Q_2(x) = x^2 - (m_2 + 2Mc^2)/(m_0 + 2M)$, where m_0 and m_2 are the moments of order 0 and 2 w.r.t. the measure μ, respectively.

The computation of the coefficients γ_n and λ_n starting from their definition is not easy. An available way is to use the SMOP $(\tilde{P}_n)$ associated with the symmetric measure $d\tilde{\mu} = d\mu + M[\delta_c + \delta_{-c}]$, because their relation with the sequence (P_n) is known (see [7]).

First, note that

$$\langle f, g \rangle = (f, g)_{\tilde{\mu}} + N[f'(c)\, g'(c) + f'(-c)\, g'(-c)].$$

From (25), we get

$$\gamma_n = \|\tilde{P}_{n+1}\|_{\tilde{\mu}}^{-2} [((x^3 - 3c^2 x)\, Q_n, \tilde{P}_{n+1})_{\tilde{\mu}} - (Q_{n+3}, \tilde{P}_{n+1})_{\tilde{\mu}}]. \tag{26}$$

Let us consider the representation of Q_n in terms of $\tilde{P}_j$:

$$Q_n(x) = \tilde{P}_n(x) + \sum_{j=0}^{n-1} \lambda_{nj} \tilde{P}_j(x).$$

Then

$$\begin{aligned}\lambda_{nj} &= \|\tilde{P}_j\|_{\tilde{\mu}}^{-2} (Q_n, P_j)_{\tilde{\mu}} \\ &= -\|\tilde{P}_j\|_{\tilde{\mu}}^{-2} N[1 + (-1)^{n+j}]\, Q_n'(c)\, \tilde{P}_j'(c)\end{aligned}$$

and so

$$Q_n(x) = \tilde{P}_n(x) - 2NQ_n'(c) \sum_{j=0}^{n-2}{}' \frac{\tilde{P}_j'(c)\, \tilde{P}_j(x)}{\|\tilde{P}_j\|_{\tilde{\mu}}^2},$$

where the symbol $\sum'$ means

$$\sum_{j=0}^{n}{}' \alpha_j = \alpha_n + \alpha_{n-2} + \cdots + \alpha_p \qquad \text{with} \quad \alpha_p = \begin{cases} \alpha_0, & \text{if } \; n \text{ even} \\ \alpha_1, & \text{if } \; n \text{ odd.} \end{cases}$$

Hence, $Q_n'(c) = \tilde{P}_n'(c)/s_{n-2}$ where we have written

$$s_{n-2} = 1 + 2N \sum_{j=0}^{n-2}{}' \frac{[\tilde{P}_j'(c)]^2}{\|\tilde{P}_j\|_{\tilde{\mu}}^2}.$$

Using the three term recurrence formula verified by the SMOP $(\tilde{P}_n)$,

$$x\tilde{P}_{n+1}(x) = \tilde{P}_{n+2}(x) + \tilde{\gamma}_{n+1} \tilde{P}_n(x)$$

we get

$$\begin{aligned}(x^3 - 3c^2 x)\, \tilde{P}_{n+1}(x) = \tilde{P}_{n+4}(x) &+ (\tilde{\gamma}_{n+1} + \tilde{\gamma}_{n+2} + \tilde{\gamma}_{n+3} - 3c^2)\, \tilde{P}_{n+2}(x) \\ &+ \tilde{\gamma}_{n+1}(\tilde{\gamma}_n + \tilde{\gamma}_{n+1} + \tilde{\gamma}_{n+2} - 3c^2)\, \tilde{P}_n(x) \\ &+ \tilde{\gamma}_{n-1} \tilde{\gamma}_n \tilde{\gamma}_{n+1} \tilde{P}_{n-2}(x).\end{aligned}$$

Taking into account that

$$\langle (x^3-3c^2x)\,Q_n, \tilde{P}_{n+1}\rangle = \langle Q_n, (x^3-3c^2x)\,\tilde{P}_{n+1}\rangle$$

and

$$Q_n(x) = \tilde{P}_n(x) - 2NQ_n'(c)\,\frac{\tilde{P}_{n-2}'(c)\,\tilde{P}_{n-2}(x)}{\|\tilde{P}_{n-2}\|_{\tilde{\mu}}^2} - 2NQ_n'(c)\sum_{j=0}^{n-4}{}' \frac{\tilde{P}_j'(c)\,\tilde{P}_j(x)}{\|\tilde{P}_j\|_{\tilde{\mu}}^2}$$

it follows

$$\begin{aligned}((x^3-3c^2x)\,Q_n, \tilde{P}_{n+1})_{\tilde{\mu}} &= (Q_n, (x^3-3c^2x)\,\tilde{P}_{n+1})_{\tilde{\mu}} = \tilde{\gamma}_{n+1}(\tilde{\gamma}_n+\tilde{\gamma}_{n+1}+\tilde{\gamma}_{n+2}-3c^2)\,\|\tilde{P}_n\|_{\tilde{\mu}}^2 \\ &\quad - 2NQ_n'(c)\,\tilde{P}_{n-2}'(c)\,\tilde{\gamma}_{n-1}\tilde{\gamma}_n\tilde{\gamma}_{n+1} \\ &= \|\tilde{P}_{n+1}\|_{\tilde{\mu}}^2\left[\tilde{\gamma}_n+\tilde{\gamma}_{n+1}+\tilde{\gamma}_{n+2}-3c^2-2N\,\frac{Q_n'(c)\,\tilde{P}_{n-2}'(c)}{\|\tilde{P}_{n-2}\|_{\tilde{\mu}}^2}\right].\end{aligned}$$

