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J. Math. Anal. Appl. 298 (2004) 171–183

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On rational transformations of linear functionals: direct problem

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Received 10 February 2004

Available online 4 August 2004

Submitted by K. Jarosz

Abstract

Let u be a quasi-definite linear functional. We find necessary and sufficient conditions in order to the linear functional v satisfying $(x-\tilde{a})u = \lambda(x-a)v$ be a quasi-definite one. Also we analyze some linear relations linking the polynomials orthogonal with respect to u and v.

Keywords: Orthogonal polynomials; Recurrence relations; Spectral transformations; Linear functionals

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1 Partially supported by MCYT Grant BFM 2003-06335-C03-03 (Spain), FEDER funds (EU), and DGA E-12/25 (Spain).
2 Partially supported by MCYT Grant BFM 2003-06335-C03-02 (Spain) and INTAS Research Network NeCCA INTAS 03-51-66378.
3 Partially supported by MCYT Grant BFM 2001-1793 (Spain), FEDER funds (EU), and DGA E-12/25 (Spain).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.04.065

1. Introduction

Let u be a linear functional in the linear space $\mathbb{P}$ of polynomials with complex coefficients and denote by $\{u_n\}_{n\geqslant 0}$ the sequence of the moments associated with u, $u_n = \langle u, x^n\rangle$, $n \geqslant 0$, where $\langle \cdot\,,\cdot\rangle$ means the duality bracket.

The linear functional u is said to be quasi-definite if the Hankel matrix $H = (u_{i+j})_{i,j=0}^{\infty}$ is quasi-definite, i.e., the principal submatrices $H_n = (u_{i+j})_{i,j=0}^{n}$, $n \in \mathbb{N} \cup \{0\}$, are nonsingular.

The linear functional δ_a given by $\langle \delta_a, P\rangle = P(a)$, for every $P \in \mathbb{P}$, is not a quasi-definite linear functional since rank $H_n = 1$ for every $n \geqslant 0$. This linear functional is said to be either the Dirac linear functional or the Dirac mass at the point a.

To the linear functional u we can associate a formal power series $S_u(z) = \sum_{n=0}^{\infty} \frac{u_n}{z^{n+1}}$ which is related with the z-transform of the sequence $\{u_n\}$ of moments of u. S_u is said to be the Stieltjes function of u. For the Dirac linear functional $u = \delta_a$ given as above, we have $S_u(z) = 1/(z-a)$ in a neighborhood of infinite.

Assuming u quasi-definite, there exists a sequence of monic polynomials $\{P_n\}_{n\geqslant 0}$ such that (see [2])

(i) $\deg P_n = n$, $n \geqslant 0$,
(ii) $\langle u, P_n P_m\rangle = k_n \delta_{n,m}$ with $k_n \neq 0$.

The sequence $\{P_n\}_{n\geqslant 0}$ is said to be the sequence of monic orthogonal polynomials (SMOP) with respect to the linear functional u.

If $\{P_n\}_{n\geqslant 0}$ is an SMOP with respect to the quasi-definite linear functional u, then it is well known (see [2]) that it satisfies a three-term recurrence relation

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geqslant 0, \tag{1.1}$$

with $\gamma_n \neq 0$ and $P_{-1}(x) = 0$, $P_0(x) = 1$.

Conversely, given a sequence of monic polynomials generated by a recurrence relation as above, there exists a unique quasi-definite linear functional u such that the family $\{P_n\}_{n\geqslant 0}$ is the corresponding SMOP. Such a result is known as the Favard theorem (see [2]).

For an SMOP $\{P_n\}_{n\geqslant 0}$ relative to u, let $\{P_n^{(1)}\}_{n\geqslant 0}$ be the sequence of monic polynomials such that

$$P_{n+1}^{(1)}(x) = (x-\beta_{n+1})P_n^{(1)}(x) - \gamma_{n+1}P_{n-1}^{(1)}(x), \quad n \geqslant 0,$$
$$P_{-1}^{(1)}(x) = 0, \qquad P_0^{(1)}(x) = 1.$$

According to the Favard theorem there exists a quasi-definite linear functional $u^{(1)}$ such that $\{P_n^{(1)}\}_{n\geqslant 0}$ is the corresponding SMOP. The family $\{P_n^{(1)}\}_{n\geqslant 0}$ is said to be the sequence of polynomials of first kind associated with the linear functional u.

Another representation of $\{P_n^{(1)}\}_{n\geqslant 0}$ is given by

$$P_n^{(1)}(y) = \frac{1}{u_0}\left\langle u, \frac{P_{n+1}(y) - P_{n+1}(x)}{y-x}\right\rangle,$$

$n \geqslant 0$ (see [2, Chapter 3]).

Notice that $P_n^{(1)}(z)/P_{n+1}(z)$ is the $(n+1)$-convergent of the continued fraction

$$\cfrac{1}{z-\beta_0-\cfrac{\gamma_1}{z-\beta_1-\ddots}}.$$

Thus

$$S_u(z) = \cfrac{u_0}{z-\beta_0-\cfrac{\gamma_1}{z-\beta_1-\ddots}} \tag{1.2}$$

from a formal point of view (see [2]).

