J. Math. Anal. Appl. 401 (2013) 182–197

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Orthogonal polynomials generated by a linear structure relation: Inverse problem

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ARTICLE INFO

Article history:
Received 27 September 2012
Available online 7 December 2012
Submitted by Kathy Driver

Keywords:
Orthogonal polynomials
Moment linear functionals
Difference equations
Inverse problems

ABSTRACT

Let $(P_n)_n$ and $(Q_n)_n$ be two sequences of monic polynomials linked by a type structure relation such as

$$Q_n(x) + r_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x),$$

where $(r_n)_n$, $(s_n)_n$ and $(t_n)_n$ are sequences of complex numbers.

First, we state necessary and sufficient conditions on the parameters such that the above relation becomes non-degenerate when both sequences $(P_n)_n$ and $(Q_n)_n$ are orthogonal with respect to regular moment linear functionals $\mathbf{u}$ and $\mathbf{v}$, respectively.

Second, assuming that the above relation is non-degenerate and $(P_n)_n$ is an orthogonal sequence, we obtain a characterization for the orthogonality of the sequence $(Q_n)_n$ in terms of the coefficients of the polynomials Φ and Ψ which appear in the rational transformation (in the distributional sense) $\Phi\mathbf{u} = \Psi\mathbf{v}$.

Some illustrative examples of the developed theory are presented.

1. Introduction

The analysis of M–N type linear structure relations involving two monic orthogonal polynomial sequences (MOPS), $(P_n)_n$ and $(Q_n)_n$, such as

$$Q_n(x) + \sum_{i=1}^{M-1} r_{i,n} Q_{n-i}(x) = P_n(x) + \sum_{i=1}^{N-1} s_{i,n} P_{n-i}(x), \quad n \geq 0,$$

where M and N are fixed positive integer numbers, and $(r_{i,n})_n$ and $(s_{i,n})_n$ are sequences of complex numbers (and empty sum equals zero), has been a subject of research interest in the last decades, both from the algebraic and the analytical point of view. For historical references, as well as a description of several aspects focused on the interest and importance of the study of structure relations involving linear combinations of two MOPSs, we refer the introductory sections in the recent works [1,4] by F. Marcellán and three of the authors of this article, as well as the references therein. Such a study is also of interest in the framework of the theory of Sobolev orthogonal polynomials, in particular in connection with the notion of coherent pair of measures and its generalizations, where linear structure relations involving derivatives of at least one of the families $(P_n)_n$ and $(Q_n)_n$ appear (see e.g. [9,11,7]).

It is known [14] that up to some natural conditions (avoiding degenerate cases) the above M–N type structure relation leads to a rational transformation

$$\Phi\mathbf{u} = \Psi\mathbf{v}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.12.004

between the regular (or quasi-definite) moment linear functionals **u** and **v** with respect to which the sequences $(P_n)_n$ and $(Q_n)_n$ are orthogonal (respectively), where Φ and Ψ are polynomials of (exact) degrees $M-1$ and $N-1$, respectively. As usual, $\langle \mathbf{w}, q\rangle$ means the action of the functional **w** over the polynomial q and the left product of a functional **w** (defined in the space of all polynomials) by a polynomial ϕ is defined in the distributional sense, i.e., $\langle \phi\mathbf{w}, p\rangle = \langle \mathbf{w}, \phi p\rangle$ for any polynomial p. In terms of the Stieltjes transforms associated with **u** and **v**, the above relation between the functionals leads to a linear spectral transformation, in the sense described and studied by A. Zhedanov [17], and by V. Spiridonov, L. Vinet, and A. Zhedanov [16]. Moreover, P. Maroni [12] gave a characterization of the relation between the MOPSs associated with two regular functionals **u** and **v** fulfilling $\Phi\mathbf{u} = \Psi\mathbf{v}$. In connection with the study of direct problems related to orthogonal polynomials associated with this kind of modifications of linear functionals (rational modifications), besides the work [2], among others we also point out the works by W. Gautschi [8], M. Sghaier and J. Alaya [15], M.I. Bueno and F. Marcellán [5], and J.H. Lee and K.H. Kwon [10], in particular, in the framework of the so-called Christoffel formula and its generalizations.

Concerning the above M–N type structure relation, most of the papers in the available literature deal with relations considering concrete values for M and N, specially $M, N \in \{1, 2, 3\}$. Indeed, the simplest relations of types 1–2 and 2–1 have been studied in [11], the 2–2 type relation in [3,2], and the more elaborated situation involving a 1–3 type relation has been studied in [4]. In addition, the 1–N type relation with constant coefficients (i.e., each $(s_{i,n})_n$ is a constant sequence) has been analyzed in [1]. In all these works a main problem stated and solved therein was the following inverse problem: assuming that $(P_n)_n$ is a MOPS and $(Q_n)_n$ only a simple set of polynomials – i.e., every Q_n is a polynomial of degree n –, to determine necessary and sufficient conditions such that $(Q_n)_n$ becomes also a MOPS. The general M–N type relations have been considered in [14], but the results were therein obtained assuming the orthogonality of both sequences $(P_n)_n$ and $(Q_n)_n$, as well as some additional assumptions ensuring non-degenerate situations. The analysis of the regularity conditions is usually a hard task, since it involves solving systems of nonlinear difference equations, and in general there are not available methods for solving them. Therefore often the success depends on the application of *ad-hoc* methods for solving such systems.

In this contribution we focus on the analysis of the M–N type relation with $M = 2$ and $N = 3$, that is

$$Q_n(x) + r_nQ_{n-1}(x) = P_n(x) + s_nP_{n-1}(x) + t_nP_{n-2}(x), \quad n \geq 0, \tag{1.1}$$

where $(r_n)_n$, $(s_n)_n$, and $(t_n)_n$ are sequences of complex numbers with the conventions $r_0 = s_0 = t_0 = t_1 = 0$.

Our aim is twofold. On the first hand, we determine whether (1.1) is a degenerate or a non-degenerate structure relation. We say that the 2–3 type relation (1.1) is degenerate if there exists another structure relation of type M–N linking $(P_n)_n$ and $(Q_n)_n$ with $M < 2$ or $N < 3$. In Theorem 2.1 we will see how some appropriate initial conditions, involving only the parameters $r_1, r_2, r_3, s_1, s_2, t_2$, and t_3, allow us to describe all the possible degenerate cases. Besides, only under the assumption $t_2 \neq r_2(s_1 - r_1)$ and $r_3t_3 \neq 0$ we really have a non-degenerate 2–3 type relation. Since all the degenerate cases have been already considered in the previous works [3,2,4,11], we will focus on the non-degenerate case. Some non-degenerate (1.1) relations have been already considered in [14], but as we will prove not all of them.

On the other hand, the following so-called inverse problem is considered: given two sequences of monic polynomials $(P_n)_n$ and $(Q_n)_n$ such that (1.1) holds, and under the assumptions that (1.1) is non-degenerate and $(P_n)_n$ is a MOPS, to find necessary and sufficient conditions such that $(Q_n)_n$ becomes also a MOPS and, under such conditions, to give the relation between the linear functionals with respect to which $(P_n)_n$ and $(Q_n)_n$ are orthogonal. In this paper, we not only show such a characterization (see Theorem 3.1), but we achieve another more developed result which describes the orthogonality of $(Q_n)_n$ in terms of some sequences which remain constant (see Theorem 3.2). Even more, the most interesting fact is that the values of these constants are precisely the coefficients of the polynomials involved in the relation between the linear functionals. We want to observe that the same property was obtained in [11, Theorem 2], [3, Theorem 2.2], and [4, Theorem 2.2] for non-degenerate relations 1–2, 2–2, and 1–3 respectively. Thus, for a general M–N type structure relation (avoiding degenerate cases) we conjecture that a deeper solution of the inverse problem can be done in terms of the existence of certain constant sequences whose values coincide with the coefficients of the polynomials of (exact) degree $M-1$ and $N-1$ which relate both regular functionals.