Besides,

$$(Q_{n+3}, \tilde{P}_{n+1})_{\tilde{\mu}} = -2N\tilde{P}_{n+1}'(c)\,Q_{n+3}'(c).$$

therefore, by using (26),

$$\gamma_n = \tilde{\gamma}_n+\tilde{\gamma}_{n+1}+\tilde{\gamma}_{n+2}-3c^2-2N\left[\frac{Q_n'(c)\,\tilde{P}_{n-2}'(c)}{\|\tilde{P}_{n-2}\|_{\tilde{\mu}}^2}-\frac{Q_{n+3}'(c)\,\tilde{P}_{n+1}'(c)}{\|\tilde{P}_{n+1}\|_{\tilde{\mu}}^2}\right].$$

On the other side

$$\begin{aligned}\lambda_n &= \frac{\langle Q_n, \tilde{P}_n\rangle}{\langle Q_{n-1}, \tilde{P}_{n-1}\rangle} \\ &= \frac{(Q_n, \tilde{P}_n)_{\tilde{\mu}} + 2NQ_n'(c)\,\tilde{P}_n'(c)}{(Q_{n-1}, \tilde{P}_{n-1})_{\tilde{\mu}} + 2NQ_{n-1}'(c)\,\tilde{P}_{n-1}'(c)} \\ &= \frac{\|\tilde{P}_n\|_{\tilde{\mu}}^2 + 2NQ_n'(c)\,\tilde{P}_n'(c)}{\|\tilde{P}_{n-1}\|_{\tilde{\mu}}^2 + 2NQ_{n-1}'(c)\,\tilde{P}_{n-1}'(c)}\end{aligned}$$

using the formula $Q_n'(c) = \tilde{P}_n'(c)/s_{n-2}$ and the definition of s_{n-2} we obtain finally

$$\lambda_n = \tilde{\gamma}_n\,\frac{s_n}{s_{n-1}}\,\frac{s_{n-3}}{s_{n-2}}.$$

The parameters s_n can be obtained explicitly because

$$2\sum_{k=0}^{n}{}' \frac{[\tilde{P}_j'(c)]^2}{\|\tilde{P}_j\|_{\tilde{\mu}}^2} = \frac{d}{dx}\left[\tilde{K}_n^{(0,1)}(x, c) + (-1)^{n-1}\tilde{K}_n^{(0,1)}(x, -c)\right]_{x=c}$$

$$= \tilde{K}_n^{(1,1)}(c, c) + (-1)^{n-1}\tilde{K}_n^{(1,1)}(c, -c),$$

where $\tilde{K}_n$ denotes the kernel associated to the sequence $(\tilde{P}_n)$.

COROLLARY 13. *The formulas*

$$(x-3c^2)\,U_n(x) = V_{n+1}(x) + \gamma_{2n} V_n(x) + \gamma_{2n-1}\lambda_{2n} V_{n-1}(x) + \lambda_{2n}\lambda_{2n-1}\lambda_{2n-2} V_{n-2}(x)$$

and

$$x(x-3c^2)\,V_n(x) = U_{n+2}(x) + \gamma_{2n+1} U_{n+1}(x) + \gamma_{2n}\lambda_{2n+1} U_n(x) + \lambda_{2n+1}\lambda_{2n}\lambda_{2n-1} U_{n-1}(x)$$

hold.

Proof. It suffices to use (8) and the recurrence relation obtained in Proposition 12. ∎

Next we show a relation between the sequences (U_n), (V_n), and the sequence $(P_n^{(c,c)})$ of MOP w.r.t. the measure μ_2 $(d\mu_2 = (x^2-c^2)^2\,d\mu)$, derived from the self-adjointness of operator multiplication by $(x^2-c^2)^2$.

PROPOSITION 14. *The sequence (Q_n) satisfies the formula*

$$Q_n(x) = P_n^{(c,c)}(x) + \alpha_n P_{n-2}^{(c,c)}(x) + \beta_n P_{n-4}^{(c,c)}(x),$$

where

$$\alpha_n = \|P_{n-2}^{(c,c)}\|_{\mu_2}^{-2} \int_I Q_n\, P_{n-2}^{(c,c)}\, d\mu_2$$

and $\beta_n = \|P_{n-4}^{(c,c)}\|_{\mu_2}^{-2} \langle Q_n, Q_n\rangle \neq 0$.