For simplicity we will assume $u_0 = 1$.

Let $\{P_n(x,\alpha)\}_{n\geqslant 0}$ be the sequence of monic polynomials satisfying (1.1) with initial conditions $P_0(x,\alpha) = 1$, $P_1(x,\alpha) = P_1(x) - \alpha$. Taking into account the Favard theorem, there exists a quasi-definite linear functional u_α such that $\{P_n(x,\alpha)\}_{n\geqslant 0}$ is the corresponding SMOP. This sequence is said to be the co-recursive SMOP of parameter α associated with the linear functional u. It is known see [2,7] that $P_n(x,\alpha) = P_n(x) - \alpha P_{n-1}^{(1)}(x)$.

From (1.2) we get

$$S_{u^{(1)}}(z) = \frac{1}{\gamma_1}\left[z-\beta_0-\frac{1}{S_u(z)}\right],$$

$$S_{u_\alpha}(z) = \left[\frac{1}{S_u(z)}-\alpha\right]^{-1} = \frac{S_u(z)}{1-\alpha S_u(z)}.$$

These two bilinear rational transforms are related to self-similar reductions and spectral transformations in the theory of nonlinear integrable systems (see [12]).

For a linear functional u, a polynomial π, and a complex number a, let πu, $(x-a)^{-1}u$, and Du be the linear functionals defined on $\mathbb{P}$ by

$$\langle \pi u, P\rangle = \langle u, \pi P\rangle,$$

$$\left\langle (x-a)^{-1}u, P\right\rangle = \left\langle u, \frac{P(x)-P(a)}{x-a}\right\rangle,$$

$$\langle Du, P\rangle = -\langle u, P'\rangle,$$

where $P \in \mathbb{P}$.

A Cauchy product of two linear functionals u, v can be defined as the linear functional uv such that $\langle uv, x^n\rangle = \sum_{h=0}^{n} u_h v_{n-h}$, $n \geqslant 0$. Obviously, $uv = vu$ and $\delta_0 u = u\delta_0 = u$. Since $u_0 = 1$, there exists a unique linear functional v such that $uv = vu = \delta_0$. This linear functional v is said to be the inverse linear functional of u and it will be denoted by u^{-1}. Notice that $(u^{-1})_0 = 1$ and $(u^{-1})_n = -\sum_{h=0}^{n-1} u_{n-h}(u^{-1})_h$, $n \geqslant 1$ (see [10]).

Since $z^2 S_{u^{-1}}(z) S_u(z) = 1$, we have $S_{u^{(1)}}(z) = \frac{1}{\gamma_1}[z-\beta_0-z^2 S_{u^{-1}}(z)]$. Taking into account $(u^{-1})_0 = 1$ and $(u^{-1})_1 = -\beta_0$, we get $u^{(1)} = -\frac{1}{\gamma_1}x^2 u^{-1}$. Concerning the linear functional u_α, it is easy to check that $u_\alpha = (u^{-1} + \alpha\delta_0')^{-1}$. This is an alternative proof of the result of [10] but notice that there the Stieltjes function has an opposite sign.

In the constructive theory of orthogonal polynomials the so-called direct problem is considered. A direct problem for linear functionals can be stated as follows: given two linear functionals u, v such that $v = F(u)$, where F is a function defined in $\mathbb{P}'$, the dual space of $\mathbb{P}$, to find necessary and sufficient conditions in order to F preserves quasi-definiteness. As a subsequent question, to find the explicit relations between the corresponding SMOP $\{P_n\}$ and $\{Q_n\}$ associated with u and v, respectively.

If u is a linear functional defined by a nonnegative measure μ on some interval I of the real line, with an infinite set of increasing points such that the moments exist, i.e., $\langle u, x^n\rangle = \int_I x^n\,d\mu < \infty$ then we can introduce the linear functional v such that

$$\langle v, x^n\rangle = \int_I x^n \frac{p(x)}{q(x)}\,d\mu, \tag{1.3}$$

where p, q are two polynomials with pairwise distinct zeros that has constant sign on I. If we assume (1.3) is finite for every n, the generalized Christoffel theorem gives the SMOP with respect to v in terms of polynomials of the SMOP with respect to u (see [4,11]). In terms of linear functionals, the above transform reads $qv = pu$. Notice that $pu = qv$ is a more general transform because of Dirac measures and derivatives of Dirac measures at the zeros of $q(x)$ can be considered for v in addition in such a general problem.

When $q(x) = 1$ and $p(x) = x - \tilde{a}$, the transform for linear functionals is said to be a Christoffel transform (see [12]). Using the Jacobi matrix J associated with the linear functional u, the shifted Darboux transform of J without free parameter yields the Jacobi matrix of v (see [6]).