The structure of the paper is the following. In Section 2, the first of the above mentioned questions, i.e., determining under which conditions the 2–3 type (1.1) relation is either degenerate or non-degenerate is solved in Theorem 2.1. Besides, a regularity characterization of the functional $\Phi\mathbf{u}$ is given in Proposition 2.2, filling out the non-degenerate relation (1.1) studied in [14, Section 5]. Also, an example of a non-degenerate relation (1.1) where the functional $\Phi\mathbf{u}$ is not regular is presented. The announced regularity (orthogonality) conditions, i.e. the solution of our inverse problem, will be stated in Theorems 3.1 and 3.2 in Section 3. Finally, in Section 4 we present a computational example illustrating the developed theory. The reader may find the basic background on orthogonal polynomials needed in the sequel in most of the articles appearing in the set of references, specially the monograph [6] by T.S. Chihara where the general theory is presented, and the paper [13] by P. Maroni concerning some algebraic aspects of the theory.

2. Degenerate and non-degenerate 2–3 type relations

Let $(P_n)_n$ and $(Q_n)_n$ be two sequences of monic polynomials orthogonal with respect to the regular functionals **u** and **v** (resp.), normalized by $\langle \mathbf{u}, 1\rangle = 1 = \langle \mathbf{v}, 1\rangle$. Let $(\beta_n)_n$ and $(\gamma_n)_n$ be the sequences of recurrence coefficients characterizing $(P_n)_n$, and $(\widetilde{\beta}_n)_n$ and $(\widetilde{\gamma}_n)_n$ the corresponding sequences characterizing $(Q_n)_n$. Suppose that these families of polynomials are related by the 2–3 type relation (1.1) with the conventions $r_0 = s_0 = t_0 = t_1 = 0$. It is known [14, Theorem 1.1] that the initial conditions $t_2 \neq r_2(s_1 - r_1)$ and $r_3 t_3 \neq 0$ yield a relation between the linear functionals $\mathbf{u}$ and $\mathbf{v}$ such as

$$\Phi \mathbf{u} = \Psi \mathbf{v},$$

where Φ and Ψ are polynomials of (exact) degrees 1 and 2, respectively.

We will show that these initial conditions are not only sufficient but also necessary to have a non-degenerate relation, that is $r_n t_n \neq 0$ for all $n \geq 3$.

First, we point out that the conditions $t_2 \neq r_2(s_1 - r_1)$ and $r_3 \neq 0$ imply that there exists a complex number c such that $\langle (x-c)\mathbf{u}, Q_3\rangle = 0$ and therefore $\langle (x-c)\mathbf{u}, Q_n\rangle = 0$ for all $n \geq 3$. Indeed, for an arbitrary $c \in \mathbb{C}$, we may write

$$\begin{aligned}\langle (x-c)\mathbf{u}, Q_3\rangle &= \langle \mathbf{u}, (x-c)(P_3 + s_3P_2 + t_3P_1 - r_3Q_2)\rangle \\ &= t_3\langle \mathbf{u}, (x-c)P_1\rangle - r_3\langle \mathbf{u}, (x-c)[P_2 + (s_2 - r_2)P_1 + t_2 - r_2(s_1 - r_1)]\rangle \\ &= [t_3 - r_3(s_2 - r_2)]\gamma_1 - r_3[t_2 - r_2(s_1 - r_1)](\beta_0 - c).\end{aligned}$$

Then there exists c such that $\langle (x-c)\mathbf{u}, Q_3\rangle = 0$, more precisely

$$c := \beta_0 - \frac{\gamma_1}{r_3}\frac{t_3 - r_3(s_2 - r_2)}{t_2 - r_2(s_1 - r_1)}. \tag{2.1}$$

For this choice of c and taking into account the 2–3 type relation (1.1) we have

$$\langle (x-c)\mathbf{u}, Q_n\rangle = -r_n\langle (x-c)\mathbf{u}, Q_{n-1}\rangle, \quad n \geq 4.$$

Thus $\langle (x-c)\mathbf{u}, Q_n\rangle = 0$ for all $n \geq 3$, as we wish to prove. Moreover, using standard results [13], we obtain the relation between the functionals $(x-c)\mathbf{u}$ and $\mathbf{v}$:

$$(x-c)\mathbf{u} = \sum_{j=0}^{2} \frac{\langle (x-c)\mathbf{u}, Q_j\rangle}{\langle \mathbf{v}, Q_j^2\rangle} Q_j\,\mathbf{v}, \tag{2.2}$$

being

$$\begin{aligned}&\langle (x-c)\mathbf{u}, Q_0\rangle = \beta_0 - c,\\ &\langle (x-c)\mathbf{u}, Q_1\rangle = \gamma_1 + (s_1 - r_1)(\beta_0 - c),\\ &\langle (x-c)\mathbf{u}, Q_2\rangle = (s_2 - r_2)\gamma_1 + (\beta_0 - c)[t_2 - r_2(s_1 - r_1)] = \frac{\gamma_1 t_3}{r_3}.\end{aligned} \tag{2.3}$$

Therefore if $t_2 \neq r_2(s_1 - r_1)$ and $r_3t_3 \neq 0$, we see that the relation between the regular functionals $\mathbf{u}$ and $\mathbf{v}$ is $(x-c)\mathbf{u} = h_2(x)\mathbf{v}$, where h_2 is a polynomial of exact degree two. Moreover, if $t_2 \neq r_2(s_1 - r_1)$, $r_3 \neq 0$, and $t_3 = 0$, then $\langle (x-c)\mathbf{u}, Q_2\rangle = 0$ and so (2.2) reduces to $(x-c)\mathbf{u} = h_1\mathbf{v}$, with h_1 a polynomial of degree less than or equal to one, so we have a degenerate case. In the next theorem we deduce all the possible degenerate cases from some appropriate initials conditions involving only the seven parameters $r_1, r_2, r_3, s_1, s_2, t_2$, and t_3.

Theorem 2.1. *Let $(P_n)_n$ and $(Q_n)_n$ be two MOPSs with respect to the regular functionals $\mathbf{u}$ and $\mathbf{v}$, respectively, normalized by $\langle \mathbf{u}, 1\rangle = 1 = \langle \mathbf{v}, 1\rangle$. Assume that there exist sequences of complex numbers $(r_n)_n$, $(s_n)_n$, and $(t_n)_n$ such that the 2–3 type relation (1.1) holds, with $r_0 = s_0 = t_0 = t_1 = 0$. We have*

(i) *If $t_2 = r_2(s_1 - r_1)$ and $s_1 = r_1$, then $t_n = 0,\ n \geq 2$ and $s_n = r_n,\ n \geq 1$. Thus (1.1) reduces to the trivial 1 − 1 type relation $Q_n = P_n, n \geq 0$.*

(ii) *If $t_2 = r_2(s_1 - r_1)$ and $s_1 \neq r_1$, then $t_n = r_n(s_{n-1} - r_{n-1}),\ n \geq 2$ and $s_n \neq r_n,\ n \geq 1$. In this case (1.1) reduces to a 1–2 type relation:*

$$Q_n = P_n + a_nP_{n-1}, \quad n \geq 0; \qquad a_n := s_n - r_n \neq 0, \quad n \geq 1.$$

(iii) *If $t_2 \neq r_2(s_1 - r_1)$ and $r_3 = 0$, then $t_n \neq r_n(s_{n-1} - r_{n-1}),\ n \geq 2$ and $r_n = 0,\ n \geq 3$. In this case (1.1) reduces to a 1–3 type relation:*

$$\begin{aligned}&Q_n = P_n + a_nP_{n-1} + b_nP_{n-2}, \quad n \geq 0;\\ &a_n := s_n - r_n, \quad n \geq 1; \qquad b_n := t_n - r_n(s_{n-1} - r_{n-1}) \neq 0, \quad n \geq 2.\end{aligned}$$

(iv) *If $t_2 \neq r_2(s_1 - r_1)$ and $r_3 \neq 0$, then $r_n \neq 0,\ n \geq 3$. In addition:*