Proof. It is a consequence of the self-adjointness of operator multiplication by $(x^2-c^2)^2$ and the symmetry of the sequences $(P_n^{(c,c)})$ and (Q_n). ∎

If we define Y_n and Y_n^* by

$$P_{2n}^{(c,c)}(x) = Y_n(x^2) \quad \text{and} \quad P_{2n+1}^{(c,c)}(x) = x\,Y_n^*(x^2)$$

from the above proposition we obtain

$$U_n(x) = Y_n(x) + \alpha_{2n} Y_{n-1}(x) + \beta_{2n} Y_{n-2}(x)$$

$$V_n(x) = Y_n^*(x) + \alpha_{2n+1} Y_{n-1}^*(x) + \beta_{2n+1} Y_{n-2}^*(x).$$

Then

COROLLARY 15. *The sequences* (U_n) *and* (V_n) *are strictly quasi-orthogonal of order* 2 *w.r.t. the measures* $v_2 = \phi(\mu_2)$ *and* $x\,dv_2$, *respectively.*

If we let $c \to 0$, we recover the remark after Proposition 3 for $s = 2$.

REFERENCES

1. M. ALFARO, F. MARCELLÁN, A. RONVEAUX, AND M. L. REZOLA, On orthogonal polynomials of Sobolev type: Algebraic properties and zeros, *SIAM J. Math. Anal.* **23** (1992), 737–757.
2. H. BAVINCK AND H. G. MEIJER, Orthogonal polynomials with respect to a symmetric inner product involving derivatives, *Appl. Anal.* **33** (1989), 103–117.
3. H. BAVINCK AND H. G. MEIJER, On orthogonal polynomials with respect to an inner product involving derivatives: Zeros and recurrence relations, *Indag. Math. (N.S.)* **1** (1990), 7–14.
4. H. BAVINCK AND H. G. MEIJER, On the zeros of orthogonal polynomials in a discrete Sobolev space with a symmetric inner product, preprint, 1991.
5. M. G. DE BRUIN, A tool for locating zeros of of orthogonal polynomials in Sobolev inner product spaces, *J. Comput. Appl. Math.* **49** (1993), 27–36.
6. T. S. CHIHARA, "An Introduction to Orthogonal Polynomials," Gordon & Breach, New York, 1978.
7. N. DRAÏDI AND P. MARONI, Sur l'adjonction de deux masses de Dirac à une forme régulière quelconque, *in* "Actas V Simp. de Pol. Ort. y sus Aplic., Vigo, 1988" (A. Cachafeiro and E. Godoy, Eds.), pp. 83–90, Santiago de Compostela, 1989.
8. D. EVANS, L. L. LITTLEJOHN, F. MARCELLÁN, C. MARKETT, AND A. RONVEAUX, On recurrence relations for Sobolev orthogonal polynomials, *SIAM J. Math. Anal.*, to appear.
9. R. KOEKOEK, Generalizations of Laguerre polynomials, *J. Math. Anal. Appl.* **153** (1990), 576–590.
10. R. KOEKOEK, "Generalizations of the Classical Laguerre Polynomials and Some *q*-Analogues," Doctoral Dissertation, Technical University of Delft, The Netherlands, 1990.
11. R. KOEKOEK AND H. G. MEIJER, A generalization of Laguerre polynomials, *SIAM J. Math. Anal.* **24** (1993), 768–782.
12. F. MARCELLÁN, M. ALFARO, AND M. L. REZOLA, Orthogonal polynomials on Sobolev spaces: Old and new directions, *J. Comput. Appl. Math.* **48** (1993), 113–132.
13. F. MARCELLÁN, T. E. PÉREZ, AND M. PIÑAR, On zeros of Sobolev type orthogonal polynomials, *Rend. Mat. Roma (7)* **12** (1992), 455–473.
14. F. MARCELLÁN AND A. RONVEAUX, On a class of polynomials orthogonal with respect to a Sobolev inner product, *Indag. Math. (N.S.)* **1** (1990), 451–464.

15. F. MARCELLÁN AND G. SANSIGRE, Orthogonal polynomials on the unit circle: Symmetrization and quadratic decomposition, *J. Approx. Theory* **65** (1991), 109–119.
16. F. MARCELLÁN AND W. VAN ASSCHE, Relative asymptotics for orthogonal polynomials witha Sobolev inner product, *J. Approx. Theory* **72** (1993), 193–209.
17. H. G. MEIJER, Zero distribution of orthogonal polynomials in a certain discrete Sobolev space, *J. Math. Anal. Appl.* **172** (1993), 520–532.
18. H. G. MEIJER, Laguerre polynomials generalized to a certain discrete Sobolev inner product space, *J. Approx. Theory* **73** (1993), 1–16.
19. H. G. MEIJER, On real and complex zeros of orthogonal polynomials in a discrete Sobolev space, *J. Comput. Appl. Math.* **49** (1993), 179–192.
20. T. E. PÉREZ AND M. PIÑAR, Global properties of the zeros for Sobolev-type orthogonal polynomials, *J. Comput. Appl. Math.* **49** (1993), 225–232.