It is known that v is quasi-definite if and only if $P_n(\tilde{a}) \neq 0$, $n \geqslant 1$, and

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) - \frac{P_{n+1}(\tilde{a})}{P_n(\tilde{a})}P_n(x)$$

as well as

$$\frac{Q_n(x)P_n(\tilde{a})}{\langle u, P_n^2\rangle} = \sum_{k=0}^{n} \frac{P_k(x)P_k(\tilde{a})}{\langle u, P_k^2\rangle}.$$

The polynomials $\{Q_n\}_{n\geqslant 0}$ are said to be the monic kernel polynomials of parameter $\tilde{a}$ associated with the linear functional u (see [2]).

If $p(x) = 1$ and $q(x) = \lambda(x - a)$ then the transform is said to be the Geronimus transform of the linear functional u (see [10,12]). The Jacobi matrix of v is the shifted Darboux transform with free parameter of the Jacobi matrix of u (see [6]).

Notice that in such a case, $v = \lambda^{-1}(x - a)^{-1}u + \delta_a$ is a quasi-definite linear functional if and only if $P_n(a, -\lambda^{-1}) \neq 0$, $n \geqslant 1$, and then

$$Q_n(x) = P_n(x) - \frac{P_n(a, -\lambda^{-1})}{P_{n-1}(a, -\lambda^{-1})}P_{n-1}(x)$$

(see [9]).

In our contribution, we analyze the direct problem stated as above for the case $p(x) = (x - \tilde{a})$ and $q(x) = \lambda(x - a)$. For $a \neq \tilde{a}$ this situation has not been studied in the literature as far as we know up to in the so-called positive definite case (see [4]).

In Section 2, given a quasi-definite linear functional u and complex numbers $a, \tilde{a}$, and λ with $a \neq \tilde{a}$ and $\lambda \neq 0$, we characterize the quasi-definiteness of the linear functional $v = \frac{1}{\lambda}(x-a)^{-1}(x-\tilde{a})u + (1-\frac{1}{\lambda})\delta_a$. Instead of the analysis of the quasi-definiteness of the linear functional v in two steps (first, the rational perturbation and, second, the addition of the Dirac linear functional) we consider the whole transformation taking into account the first one cannot preserve the quasi-definiteness of the linear functional u. Indeed in [4] this constraint must be emphasized when polynomial perturbations are introduced. Further, we show that $(x-\tilde{a})Q_n$ is a linear combination of three consecutive polynomials of the SMOP $\{P_n\}_{n\geqslant 0}$.

Notice that the confluent case $a = \tilde{a}$ yields a perturbation of u via the addition of a Dirac mass at the point $x = a$. This corresponds to the Uvarov transform of the linear functional u (see [12]). The direct problem has been solved in [8]. We point out that the results for $a \neq \tilde{a}$ extend in a natural way those already known for $a = \tilde{a}$.

In Section 3, under the thesis of Section 2 we characterize when the relation between $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, obtained there, can be reduced to a relation $P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x)$ with $s_n t_n \neq 0$ for every $n \geqslant 1$, and $s_1 \neq t_1$. This last type of relation, as an inverse problem, has been analyzed in [1]. The motivation for such a kind of problems is reflected in [3] when an extension of the concept of coherent pairs of measures associated with Sobolev inner products is considered.

We also observe that there is an important difference for the cases $a = \tilde{a}$ and $a \neq \tilde{a}$. Namely, if $a = \tilde{a}$ then $s_n \neq t_n$ for every $n \geqslant 1$ while if $a \neq \tilde{a}$ both situations, i.e., either $s_n \neq t_n$ for every $n \geqslant 1$ or $s_n = t_n$ for some values of n, can appear as we show in some examples.

2. Direct problem

In this section, we study the direct problem for $v = \frac{1}{\lambda}(x-a)^{-1}(x-\tilde{a})u + (1-\frac{1}{\lambda})\delta_a$ where u is a given quasi-definite linear functional, and $a, \tilde{a}, \lambda \in \mathbb{C}$ with $a \neq \tilde{a}$, $\lambda \neq 0$.

Theorem 2.1. *Let u, v be two linear functionals related by*

$$(x-\tilde{a})u = \lambda(x-a)v, \quad a, \tilde{a}, \lambda \in \mathbb{C}. \tag{2.1}$$

Assume $u_0 = 1 = v_0$ and $a \neq \tilde{a}$. If u is a quasi-definite linear functional with corresponding SMOP $\{P_n\}_{n\geqslant 0}$ then, the linear functional v is quasi-definite if and only if

$$\Delta_n = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_n(a) & R_{n-1}(a) \end{vmatrix} \neq 0, \quad n \geqslant 1,$$

where $R_n(x) = (\lambda-1)P_n(x) + (a-\tilde{a})P^{(1)}_{n-1}(x)$. Furthermore, if $\{Q_n\}_{n\geqslant 0}$ is the SMOP associated with v then

$$(x-\tilde{a})Q_n(x) = \Delta_n^{-1} \begin{vmatrix} P_{n+1}(x) & P_n(x) & P_{n-1}(x) \\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geqslant 1. \tag{2.2}$$

Proof. Assume v is a quasi-definite linear functional and $\{Q_n\}_{n\geqslant 0}$ is its corresponding SMOP.