(iv-a) *If $t_3 = 0$ and $t_2 = s_2(s_1 - r_1)$, then $t_n = 0 = s_n,\ n \geq 3$. In this case (1.1) reduces to a 2–1 type relation:*

$$Q_n + c_nQ_{n-1} = P_n, \quad n \geq 0; \qquad c_n := r_n - s_n \neq 0, \quad n \geq 1.$$

(iv-b) *If $t_3 = 0$ and $t_2 \neq s_2(s_1 - r_1)$, then $t_n = 0,\ n \geq 3$ and $s_n \neq 0,\ n \geq 3$. In this case (1.1) reduces to a 2–2 type relation. More precisely:*

- *If $s_1 \neq r_1$, then (1.1) becomes*
$$Q_n + c_nQ_{n-1} = P_n + d_nP_{n-1}, \quad n \geq 0;$$
$$c_1 - d_1 := r_1 - s_1$$
$$c_2 := r_2 - t_2/(s_1 - r_1), \qquad d_2 := s_2 - t_2/(s_1 - r_1)$$
$$c_n := r_n, \quad n \geq 3, \qquad d_n := s_n, \quad n \geq 3$$
so that $c_nd_n \neq 0,\ n \geq 1$.
- *If $s_1 = r_1$, then*
$$Q_1 = P_1;$$
$$Q_2 + r_2Q_1 = P_2 + s_2P_1 + t_2;$$
$$Q_n + r_nQ_{n-1} = P_n + s_nP_{n-1}, \quad n \geq 3.$$

(v) *If $t_2 \neq r_2(s_1 - r_1)$ and $r_3t_3 \neq 0$, then $r_nt_n \neq 0,\ n \geq 3$. Thus in this case (1.1) is a non-degenerate 2–3 type relation.*

Proof. From (1.1) it follows that

$$\begin{aligned} &\langle \mathbf{u}, Q_1\rangle = s_1 - r_1, \qquad \langle \mathbf{u}, Q_2\rangle = t_2 - r_2(s_1 - r_1),\\ &\langle \mathbf{u}, Q_n\rangle = -r_n \langle \mathbf{u}, Q_{n-1}\rangle, \quad n \geq 3. \end{aligned} \tag{2.4}$$

(i) If $t_2 = r_2(s_1 - r_1)$ and $s_1 = r_1$, (2.4) implies $\langle \mathbf{u}, Q_n\rangle = 0,\ n \geq 1$, so $\mathbf{u} = \mathbf{v}$. Thus $P_n = Q_n$ for all $n \geq 0$ and the relation (1.1) derives $s_n = r_n,\ n \geq 1$, and $t_n = 0,\ n \geq 2$.

(ii) If $t_2 = r_2(s_1 - r_1)$ and $s_1 \neq r_1$, then from (2.4) we have $\langle \mathbf{u}, Q_1\rangle \neq 0$ and $\langle \mathbf{u}, Q_n\rangle = 0,\ n \geq 2$. Hence, the relation between the two functionals is $\mathbf{u} = h(x)\mathbf{v}$ where h is a polynomial of degree one, and so $Q_n(x) = P_n(x) + a_nP_{n-1}(x)$ for all $n \geq 0$, with $a_n \neq 0,\ n \geq 1$. Then the relation (1.1) yields $s_n = a_n + r_n,\ n \geq 1$, and $t_n = r_na_{n-1} = r_n(s_{n-1} - r_{n-1}),\ n \geq 2$. Observe that we obtain a degenerate case, namely a 1–2 type relation.

(iii) If $t_2 \neq r_2(s_1 - r_1)$ and $r_3 = 0$, from (2.4) we deduce $\langle \mathbf{u}, Q_2\rangle \neq 0$ and $\langle \mathbf{u}, Q_n\rangle = 0,\ n \geq 3$, so there exists a polynomial h of degree two such that $\mathbf{u} = h(x)\mathbf{v}$. Thus, $Q_n(x) = P_n(x) + a_nP_{n-1}(x) + b_nP_{n-2}(x)$ for all $n \geq 0$, with $b_n \neq 0,\ n \geq 2$. Again, the relation (1.1) leads to $s_n = a_n + r_n,\ n \geq 1,\ t_n = b_n + r_na_{n-1},\ n \geq 2$, and $r_nb_{n-1} = 0,\ n \geq 3$, so $r_n = 0,\ n \geq 3$, and $t_n \neq 0,\ n \geq 3$. Then we have another degenerate case, namely a 1–3 type relation.

(iv) If $t_2 \neq r_2(s_1 - r_1)$ and $r_3 \neq 0$, from (2.4) we see that $\langle \mathbf{u}, Q_2\rangle \neq 0,\ \langle \mathbf{u}, Q_3\rangle \neq 0$, and for each $n \geq 4$ we have $\langle \mathbf{u}, Q_n\rangle = 0$ if $r_n = 0$. Assuming that there exists $n \geq 4$ such that $r_n = 0$, let $n_0 := \min\{n \in \mathbb{N} \mid n \geq 4, r_n = 0\}$. Then $\langle \mathbf{u}, Q_n\rangle = 0,\ n \geq n_0$ and $\langle \mathbf{u}, Q_n\rangle \neq 0,\ 2 \leq n \leq n_0 - 1$. Hence, $\mathbf{u} = h(x)\mathbf{v}$, with h a polynomial of degree $n_0 - 1$, and so

$$Q_n(x) = P_n(x) + \sum_{j=1}^{n_0-1} a_n^{(j)} P_{n-j}(x),$$

with $a_n^{(n_0-1)} \neq 0,\ n \geq n_0 - 1$. Taking into account (1.1) we easily see that this is not possible, so $r_n \neq 0,\ n \geq 3$. Moreover:

(iv-a) If $t_3 = 0$ and $t_2 = s_2(s_1 - r_1)$, then (1.1) implies $\langle \mathbf{v}, P_n\rangle = 0,\ n \geq 2$ and $\langle \mathbf{v}, P_1\rangle \neq 0$, so $\mathbf{v} = h(x)\mathbf{u}$ with h a polynomial of degree one. Then working in the same way as in (ii) we get $t_n = 0 = s_n,\ n \geq 3$. Note that, in this case (iv-a) we have another degenerate case, namely a 2–1 type relation.

(iv-b) If $t_3 = 0$ and $t_2 \neq s_2(s_1 - r_1)$, then by (2.3) and (2.4) we can obtain

$$\langle (x - c)\mathbf{u}, Q_1\rangle = \gamma_1 \frac{t_2 - s_2(s_1 - r_1)}{t_2 - r_2(s_1 - r_1)} \neq 0, \qquad \langle (x - c)\mathbf{u}, Q_n\rangle = 0, \quad n \geq 2,$$

so there exists a polynomial h of degree one such that $(x - c)\mathbf{u} = h(x)\mathbf{v}$. Applying the auxiliary functional $(x - c)^{n-2}\mathbf{u}$ to the main relation (1.1) we obtain for $n \geq 3$

$$\begin{aligned} t_n\langle \mathbf{u}, P_{n-2}^2\rangle &= \langle (x - c)^{n-2}\mathbf{u}, P_n + s_nP_{n-1} + t_nP_{n-2}\rangle\\ &= \langle (x - c)^{n-2}\mathbf{u}, Q_n + r_nQ_{n-1}\rangle\\ &= \langle \mathbf{v}, (x - c)^{n-3}\, h(x)\, (Q_n + r_nQ_{n-1})\rangle = 0. \end{aligned}$$

Then the condition $t_n = 0,\ n \geq 3$, holds and (1.1) becomes

$$Q_n(x) + r_nQ_{n-1}(x) = P_n(x) + s_nP_{n-1}(x), \quad n \geq 3.$$

On the other hand, to analyze the parameters s_n, we see that for $n \geq 3$

$$\begin{aligned} s_n\langle \mathbf{u}, P_{n-1}^2\rangle &= \langle (x - c)^{n-1}\mathbf{u}, P_n + s_nP_{n-1}\rangle = \langle (x - c)^{n-1}\mathbf{u}, Q_n + r_nQ_{n-1}\rangle\\ &= \langle \mathbf{v}, (x - c)^{n-2}\, h(x)\, (Q_n + r_nQ_{n-1})\rangle = kr_n\langle \mathbf{v}, Q_{n-1}^2\rangle, \end{aligned}$$

where k is the leading coefficient of the polynomial h, and so we obtain $s_n \neq 0,\ n \geq 3$. Thus another degenerate case appears, namely a 2–2 type relation.