Consider the Fourier expansion of $(x-\tilde{a})Q_n$ in terms of the polynomials P_n, that is

$$(x-\tilde{a})Q_n(x) = P_{n+1}(x) + \sum_{j=0}^{n} \alpha_{n,j} P_j(x), \quad n \geqslant 1,$$

where $\alpha_{nj} = \langle u, P_j^2\rangle^{-1}\langle u, (x-\tilde{a})Q_n P_j\rangle$. From formula (2.1) we get

$$(x-\tilde{a})Q_n(x) = P_{n+1}(x) + \alpha_{n,n}P_n(x) + \alpha_{n,n-1}P_{n-1}(x) \tag{2.3}$$

with $\alpha_{n,n-1} = \lambda\frac{\langle v, Q_n^2\rangle}{\langle u, P_{n-1}^2\rangle} \neq 0$.

For $x = \tilde{a}$

$$0 = P_{n+1}(\tilde{a}) + \alpha_{n,n}P_n(\tilde{a}) + \alpha_{n,n-1}P_{n-1}(\tilde{a}). \tag{2.4}$$

On the other hand,

$$(a-\tilde{a})Q_n(a) = P_{n+1}(a) + \alpha_{n,n}P_n(a) + \alpha_{n,n-1}P_{n-1}(a). \tag{2.5}$$

Subtracting (2.5) to (2.3) and dividing by $x-a$, we can apply u in order to get

$$\left\langle u, \frac{(x-\tilde{a})Q_n(x) - (a-\tilde{a})Q_n(a)}{x-a}\right\rangle = P_n^{(1)}(a) + \alpha_{n,n}P_{n-1}^{(1)}(a) + \alpha_{n,n-1}P_{n-2}^{(1)}(a). \tag{2.6}$$

The left-hand side becomes

$$\left\langle u, (x-\tilde{a})\frac{Q_n(x)-Q_n(a)}{x-a}\right\rangle + Q_n(a) = \lambda\langle v, Q_n(x) - Q_n(a)\rangle + Q_n(a) = (1-\lambda)Q_n(a)$$

and therefore

$$(1-\lambda)Q_n(a) = P_n^{(1)}(a) + \alpha_{n,n}P_{n-1}^{(1)}(a) + \alpha_{n,n-1}P_{n-2}^{(1)}(a). \tag{2.7}$$

Thus, (2.5) and (2.7) yield

$$0 = R_{n+1}(a) + \alpha_{n,n}R_n(a) + \alpha_{n,n-1}R_{n-1}(a). \tag{2.8}$$

Since the system of Eqs. (2.4) and (2.8) in $\alpha_{n,n}$ and $\alpha_{n,n-1}$ has a non-zero solution, then we get $\Delta_n \neq 0$ for every $n \geqslant 1$.

Besides, from (2.3), (2.4), and (2.8) we obtain (2.2).

Conversely, if $\Delta_n \neq 0$ for every $n \geqslant 1$ we will prove that the polynomials Q_n defined by

$$(x-\tilde{a})Q_n(x) = \Delta_n^{-1}\begin{vmatrix} P_{n+1}(x) & P_n(x) & P_{n-1}(x) \\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_{n+1}(a) & R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geqslant 1,$$

are orthogonal with respect to v. Indeed, for $0 \leqslant j \leqslant n-2$,

$$\lambda\langle v, Q_n(x)(x-a)P_j(x)\rangle = \langle u, (x-\tilde{a})Q_n(x)P_j(x)\rangle = 0$$

and for $j = n-1$,

$$\lambda\big\langle v, Q_n(x)(x-a)P_{n-1}(x)\big\rangle = \big\langle u, (x-\tilde{a})Q_n(x)P_{n-1}(x)\big\rangle = \Delta_{n+1}\Delta_n^{-1}\big\langle u, P_{n-1}^2\big\rangle \neq 0.$$

Thus, we only need to prove that $\langle v, Q_n\rangle = 0$ for every $n \geqslant 1$. In order to do this, observe that

$$\begin{aligned}\lambda\langle v, Q_n\rangle &= \lambda\left[\left\langle v, (x-a)\frac{Q_n(x)-Q_n(a)}{x-a}\right\rangle + Q_n(a)\right]\\ &= \left\langle (x-\tilde{a})u, \frac{Q_n(x)-Q_n(a)}{x-a}\right\rangle + \lambda Q_n(a)\\ &= \left\langle u, \frac{(x-\tilde{a})Q_n(x)-(a-\tilde{a})Q_n(a)}{x-a}\right\rangle + (\lambda-1)Q_n(a).\end{aligned}$$

Applying the expression of $(x-\tilde{a})Q_n(x)$ in terms of the polynomials $P_n(x)$ and (2.7) we get

$$\begin{aligned}&\left\langle u, \frac{(x-\tilde{a})Q_n(x)-(a-\tilde{a})Q_n(a)}{x-a}\right\rangle\\ &\quad = \Delta_n^{-1}\begin{vmatrix} P_n^{(1)}(a) & P_{n-1}^{(1)}(a) & P_{n-2}^{(1)}(a)\\ P_{n+1}(\tilde{a}) & P_n(\tilde{a}) & P_{n-1}(\tilde{a})\\ R_{n+1}(a) & R_n(a) & R_{n-1}(a)\end{vmatrix} = (1-\lambda)Q_n(a).\end{aligned}$$

So $\langle v, Q_n\rangle = 0$ for every $n \geqslant 1$.