(v) If $t_2 \neq r_2(s_1-r_1)$ and $r_3t_3 \neq 0$, then as we have seen just before the statement of this theorem, there exists a constant c such that $(x-c)\mathbf{u} = h_2(x)\mathbf{v}$, with h_2 a polynomial of degree two and so, by (1.1), we obtain

$$\begin{aligned} t_n\langle \mathbf{u}, P_{n-2}^2\rangle &= \langle (x-c)\mathbf{u}, (P_n + s_nP_{n-1} + t_nP_{n-2})Q_{n-3}\rangle \\ &= \langle \mathbf{v}, h_2(x)(Q_n + r_nQ_{n-1})Q_{n-3}\rangle = k_2 r_n \langle \mathbf{v}, Q_{n-1}^2\rangle, \quad n \geq 3, \end{aligned}$$

where k_2 is the leading coefficient of the polynomial h_2. Now, it is enough to apply (iv) to obtain $r_n \neq 0$, $n \geq 3$, and so also $t_n \neq 0$, $n \geq 3$. Thus the proof is concluded. □

Remark 2.1. Observe that (v) is the unique case where the relation (1.1) between the two families $(P_n)_n$ and $(Q_n)_n$ is a non-degenerate 2–3 type relation. All the degenerate cases, except the case (iv-b) with $r_1 = s_1$, have already been considered in the previous works [3,2,4,11]. The case (iv-b) with $r_1 = s_1$ can be studied and solved in a similar way as in [3]. Then from now on we will concentrate on the analysis of the non-degenerate case.

The non-degenerate 2–3 type relations have already been considered in [14, Section 5]. However, there some additional hypothesis about the parameters involved in the relation (1.1) were imposed, namely

$$t_n \neq r_n(s_{n-1} - r_{n-1}), \quad n \geq 3.$$

In the following proposition we prove that to impose these conditions, together with the conditions $r_3t_3 \neq 0$ and $t_2 \neq r_2(s_1 - r_1)$, is equivalent to assume that the functional $(x-c)\mathbf{u}$ is regular.

Proposition 2.2. *Let $(P_n)_n$ and $(Q_n)_n$ be two MOPSs with respect to the regular functionals $\mathbf{u}$ and $\mathbf{v}$, respectively, normalized by $\langle \mathbf{u}, 1\rangle = 1 = \langle \mathbf{v}, 1\rangle$. Assume that there exist sequences of complex numbers $(r_n)_n$, $(s_n)_n$, and $(t_n)_n$ such that the 2–3 type relation (1.1) holds, with $r_0 = s_0 = t_0 = t_1 = 0$ and the initial conditions $t_2 \neq r_2(s_1 - r_1)$ and $r_3t_3 \neq 0$. Then the following statements are equivalent:*

(i) *The functional $(x-c)\mathbf{u}$ is regular.*
(ii) *$t_n \neq r_n(s_{n-1} - r_{n-1})$ for all $n \geq 2$.*

Proof. Multiplying both sides of (1.1) by P_{n-1} and applying $\mathbf{u}$, we find

$$\langle \mathbf{u}, Q_nP_{n-1}\rangle = (s_n - r_n)\langle \mathbf{u}, P_{n-1}^2\rangle, \quad n \geq 1. \tag{2.5}$$

Moreover, multiplying both sides of (1.1) by P_{n-2}, then applying $\mathbf{u}$ and taking into account (2.5), we get

$$\langle \mathbf{u}, Q_nP_{n-2}\rangle = [t_n - r_n(s_{n-1} - r_{n-1})]\langle \mathbf{u}, P_{n-1}^2\rangle, \quad n \geq 2. \tag{2.6}$$

Thus

$$t_n \neq r_n(s_{n-1} - r_{n-1}) \Longleftrightarrow \langle \mathbf{u}, Q_nP_{n-2}\rangle \neq 0, \quad n \geq 2.$$

On the other hand, it is well known that $(x-c)\mathbf{u}$ is a regular functional if and only if $P_n(c) \neq 0$ for all $n \geq 0$. Therefore we only need to show that

$$\langle \mathbf{u}, Q_{n+2}P_n\rangle \neq 0 \Longleftrightarrow P_n(c) \neq 0, \quad n \geq 0.$$

Indeed, since $P_n(x) = \sum_{j=0}^{n} a_j^n (x-c)^j$ with $a_n^n = 1$ and $a_0^n = P_n(c)$, and (as we have seen just before the statement of Theorem 2.1) the relation between the regular functionals $\mathbf{u}$ and $\mathbf{v}$ is $(x-c)\mathbf{u} = h_2(x)\mathbf{v}$, where h_2 is a polynomial of degree two, we obtain for all $n \geq 1$

$$\begin{aligned} \langle \mathbf{u}, Q_{n+2}P_n\rangle &= \left\langle (x-c)\mathbf{u}, Q_{n+2}\left[(x-c)^{n-1} + \sum_{j=1}^{n-1} a_j^n (x-c)^{j-1}\right]\right\rangle + P_n(c)\langle \mathbf{u}, Q_{n+2}\rangle \\ &= \left\langle h_2(x)\mathbf{v}, Q_{n+2}\left[(x-c)^{n-1} + \sum_{j=1}^{n-1} a_j^n (x-c)^{j-1}\right]\right\rangle + P_n(c)\langle \mathbf{u}, Q_{n+2}\rangle \\ &= P_n(c)\langle \mathbf{u}, Q_{n+2}\rangle, \quad n \geq 0. \end{aligned}$$

To conclude the proof it suffices to observe that from (v) in Theorem 2.1 we have $r_n \neq 0$, $n \geq 3$, and therefore taking into account (2.4) we obtain $\langle \mathbf{u}, Q_n\rangle \neq 0$ for all $n \geq 2$. □

Next, we are going to present an example showing that a non-degenerate 2–3 type relation (1.1) may occur even when the functional $(x-c)\mathbf{u}$ is not regular. This example shows that Theorem 5.1 in [14] does not give a full description of the non-degenerate 2–3 type relations. The full characterization of these relations (including the determination of the orthogonality conditions) is the main purpose of our study in the next section.

Example. In the sequel we denote by $\mathbf{w_2}$, $\mathbf{w_3}$, and $\mathbf{w_4}$ the regular functionals associated with the Chebyshev polynomials of the second, third, and fourth kind, which are represented (up to suitable normalizing constants) by the weight functions $(1-x)^{1/2}(1+x)^{1/2}$, $(1-x)^{-1/2}(1+x)^{1/2}$, and $(1-x)^{1/2}(1+x)^{-1/2}$, respectively. Consider also the regular functional

$$\mathbf{u} = -\frac{1}{3}\, x\, \mathbf{w_3} + \delta_1,$$

where δ_ξ means the Dirac functional at a point $\xi \in \mathbb{C}$, so that $\langle \delta_\xi, p\rangle := p(\xi)$ for every polynomial p. The regularity of $\mathbf{u}$ has been stated in an example presented in [2, pp. 181–182].

Denote by $(P_n)_n$, $(Q_n)_n$ and $(R_n)_n$ the MOPSs with respect to $\mathbf{u}$, $\mathbf{w_4}$, and $\mathbf{w_2}$ (respectively). These functionals satisfy the following relations

$$\mathbf{w_2} = (1+x)\mathbf{w_4}, \qquad 3(x-1)\mathbf{u} = x\mathbf{w_2} = x(1+x)\mathbf{w_4}.$$

As a consequence of these relations and the results in [2], we have that the relation

$$Q_n(x) = R_n(x) + \lambda_n R_{n-1}(x), \quad n \geq 0$$

with $\lambda_n = \frac{1}{2},\ n \geq 1$, holds as well as the following 2–2 type relation

$$P_n(x) + a_n P_{n-1}(x) = R_n(x) + b_n R_{n-1}(x), \quad n \geq 1,$$

where

$$a_{2n} = b_{2n} = -\frac{4n+1}{2(4n-1)}, \quad n \geq 1,$$

$$a_{2n+1} = \frac{4n-1}{2(4n+1)}, \qquad b_{2n+1} = -\frac{4n+3}{2(4n+1)}, \quad n \geq 0.$$

Moreover, since $x\mathbf{w_2}$ is not a regular functional, then $(x-1)\mathbf{u}$ is not a regular functional. Nevertheless, a non-degenerate 2–3 type relation between $(P_n)_n$ and $(Q_n)_n$ holds. Indeed, noticing that $b_n \neq \lambda_n,\ n \geq 1$, we may define parameters s_n, t_n, and r_n, as

$$s_n = a_n + \lambda_{n-1}\,\frac{b_n - \lambda_n}{b_{n-1} - \lambda_{n-1}}, \quad n \geq 2,$$

$$t_n = a_{n-1}\,\lambda_{n-1}\,\frac{b_n - \lambda_n}{b_{n-1} - \lambda_{n-1}} \neq 0, \quad n \geq 2,$$

$$r_n = b_{n-1}\,\frac{b_n - \lambda_n}{b_{n-1} - \lambda_{n-1}} \neq 0, \quad n \geq 2$$

and then, by straightforward computations, we check that formula (1.1) is satisfied for all $n \geq 2$. For $n = 1$ we have $P_1(x) + a_1 = R_1(x) + b_1 = Q_1(x) - \lambda_1 + b_1$, hence $s_1 - r_1 = a_1 - b_1 + \lambda_1 = \frac{3}{2} \neq 0$. Moreover, $t_2 = -\frac{1}{6} \neq -\frac{3}{2} = r_2(s_1 - r_1)$. Finally, we observe that $t_{2n+1} = r_{2n+1}(s_{2n} - r_{2n})$ for all $n \geq 1$, hence the conditions $t_n \neq r_n(s_{n-1} - r_{n-1})$ for all $n \geq 3$ are not satisfied.

3. Orthogonality characterizations

From now on, $(P_n)_n$ denotes a MOPS with respect to a regular functional $\mathbf{u}$, and $(\beta_n)_n$ and $(\gamma_n)_n$ the corresponding sequences of recurrence coefficients, so that

$$\begin{aligned} &P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geq 0, \\ &P_0(x) = 1, \qquad P_{-1}(x) = 0, \end{aligned} \tag{3.1}$$

with $\gamma_n \neq 0$ for all $n \geq 1$. In this section we give two characterizations of the orthogonality of a sequence $(Q_n)_n$ of monic polynomials defined by a non-degenerate type relation (1.1). We already know from Theorem 2.1 that in order to have a non-degenerate 2–3 type relation with $(P_n)_n$ and $(Q_n)_n$ MOPSs, the conditions

$$r_3 t_3 \neq 0, \qquad t_2 \neq r_2(s_1 - r_1)$$

must hold, and these conditions imply $r_n t_n \neq 0$ for all $n \geq 3$.

The first characterization of the orthogonality of the sequence $(Q_n)_n$ is the following.

Theorem 3.1. *Let $(P_n)_n$ be a MOPS and $(\beta_n)_n$ and $(\gamma_n)_n$ the corresponding sequences of recurrence coefficients. We define recursively a sequence $(Q_n)_n$ of monic polynomials by formula (1.1), i.e.,*

$$Q_n(x) + r_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x), \quad n \geq 0,$$

where $(r_n)_n$, $(s_n)_n$, and $(t_n)_n$ are sequences of complex numbers fulfilling the conventions $r_0 = s_0 = t_0 = t_1 = 0$, and such that

$$t_2 \neq r_2(s_1 - r_1), \qquad r_n t_n \neq 0, \quad n \geq 3.$$

Then $(Q_n)_n$ is a MOPS with recurrence coefficients $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$, where

$$\widetilde{\beta}_n := \beta_n + s_n - s_{n+1} - r_n + r_{n+1}, \quad n \geq 0 \tag{3.2}$$

$$\widetilde{\gamma}_n := \gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1}) - r_n(r_{n+1} - r_n - \widetilde{\beta}_n + \widetilde{\beta}_{n-1}), \quad n \geq 1 \tag{3.3}$$

if and only if $\tilde{\gamma}_1\tilde{\gamma}_2 \neq 0$ and the following equations hold:

$$b_2 - d_2 = a_2(s_1 - r_1), \tag{3.4}$$

$$b_3 - d_3 = a_3(s_2 - r_2), \tag{3.5}$$

$$c_3 - b_3(s_1 - r_1) = a_3[t_2 - s_2(s_1 - r_1)], \tag{3.6}$$

$$b_n = a_n s_{n-1}, \quad n \geq 4, \tag{3.7}$$

$$c_n = a_n t_{n-1}, \quad n \geq 4, \tag{3.8}$$

$$d_n = a_n r_{n-1}, \quad n \geq 4, \tag{3.9}$$

where

$$a_n := \gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1}), \quad n \geq 1, \tag{3.10}$$

$$b_n := s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2}), \quad n \geq 2, \tag{3.11}$$

$$c_n := t_n\gamma_{n-2}, \quad n \geq 3, \tag{3.12}$$

$$d_n := r_n\widetilde{\gamma}_{n-1}, \quad n \geq 2. \tag{3.13}$$

Proof. From the definition of Q_n we get

$$Q_{n+1}(x) = P_{n+1}(x) + s_{n+1}P_n(x) + t_{n+1}P_{n-1}(x) - r_{n+1}Q_n(x), \quad n \geq 0. \tag{3.14}$$

Inserting formula (3.1) in (3.14), applying (1.1) to $xP_n(x)$, and then substituting $xP_{n-1}(x)$ and $xP_{n-2}(x)$ using again (3.1), we get

$$\begin{aligned} Q_{n+1}(x) &= xQ_n(x) + (s_{n+1} - \beta_n - s_n)P_n(x) - r_{n+1}Q_n(x) + r_n xQ_{n-1}(x) + (t_{n+1} - \gamma_n - s_n\beta_{n-1} - t_n)P_{n-1}(x) \\ &\quad - (s_n\gamma_{n-1} + t_n\beta_{n-2})P_{n-2}(x) - t_n\gamma_{n-2}P_{n-3}(x), \quad n \geq 0, \end{aligned}$$

with the usual convention that polynomials with negative index are zero. Now, Eq. (1.1) applied to $P_n(x)$ and the definition (3.2) of $\widetilde{\beta}_n$ yield

$$\begin{aligned} Q_{n+1}(x) &= (x - \tilde{\beta}_n)Q_n(x) + r_n(r_{n+1} - r_n - \tilde{\beta}_n)Q_{n-1}(x) + [t_{n+1} - \gamma_n - t_n - s_n(s_{n+1} \\ &\quad - s_n - \beta_n + \beta_{n-1})]P_{n-1}(x) - [s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2})]P_{n-2}(x) - t_n\gamma_{n-2}P_{n-3}(x) \\ &\quad - r_n[Q_n(x) - xQ_{n-1}(x)], \quad n \geq 0. \end{aligned}$$

So $(Q_n)_n$ is a MOPS if and only if $\tilde{\gamma}_n \neq 0$ for all $n \geq 1$ and

$$\begin{aligned} & r_n(r_{n+1} - r_n - \tilde{\beta}_n)Q_{n-1}(x) + [t_{n+1} - \gamma_n - t_n - s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1})]P_{n-1}(x) \\ &\quad - [s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2})]P_{n-2}(x) - t_n\gamma_{n-2}P_{n-3}(x) \\ &\quad - r_n[Q_n(x) - xQ_{n-1}(x)] = -\tilde{\gamma}_nQ_{n-1}(x), \quad n \geq 0. \end{aligned} \tag{3.15}$$

Moreover, $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$ are the corresponding sequences of recurrence coefficients of $(Q_n)_n$.