As a conclusion, $\langle v, Q_n^2\rangle = \langle v, Q_n(x-a)P_{n-1}\rangle \neq 0$, and $\langle v, Q_n p\rangle = 0$ for every polynomial p of degree less than n. □

Corollary 2.2. *Under the conditions of Theorem* 2.1 *the linear functional* v *is quasi-definite if and only if* $1 + \sum_{j=0}^{n-1} \frac{P_j(\tilde{a})R_j(a)}{\langle u, P_j^2\rangle} \neq 0$*, for every* $n \geqslant 1$.

Furthermore, we have

$$(x-\tilde{a})Q_n(x) = P_{n+1}(x) + a_n(a,\tilde{a})P_n(x) + b_n(a,\tilde{a})P_{n-1}(x), \quad n \geqslant 1 \tag{2.9}$$

with

$$a_n(a,\tilde{a}) = \beta_n - \tilde{a} + (a-\tilde{a})\Delta_n^{-1}P_{n-1}(\tilde{a})R_n(a) \tag{2.10}$$

and

$$b_n(a,\tilde{a}) = \gamma_n + (\tilde{a}-a)\Delta_n^{-1}P_n(\tilde{a})R_n(a). \tag{2.11}$$

Proof. From the expression of Δ_n, using the Christoffel–Darboux formula (see [2]), we have for $n \geqslant 1$

$$\Delta_n = (a-\tilde{a})\big[(1-\lambda)K_{n-1}(a,\tilde{a};u)\big\langle u, P_{n-1}^2\big\rangle + B_n(a,\tilde{a})\big],$$

where $K_n(x,y;u)$ denotes the reproducing kernel of degree n associated with u and

$$B_n(a,\tilde{a}) = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a})\\ P_{n-1}^{(1)}(a) & P_{n-2}^{(1)}(a)\end{vmatrix}.$$

Inserting the three-term recurrence relation for both polynomials P_n and $P_{n-1}^{(1)}$, we get

$$\frac{B_n(a,\tilde{a})}{\langle u, P_{n-1}^2\rangle} = (\tilde{a}-a)\frac{P_{n-1}(\tilde{a})P_{n-2}^{(1)}(a)}{\langle u, P_{n-1}^2\rangle} + \frac{B_{n-1}(a,\tilde{a})}{\langle u, P_{n-2}^2\rangle}, \quad n \geqslant 2.$$

Iteration yields

$$\frac{B_n(a,\tilde{a})}{\langle u, P_{n-1}^2\rangle} = (\tilde{a}-a)\sum_{j=0}^{n-1}\frac{P_j(\tilde{a})P_{j-1}^{(1)}(a)}{\langle u, P_j^2\rangle} - 1, \quad n \geqslant 1. \tag{2.12}$$

Therefore

$$\begin{aligned}\Delta_n &= (\tilde{a}-a)\langle u, P_{n-1}^2\rangle\left[1 + (\lambda-1)K_{n-1}(a,\tilde{a};u) + (a-\tilde{a})\sum_{j=0}^{n-1}\frac{P_j(\tilde{a})P_{j-1}^{(1)}(a)}{\langle u, P_j^2\rangle}\right]\\ &= (\tilde{a}-a)\langle u, P_{n-1}^2\rangle\left[1 + \sum_{j=0}^{n-1}\frac{P_j(\tilde{a})R_j(a)}{\langle u, P_j^2\rangle}\right], \end{aligned} \tag{2.13}$$

and the first part of the corollary follows from Theorem 2.1.

On the other hand, we can write formula (2.2) as follows

$$(x-\tilde{a})Q_n(x) = P_{n+1}(x) + a_n(a,\tilde{a})P_n(x) + b_n(a,\tilde{a})P_{n-1}(x), \quad n \geqslant 1.$$

Using the three-term recurrence relation for $P_{n+1}(\tilde{a})$ and $R_{n+1}(a)$ we get

$$\begin{aligned} a_n(a,\tilde{a}) &= \beta_n - \Delta_n^{-1}\left[\tilde{a}P_n(\tilde{a})R_{n-1}(a) - aP_{n-1}(\tilde{a})R_n(a)\right]\\ &= \beta_n - \tilde{a} + (a-\tilde{a})\Delta_n^{-1}P_{n-1}(\tilde{a})R_n(a). \end{aligned}$$

Besides, from (2.13) we obtain

$$\frac{\Delta_{n+1}}{\langle u, P_n^2\rangle} = \frac{\Delta_n}{\langle u, P_{n-1}^2\rangle} + (\tilde{a}-a)\frac{P_n(\tilde{a})R_n(a)}{\langle u, P_n^2\rangle}$$

and, since $b_n(a,\tilde{a}) = \Delta_{n+1}/\Delta_n$ and $\gamma_n = \langle u, P_n^2\rangle/\langle u, P_{n-1}^2\rangle$, then

$$b_n(a,\tilde{a}) = \gamma_n + (\tilde{a}-a)\Delta_n^{-1}P_n(\tilde{a})R_n(a). \qquad \square$$