Next, we are going to see that $(Q_n)_n$ is a MOPS with recurrence coefficients $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$ if and only if $\tilde{\gamma}_n \neq 0$ for all $n \geq 1$ and the relation

$$\begin{aligned} & \left[\tilde{\gamma}_n + r_n(r_{n+1} - r_n - \tilde{\beta}_n + \tilde{\beta}_{n-1})\right]Q_{n-1}(x) + r_n\tilde{\gamma}_{n-1}Q_{n-2}(x) \\ &\quad = [\gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1})]P_{n-1}(x) \\ &\qquad + [s_n\gamma_{n-1} + t_n(s_{n+1} - s_n - \beta_n + \beta_{n-2})]P_{n-2}(x) + t_n\gamma_{n-2}P_{n-3}(x) \end{aligned} \tag{3.16}$$

holds for every $n \geq 1$.

Suppose first that $(Q_n)_n$ is a MOPS with recurrence coefficients $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$. Then

$$\begin{aligned} & Q_{n+1}(x) = (x - \widetilde{\beta}_n)Q_n(x) - \widetilde{\gamma}_nQ_{n-1}(x), \quad n \geq 0, \\ & Q_0(x) = 1, \qquad Q_{-1}(x) = 0 \end{aligned} \tag{3.17}$$

and $\widetilde{\gamma}_n \neq 0$ for all $n \geq 1$, and so it is enough to substitute the expression for $Q_n(x) - xQ_{n-1}(x)$ obtained from this three-term recurrence relation (after replacing n by $n-1$) in formula (3.15) to obtain (3.16).

Conversely, if (3.16) is satisfied and $\tilde{\gamma}_n \neq 0$ for all $n \geq 1$, then we show that the sequence $(Q_n)_n$ satisfies the three-term recurrence relation (3.17), that is, $(Q_n)_n$ is a MOPS with recurrence coefficients $(\tilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$. Indeed, applying (3.1) in

(3.16), and by the definition of $\tilde{\beta}_n$, for $n \geq 1$ we obtain

$$\begin{aligned} r_n\left(\tilde{\beta}_{n-1}Q_{n-1}(x)+\tilde{\gamma}_{n-1}Q_{n-2}(x)\right) &= \gamma_n P_{n-1}(x)+(r_{n+1}-r_n-\tilde{\beta}_n)\,[s_nP_{n-1}(x)+t_nP_{n-2}(x)-r_nQ_{n-1}(x)] \\ &\quad + s_n\,[xP_{n-1}(x)-P_n(x)]+t_nxP_{n-2}(x)-t_{n+1}P_{n-1}(x)-\tilde{\gamma}_nQ_{n-1}(x) \\ &= \gamma_nP_{n-1}(x)+(r_{n+1}-r_n-\tilde{\beta}_n)Q_n(x)-(s_{n+1}-\beta_n)P_n(x) \\ &\quad + x[s_nP_{n-1}(x)+t_nP_{n-2}(x)]-t_{n+1}P_{n-1}(x)-\tilde{\gamma}_nQ_{n-1}(x), \end{aligned}$$

where the last equality follows from (1.1). Applying (1.1) in $s_nP_{n-1}(x)+t_nP_{n-2}(x)$ as well as the recurrence relation for $(P_n)_n$, we get

$$\begin{aligned} r_n\left(\tilde{\beta}_{n-1}Q_{n-1}(x)+\tilde{\gamma}_{n-1}Q_{n-2}(x)\right) &= r_n[xQ_{n-1}(x)-Q_n(x)]-P_{n+1}(x)+r_{n+1}Q_n(x)-s_{n+1}P_n(x) \\ &\quad - t_{n+1}P_{n-1}(x)-\tilde{\beta}_nQ_n(x)+xQ_n(x)-\tilde{\gamma}_nQ_{n-1}(x). \end{aligned}$$

Hence, by definition of Q_{n+1} we have

$$Q_{n+1}(x)-(x-\tilde{\beta}_n)Q_n(x)+\tilde{\gamma}_nQ_{n-1}(x)=-r_n[\,Q_n(x)-(x-\tilde{\beta}_{n-1})Q_{n-1}(x)+\tilde{\gamma}_{n-1}Q_{n-2}(x)\,],\quad n\geq 1.$$

Therefore, since by (1.1) $Q_1(x)=P_1(x)+s_1-r_1=x-\beta_0+s_1-r_1=x-\widetilde{\beta}_0$, we deduce recursively

$$Q_{n+1}(x)=(x-\tilde{\beta}_n)Q_n(x)-\tilde{\gamma}_nQ_{n-1}(x),\quad n\geq 0,$$

and so $(Q_n)_n$ is a MOPS with recurrence coefficients $(\widetilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$.

Now, observe that from (1.1) and (3.3), it follows that (3.16) is equivalent to

$$(d_n-r_{n-1}a_n)Q_{n-2}(x)=(b_n-s_{n-1}a_n)P_{n-2}(x)+(c_n-t_{n-1}a_n)P_{n-3}(x), \tag{3.18}$$

for $n\geq 2$, where a_n, b_n, c_n, and d_n are defined by (3.10)–(3.13).

To conclude we show that (3.18) holds with $\widetilde{\gamma}_n\neq 0$ for all $n\geq 1$ if and only if $\widetilde{\gamma}_1\widetilde{\gamma}_2\neq 0$ and formulas (3.4)–(3.9) hold.

Comparing coefficients in both sides of (3.18) for $n=2$ and $n=3$, we obtain (3.4)–(3.6), respectively. Moreover, it is easy to verify that (3.7)–(3.9) hold for $n\geq 4$. Indeed, on the first hand, since by hypothesis $t_2\neq r_2(s_1-r_1)$ and $r_n\neq 0$ for every $n\geq 3$, then by (2.4) we deduce $\langle \mathbf{u}, Q_n\rangle\neq 0$ for all $n\geq 2$. On the other hand, applying $\mathbf{u}$ to both sides of (3.18) we obtain $(d_n-r_{n-1}a_n)\langle \mathbf{u}, Q_{n-2}\rangle=0$ for $n\geq 4$. Thus $d_n-r_{n-1}a_n=0$ for every $n\geq 4$, and this proves (3.9). Therefore, taking into account (3.18) again, we immediately obtain (3.7) and (3.8).

Conversely, notice that, from (3.8) and (3.9), we have

$$\tilde{\gamma}_{n-1}=\frac{r_{n-1}}{r_n}\frac{t_n}{t_{n-1}}\gamma_{n-2},\quad n\geq 4.$$

Thus, $\tilde{\gamma}_n\neq 0$ for every $n\geq 3$, hence the conditions (3.4)–(3.9) together with $\widetilde{\gamma}_1\widetilde{\gamma}_2\neq 0$ imply that (3.18) holds and $\widetilde{\gamma}_n\neq 0$ for all $n\geq 1$. This concludes the proof. □

Remark 3.1. Using the same techniques, it can be proved a similar result exchanging the role of the sequences $(P_n)_n$ and $(Q_n)_n$. More precisely: *given two sequences of monic polynomials $(P_n)_n$ and $(Q_n)_n$ linked by a non-degenerate relation (1.1), where $(Q_n)_n$ is a MOPS with $(\widetilde{\beta}_n)_n$ and $(\tilde{\gamma}_n)_n$ the corresponding sequences of recurrence coefficients, then $(P_n)_n$ is a MOPS with recurrence coefficients $(\beta_n)_n$* and $(\gamma_n)_n$, *satisfying (3.2) and (3.3), if and only if $\gamma_1\gamma_2\neq 0$ and* Eqs. (3.4)–(3.9) *hold*.

Now, we show that the orthogonality of the sequence $(Q_n)_n$ can be also characterized by the fact that there are three sequences (depending on the parameters s_n, t_n, r_n and the recurrence coefficients) which remain constant. Note that this new characterization of the orthogonality of $(Q_n)_n$ is more interesting than the previous one, giving an implicit solution of the system of Eqs. (3.4)–(3.9).

Theorem 3.2. *Let $(P_n)_n$ be a MOPS and $(\beta_n)_n$ and $(\gamma_n)_n$ the corresponding sequences of recurrence coefficients. Let $(Q_n)_n$ be a simple set of polynomials such that the structure relation (1.1) holds, i.e.,*

$$Q_n(x)+r_nQ_{n-1}(x)=P_n(x)+s_nP_{n-1}(x)+t_nP_{n-2}(x),\quad n\geq 0,$$

where $(r_n)_n$, $(s_n)_n$ and $(t_n)_n$ are sequences of complex numbers such that

$$t_2\neq r_2(s_1-r_1),\qquad r_nt_n\neq 0,\quad n\geq 3,$$

with $r_0=s_0=t_0=t_1=0$. Let $(\widetilde{\beta}_n)_n$ and $(\widetilde{\gamma}_n)_n$ be defined by (3.2) and (3.3). Then the following two statements are equivalent:

(i) *$(Q_n)_n$ is a MOPS with $(\widetilde{\beta}_n)_n$ and $(\widetilde{\gamma}_n)_n$ the corresponding sequences of recurrence coefficients.*

(ii) *It holds $\widetilde{\gamma}_1\widetilde{\gamma}_2\neq 0$ together with the initial conditions (3.4)–(3.6), and*

$$t_4\gamma_2=a_4t_3, \tag{3.19}$$

and the following three sequences remain constant

$$A_n := \frac{s_n a_{n+1}}{t_{n+1}} - \beta_{n-1} - \beta_n + s_{n+1} = A, \quad n \geq 3 \tag{3.20}$$

$$B_n := \frac{a_n a_{n+1}}{t_{n+1}} + (s_n - \beta_{n-1})\left(\frac{s_n a_{n+1}}{t_{n+1}} - \beta_n - s_n + s_{n+1}\right) + t_n - a_n - \gamma_{n-1} = B, \quad n \geq 3, \tag{3.21}$$

$$C_n := \widetilde{\beta}_n - r_{n+1} - \frac{\widetilde{\gamma}_n}{r_n} = C \quad n \geq 3, \tag{3.22}$$

where $(a_n)_n$ *is defined by* (3.10).

Proof. To prove this theorem, we introduce the auxiliary coefficients β_n^* and γ_n^*, namely

$$\beta_n^* = \beta_n + s_n - s_{n+1}, \quad n \geq 0, \tag{3.23}$$

$$\gamma_n^* = \gamma_n + t_n - t_{n+1} + s_n(s_{n+1} - s_n - \beta_n + \beta_{n-1}), \quad n \geq 1, \tag{3.24}$$

so we have $\gamma_n^* = a_n$ for all $n \geq 1$. Now, observe that the conditions (3.7)–(3.9) in Theorem 3.1 may be rewritten as

$$s_{n-1}\gamma_n^* = s_n\gamma_{n-1} + t_n(\beta_{n-2} - \beta_n^*), \quad n \geq 4; \tag{3.25}$$

$$t_{n-1}\gamma_n^* = t_n\gamma_{n-2}, \quad n \geq 4; \tag{3.26}$$

$$r_{n-1}\gamma_n^* = r_n\widetilde{\gamma}_{n-1}, \quad n \geq 4; \tag{3.27}$$

and therefore $(Q_n)_n$ is a MOPS if and only if the condition $\widetilde{\gamma}_1\widetilde{\gamma}_2 \neq 0$, the initial conditions (3.4)–(3.6) and the above Eqs. (3.25)–(3.27) hold.

First, since

$$\gamma_n^* = a_n = \widetilde{\gamma}_n + r_n(r_{n+1} - r_n - \widetilde{\beta}_n + \widetilde{\beta}_{n-1}), \quad n \geq 1,$$

we have

$$r_n\widetilde{\gamma}_{n-1} = r_{n-1}a_n = r_{n-1}[\widetilde{\gamma}_n + r_n(r_{n+1} - r_n - \widetilde{\beta}_n + \widetilde{\beta}_{n-1})], \quad n \geq 4,$$

hence, dividing the left and the right hand sides by $r_n r_{n-1}$, we immediately deduce that (3.27) holds if and only if there exists a constant C (independent of n) such that (3.22) holds.

To conclude the proof we need to show that Eqs. (3.25) and (3.26) are equivalent to (3.19)–(3.21). To do this, we notice that relations (3.25) and (3.26) are formally the same as (2.3) and (2.4) for $n \geq 4$ appearing in [4], after replacing $\widetilde{\beta}_n$ and $\widetilde{\gamma}_n$ by β_n^* and γ_n^*, respectively.

- We first prove that (3.25) and (3.26) $\Longrightarrow$ (3.19)–(3.21):

Observe that (3.26) for $n = 4$ is the condition (3.19). Moreover, from (3.25) and (3.26), by making exactly the same algebraic manipulations that have been made in the proof of (i) ⇒ (ii) in [4, Theorem 2.2] we deduce that the analogue of relation (2.10) and (2.11) for $n \geq 4$ in [4] holds. Thus, by straightforward computations, using the definition of β_n^* and γ_n^*, we obtain (3.20) and (3.21).

- Next we show that (3.19)–(3.21) $\Longrightarrow$ (3.25) and (3.26):

As before, by making the same algebraic manipulations that have been made in the proof of (ii) ⇒ (i) in [4, Theorem 2.2], we deduce that an equation analogous to (2.12) in [4] holds, i.e.,

$$\frac{\gamma_{n+1}^*}{t_{n+1}}\left(\gamma_n - \frac{t_{n+1}}{t_{n+2}}\gamma_{n+2}^*\right) = \gamma_{n-1} - \frac{t_n}{t_{n+1}}\gamma_{n+1}^*, \quad n \geq 3.$$

This relation ensures that $\gamma_{n+1}^* \neq 0$ for all $n \geq 3$, and so, taking into account hypothesis (3.19), i.e., $\gamma_{n-1} - \frac{t_n}{t_{n+1}}\gamma_{n+1}^* = 0$ for $n = 3$, using induction we may conclude that

$$\frac{\gamma_{n-1}}{t_n} = \frac{\gamma_{n+1}^*}{t_{n+1}}, \quad n \geq 3. \tag{3.28}$$

This proves (3.26). To prove (3.25), notice first that by (3.20) we have

$$\frac{s_{n-1}\gamma_n^*}{t_n} - \beta_{n-2}^* + s_{n-2} = \frac{s_n\gamma_{n+1}^*}{t_{n+1}} - \beta_n^* + s_{n-1}, \quad n \geq 4.$$

As a consequence, taking into account (3.28),

$$\frac{s_{n-1}\gamma_n^*}{t_n} = \frac{s_n\gamma_{n-1}}{t_n} + \beta_{n-2}^* - s_{n-2} + s_{n-1} - \beta_n^* = \frac{s_n\gamma_{n-1}}{t_n} + \beta_{n-2} - \beta_n^*, \quad n \geq 4,$$

and (3.25) holds. Thus the proof is finished. □

Next, to conclude this section, we will see that the constants A, B, and C appearing in the above Theorem 3.2 are, respectively, the coefficients a, b, and c of the polynomials which relate the two regular linear functionals, that is

$$\lambda(x-c)\mathbf{u} = (x^2+ax+b)\mathbf{v}. \tag{3.29}$$

First of all, we observe that the values of a, b, c, and λ may be computed from the following formulas:

$$a = -\widetilde{\beta}_0 - \widetilde{\beta}_1 + \frac{\widetilde{\gamma}_2}{t_3}\frac{r_3t_2 + (t_3 - r_3s_2)(s_1 - r_1)}{t_2 - r_2(s_1 - r_1)},$$
$$b = \widetilde{\beta}_0\widetilde{\beta}_1 - \widetilde{\gamma}_1 - \frac{\widetilde{\beta}_0\widetilde{\gamma}_2}{t_3}\frac{r_3t_2 + (t_3 - r_3s_2)(s_1 - r_1)}{t_2 - r_2(s_1 - r_1)} + \frac{\widetilde{\gamma}_1\widetilde{\gamma}_2}{t_3}\frac{t_3 - r_3(s_2 - r_2)}{t_2 - r_2(s_1 - r_1)},$$
$$c = \beta_0 - \frac{\gamma_1}{r_3}\frac{t_3 - r_3(s_2 - r_2)}{t_2 - r_2(s_1 - r_1)}, \qquad \lambda = \frac{r_3}{t_3}\frac{\widetilde{\gamma}_1\widetilde{\gamma}_2}{\gamma_1}.$$