In Theorem 2.1 and Corollary 2.2 we have assumed $a \neq \tilde{a}$. Notice that if $a = \tilde{a}$ the relation (2.1) between the linear functionals u and v becomes $u = \lambda v + (1-\lambda)\delta_a$. In this situation it is well known (see [8]) that v is quasi-definite if and only if for every $n \geqslant 1$

$$1 + (\lambda-1)K_n(a,a;u) \neq 0$$

and then

$$(x-a)Q_n(x) = P_{n+1}(x) + a_n(a)P_n(x) + b_n(a)P_{n-1}(x), \quad n \geqslant 1, \tag{2.14}$$

holds, where

$$a_n(a) = \beta_n - a - \frac{(\lambda-1)P_{n-1}(a)P_n(a)}{\langle u, P_{n-1}^2\rangle[1 + (\lambda-1)K_{n-1}(a,a;u)]}$$

and

$$b_n(a) = \gamma_n \frac{1 + (\lambda - 1)K_n(a, a; u)}{1 + (\lambda - 1)K_{n-1}(a, a; u)}.$$

Notice that, these results can be recovered from Corollary 2.2, when $\tilde{a}$ tends to a.

3. Linear relations between the polynomials $\{P_n\}$ and $\{Q_n\}$

Let u and v be quasi-definite linear functionals with corresponding SMOP $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$, respectively. In Section 2, we have obtained that if u and v satisfy the relation $(x - \tilde{a})u = \lambda(x - a)v$ with $a, \tilde{a}, \lambda \in \mathbb{C}$ then an expression of the form

$$(x - \tilde{a})Q_n(x) = P_{n+1}(x) + a_n P_n(x) + b_n P_{n-1}(x), \quad n \geqslant 1, \tag{3.1}$$

holds (see formulas (2.9) and (2.14)). That is, a linear combination of three consecutive polynomials P_n coincides with a linear combination of three consecutive polynomials Q_n.

On the other hand, in [1], it was proved that if the linear functionals u and v are quasi-definite and they are related as above, then there exists a relation $P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x)$ with $s_n t_n \neq 0$, $n \geqslant 1$, and $s_1 \neq t_1$ if and only if for every $n \geqslant 1$, $P_n \neq Q_n$.

Thus, at the present, we have two expressions linking the polynomials P_n and Q_n, the last quoted and the one given in formula (3.1).

We see below that if $P_n \neq Q_n$, $n \geqslant 1$, then both formulas are not independent. In fact, one of them can be reduced to the other.

Theorem 3.1. *Let u, v be two different quasi-definite linear functionals normalized by $u_0 = 1 = v_0$ and related by*

$$(x - \tilde{a})u = \lambda(x - a)v, \quad a, \tilde{a}, \lambda \in \mathbb{C}.$$

Let $\{P_n\}_{n\geqslant 0}$ and $\{Q_n\}_{n\geqslant 0}$ be their corresponding SMOP. The following conditions are equivalent:

(i) *Formula* (3.1) *can be reduced to an expression*

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x) \tag{3.2}$$

with $s_n t_n \neq 0$ for every $n \geqslant 1$ and $s_1 \neq t_1$.

(ii) *For all $n \geqslant 1$, $R_n(a) = (\lambda - 1)P_n(a) + (a - \tilde{a})P^{(1)}_{n-1}(a) \neq 0$.*

Proof. Suppose that (i) holds. In [1, Theorem 2.4] it has been proved that whenever such a relation (3.2) is satisfied then $P_n \neq Q_n$, for every n, and besides $P_n(x) = Q_n(x) + \lambda^{-1}R_n(a)K_{n-1}(x, a; v)$, $n \geqslant 1$ (see formula (2.24) in [1]). So, (ii) follows.

In order to derive the converse result we will first consider the case $a \neq \tilde{a}$. Inserting the three-term recurrence relation in (3.1) successively for P_{n+1} and P_n we get, for $n \geqslant 2$,

$$(x - \tilde{a})Q_n(x) = (x - \tilde{a})P_n(x) + (\tilde{a} - \beta_n + a_n)P_n(x) + (b_n - \gamma_n)P_{n-1}(x)$$

$$= (x-\tilde{a})\big[P_n(x) + (\tilde{a}-\beta_n+a_n)P_{n-1}(x)\big] + \big[(\tilde{a}-\beta_n+a_n)(\tilde{a}-\beta_{n-1}) + b_n - \gamma_n\big]P_{n-1}(x) - \gamma_{n-1}(\tilde{a}-\beta_n+a_n)P_{n-2}(x). \tag{3.3}$$

The first part of the formula (3.3) for $n-1$ reads:

$$(x-\tilde{a})Q_{n-1}(x) = (x-\tilde{a})P_{n-1}(x) + (\tilde{a}-\beta_{n-1}+a_{n-1})P_{n-1}(x) + (b_{n-1}-\gamma_{n-1})P_{n-2}(x). \tag{3.4}$$

Taking into account (2.10) and (2.11), the above two formulas can be written

$$(x-\tilde{a})Q_n(x) = (x-\tilde{a})\left[P_n(x) + \frac{(a-\tilde{a})}{\Delta_n}R_n(a)P_{n-1}(\tilde{a})P_{n-1}(x)\right] + \frac{(a-\tilde{a})}{\Delta_n}R_n(a)\gamma_{n-1}\big[P_{n-2}(\tilde{a})P_{n-1}(x) - P_{n-1}(\tilde{a})P_{n-2}(x)\big],$$

$$(x-\tilde{a})Q_{n-1}(x) = (x-\tilde{a})P_{n-1}(x) + \frac{(a-\tilde{a})}{\Delta_{n-1}}R_{n-1}(a)\big[P_{n-2}(\tilde{a})P_{n-1}(x) - P_{n-1}(\tilde{a})P_{n-2}(x)\big].$$

Thus, for any $t_n \in \mathbb{R}$, $n \geqslant 2$

$$(x-\tilde{a})\big[Q_n(x) + t_nQ_{n-1}(x)\big] = (x-\tilde{a})\left[P_n(x) + \left(\frac{(a-\tilde{a})}{\Delta_n}R_n(a)P_{n-1}(\tilde{a}) + t_n\right)P_{n-1}(x)\right] + (a-\tilde{a})\left[\frac{R_n(a)}{\Delta_n}\gamma_{n-1} + \frac{R_{n-1}(a)}{\Delta_{n-1}}t_n\right]\big[P_{n-2}(\tilde{a})P_{n-1}(x) - P_{n-1}(\tilde{a})P_{n-2}(x)\big].$$

Now, since by hypothesis $R_n(a) \neq 0$ for all n, if we take

$$t_n = -\frac{R_n(a)}{R_{n-1}(a)}\frac{\Delta_{n-1}}{\Delta_n}\gamma_{n-1}, \quad n \geqslant 2,$$

we get $t_n \neq 0$ as well as

$$Q_n(x) + t_nQ_{n-1}(x) = P_n(x) + s_nP_{n-1}(x),$$

where $s_n = (a-\tilde{a})\Delta_n^{-1}R_n(a)P_{n-1}(\tilde{a}) + t_n$.

Observe that, using (2.11), we can obtain

$$s_n = -\frac{R_n(a)}{R_{n-1}(a)} \neq 0, \quad n \geqslant 2.$$

For $n=1$, from the values of a_1 and b_1, the first part of formula (3.3) becomes $Q_1(x) = P_1(x) + \frac{(a-\tilde{a})}{\Delta_1}R_1(a)$. Then $P_1(x) + s_1 = Q_1(x) + t_1$ holds with $s_1t_1 \neq 0$ and $s_1 - t_1 \neq 0$.

Finally, notice that the case $a = \tilde{a}$ can be derived in a similar way. □

Remarks. (1) In Section 2, we have seen that the linear functional v is quasi-definite if and only if $1 + \sum_{j=0}^{n}\frac{P_j(\tilde{a})R_j(a)}{\langle u, P_j^2\rangle} \neq 0$, $n \geqslant 1$. It is worth noticing that the parameters $\{R_n(a)\}_{n\geqslant 0}$, which appear in the above result, also characterize the existence of formula (3.2).

(2) In terms of the linear functionals, we have that $R_n(a) \neq 0$ ($n \geqslant 1$) if and only if the linear functional $(x-a)w$ is quasi-definite, where w is either the linear functional u (case $a = \tilde{a}$, $\lambda \neq 1$), or the linear functional $u^{(1)}$ (case $a \neq \tilde{a}$, $\lambda = 1$) or the linear functional associated with the co-recursive polynomials (case $a \neq \tilde{a}$, $\lambda \neq 1$).

(3) If $a \neq \tilde{a}$ and $\lambda \neq 1$ it was proved in [9] that $R_n(a) \neq 0$ for every $n \geqslant 1$ if and only if the linear functional $\frac{a-\tilde{a}}{\lambda-1}(x-a)^{-1}u + \delta_a$ is quasi-definite. When u and v are related as in Theorem 3.1, this last condition is equivalent to the quasi-definiteness of the linear functional $\lambda v - u$. Moreover, in this case the SMOP associated with $\lambda v - u$ is $\{P_n - \frac{R_n(a)}{R_{n-1}(a)} P_{n-1}\}_{n \geqslant 0}$.

Next, we want to point out that a difference appears between the cases $a = \tilde{a}$ and $a \neq \tilde{a}$ with respect to the parameters s_n and t_n in formula (3.2).

In Theorem 3.1, it has been shown that there exists a relation of the form

$$P_n(x) + s_n P_{n-1}(x) = Q_n(x) + t_n Q_{n-1}(x) \tag{3.5}$$

with $s_n t_n \neq 0$, $n \geqslant 1$, and $s_1 \neq t_1$ if and only if $R_n(a) \neq 0$, $n \geqslant 1$. Moreover, we get for every $n \geqslant 1$

$$t_n - s_n = \frac{P_{n-1}(\tilde{a}) R_n(a)}{\langle u, P_{n-1}^2 \rangle [1 + \sum_{j=0}^{n-1} \frac{P_j(\tilde{a}) R_j(a)}{\langle u, P_j^2 \rangle}]}.$$

Then, whenever $a = \tilde{a}$ and $\lambda \neq 1$, (3.5) holds if and only if the linear functional $(x - \tilde{a})u$ is quasi-definite. Besides $s_n \neq t_n$, for $n \geqslant 1$.

However, if $a \neq \tilde{a}$, even if the condition $R_n(a) \neq 0$ is satisfied for all $n \geqslant 1$ then both situations either $(x - \tilde{a})u$ is quasi-definite or $(x - \tilde{a})u$ is not quasi-definite can appear. In fact, an example of the first situation was given in [1] being u and v the Jacobi linear functionals with parameters $\alpha - 1, \beta$ and $\alpha, \beta - 1$ ($\alpha, \beta > 0$), respectively, and $a = -1$, $\tilde{a} = 1$, $\lambda = -\alpha\beta^{-1}$. In this case, also $s_n \neq t_n$ for every $n \geqslant 1$.

Next, we are going to show an example of the second situation, that is, when the linear functional $(x - \tilde{a})u$ is not quasi-definite and, as a consequence, the condition $s_n \neq t_n$ is not satisfied for every $n \geqslant 1$.

Let u be the Chebyshev linear functional of second kind, that is, the Jacobi linear functional with parameters $\alpha = \beta = 1/2$, and take $a = 1$, $\tilde{a} = 0$, and $\lambda = 3$. We denote by $\{P_n\}$ the monic polynomials associated with u whose recurrence coefficients are $\beta_n = 0$ and $\gamma_n = 1/4$ (see [2]). Observe that the linear functional xu is not quasi-definite.

With these conditions the co-recursive polynomials R_n are given by

$$R_n(x) = 2\left[P_n(x) + \frac{1}{2} P_{n-1}(x)\right]. \tag{3.6}$$

Notice that $\frac{1}{2} R_n(x)$ are the monic Chebyshev polynomials of fourth kind, that is the monic Jacobi polynomials with parameters $\alpha = 1/2$ and $\beta = -1/2$, see [5].

First, we check that the linear functional v defined by $xu = 3(x-1)v$ is quasi-definite. As we have introduced in Theorem 2.1

$$\Delta_n = \begin{vmatrix} P_n(\tilde{a}) & P_{n-1}(\tilde{a}) \\ R_n(a) & R_{n-1}(a) \end{vmatrix}, \quad n \geqslant 1,$$

and since $P_{2n}(0) = (-1)^n/4^n$, $P_{2n+1}(0) = 0$, and $R_n(1) = (2n+1)/2^{n-1}$ we get

$$\Delta_{2n} = (-1)^n \frac{4n-1}{4^{2n-1}} \quad \text{and} \quad \Delta_{2n+1} = (-1)^{n+1} \frac{4n+3}{4^{2n}}.$$

Therefore, $\Delta_n \neq 0$ for every $n \geqslant 1$, and thus v is quasi-definite. Observe that $v = -\frac{x}{3}w + \delta_1$ where w denotes the Chebyshev linear functional of third kind.

As $R_n(1) \neq 0$, for $n \geqslant 1$, from Theorem 3.1 a relation of the form (3.5) holds with

$$s_n = -\frac{R_n(1)}{R_{n-1}(1)} = -\frac{2n+1}{2(2n-1)}, \quad n \geqslant 2,$$

and

$$t_n = \frac{\Delta_{n-1}}{4\Delta_n} s_n, \quad n \geqslant 2.$$

Therefore, taking into account $P_1(x) = Q_1(x) + 1$, we deduce

$$P_{2n}(x) - \frac{4n+1}{2(4n-1)} P_{2n-1}(x) = Q_{2n}(x) - \frac{4n+1}{2(4n-1)} Q_{2n-1}(x), \quad n \geqslant 1,$$

$$P_{2n+1}(x) - \frac{4n+3}{2(4n+1)} P_{2n}(x) = Q_{2n+1}(x) + \frac{4n-1}{2(4n+1)} Q_{2n}(x), \quad n \geqslant 0.$$

Notice that in this case $s_{2n} = t_{2n}$, $n \geqslant 1$.

Eventually, from the values of the recurrence coefficients of $\{P_n\}$ and Theorem 2.2 in [1], we can deduce that the recurrence parameters for $\{Q_n\}$ are $\tilde{\beta}_n = (-1)^n$, $n \geqslant 0$, and

$$\tilde{\gamma}_{2n+1} = -\frac{4n-1}{4(4n+3)}, \quad n \geqslant 0, \quad \text{and} \quad \tilde{\gamma}_{2n} = -\frac{4n+3}{4(4n-1)}, \quad n \geqslant 1.$$

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