Indeed, making both sides of (3.29) acting on the polynomials Q_0, Q_1, and Q_2, and taking into account Eqs. (2.3), we obtain the relations

$$\lambda(\beta_0 - c) = \widetilde{\beta}_0^2 + \widetilde{\beta}_0 a + b + \widetilde{\gamma}_1, \tag{3.30}$$
$$\lambda[\gamma_1 + (\beta_0 - c)(s_1 - r_1)] = (\widetilde{\beta}_0 + \widetilde{\beta}_1 + a)\widetilde{\gamma}_1, \tag{3.31}$$
$$\lambda\{(s_2 - r_2)\gamma_1 + (\beta_0 - c)[t_2 - r_2(s_1 - r_1)]\} = \widetilde{\gamma}_1\widetilde{\gamma}_2. \tag{3.32}$$

The expression for c has been already determined in Section 2; see (2.1). Hence we deduce successively λ, a, and b from Eqs. (3.32), (3.31) and (3.30), respectively. Note that these expressions can be also achieved by applying the proof of Theorem 1.1 in [14] (which is constructive) to the particular 2–3 type relation considered here.

Theorem 3.3. *Let $(P_n)_n$ and $(Q_n)_n$ be two MOPSs with respect to the regular functionals $\mathbf{u}$ and $\mathbf{v}$ respectively, normalized by $\langle \mathbf{u}, 1\rangle = 1 = \langle \mathbf{v}, 1\rangle$. Let $(\beta_n, \gamma_{n+1})_{n\geq 0}$ and $(\widetilde{\beta}_n, \widetilde{\gamma}_{n+1})_{n\geq 0}$ be the corresponding sets of recurrence coefficients, respectively. Suppose that $(P_n)_n$ and $(Q_n)_n$ are linked by a non-degenerate 2–3 type relation such as (1.1) and therefore the relation between the moment linear functionals is*

$$\lambda(x-c)\mathbf{u} = (x^2+ax+b)\mathbf{v}.$$

Then the constants A, B, and C appearing in Theorem 3.2 coincide, respectively, with the constants a, b, and c.

Proof. (i) $C = c$.

From (3.22), $C = C_3$ and using the definition of a_3 in terms of $\widetilde{\gamma}_3$ (see (3.10) and (3.3)), we have

$$C = C_3 = \widetilde{\beta}_3 - r_4 - \frac{\widetilde{\gamma}_3}{r_3} = \widetilde{\beta}_2 - r_3 - \frac{a_3}{r_3}.$$

Therefore we want to prove that

$$C = \widetilde{\beta}_2 - r_3 - \frac{a_3}{r_3} = \beta_0 - \frac{\gamma_1}{r_3}\frac{t_3 - r_3(s_2 - r_2)}{t_2 - r_2(s_1 - r_1)} = c.$$

Indeed, by Theorem 3.1, the initial conditions (3.4)–(3.6) hold. Using these second and third initial conditions we obtain

$$a_3[t_2 - s_2(s_1 - r_1)] = t_3\gamma_1 - (s_1 - r_1)[a_3(s_2 - r_2) + r_3\tilde{\gamma}_2]$$

that is

$$t_3\gamma_1 = a_3[t_2 - r_2(s_1 - r_1)] + r_3(s_1 - r_1)\tilde{\gamma}_2. \tag{3.33}$$

The first initial condition (3.4) yields

$$a_2(s_1 - r_1) = s_2\gamma_1 + t_2(s_3 - s_2 - \beta_2 + \beta_0) - r_2\tilde{\gamma}_1.$$

Handling adequately this expression we can obtain, see ((3.10) and (3.3))

$$(s_2 - r_2)\gamma_1 = (s_1 - r_1)\tilde{\gamma}_2 - (r_3 - \widetilde{\beta}_2 + \beta_0)[t_2 - r_2(s_1 - r_1)]. \tag{3.34}$$

Multiplying both sides of (3.34) by r_3, and then subtracting the resulting equation from (3.33), we obtain

$$\gamma_1\frac{t_3 - r_3(s_2 - r_2)}{t_2 - r_2(s_1 - r_1)} = a_3 + r_3(r_3 - \widetilde{\beta}_2 + \beta_0), \tag{3.35}$$

and then $c = C$.

(ii) $A = a$.

First notice that from (3.20) and (3.19), and using the definition of b_3 (see (3.11)) we have

$$A = A_3 = \frac{s_3\gamma_2}{t_3} - \beta_2 - \beta_3 + s_4 = \frac{b_3}{t_3} - \beta_1 - \beta_2 + s_3.$$

We want to prove

$$A = \frac{b_3}{t_3} - \beta_1 - \beta_2 + s_3 = -\widetilde{\beta}_0 - \widetilde{\beta}_1 + \frac{\widetilde{\gamma}_2}{t_3}\frac{r_3t_2 - (t_3 - r_3s_2)(r_1 - s_1)}{t_2 - r_2(s_1 - r_1)} = a.$$

Taking into account the expression of $\widetilde{\beta}_n$ (see (3.2)) and formula (3.35) we get

$$A + \widetilde{\beta}_0 + \widetilde{\beta}_1 = \frac{b_3}{t_3} + r_3 - \widetilde{\beta}_2 + \beta_0 = \frac{b_3}{t_3} + \frac{\gamma_1}{r_3}\frac{t_3 - r_3(s_2 - r_2)}{t_2 - r_2(s_1 - r_1)} - \frac{a_3}{r_3}. \tag{3.36}$$

Now, applying successively the second initial condition (3.5), the definition of d_3 (see (3.13)), and (3.33) we deduce

$$\begin{aligned} A + \widetilde{\beta}_0 + \widetilde{\beta}_1 &= \frac{a_3(s_2 - r_2) + r_3\widetilde{\gamma}_2}{t_3} - \frac{a_3}{r_3} + \frac{\gamma_1}{r_3}\frac{t_3 - r_3(s_2 - r_2)}{t_2 - r_2(s_1 - r_1)} \\ &= \frac{r_3}{t_3}\widetilde{\gamma}_2 - \frac{a_3}{r_3t_3}(t_3 - r_3(s_2 - r_2)) + \frac{\gamma_1}{r_3}\frac{t_3 - r_3(s_2 - r_2)}{t_2 - r_2(s_1 - r_1)} \\ &= \frac{t_3 - r_3(s_2 - r_2)}{r_3t_3}\left\{\frac{t_3\gamma_1}{t_2 - r_2(s_1 - r_1)} - a_3\right\} + \frac{r_3}{t_3}\widetilde{\gamma}_2 \\ &= \frac{t_3 - r_3(s_2 - r_2)}{r_3t_3}\frac{r_3(s_1 - r_1)\widetilde{\gamma}_2}{t_2 - r_2(s_1 - r_1)} + \frac{r_3}{t_3}\widetilde{\gamma}_2, \end{aligned}$$

hence $A = a$.

(iii) $B = b$.

From (3.21) and (3.19) we have

$$B = B_3 = \frac{a_3\gamma_2}{t_3} + t_3 - a_3 - \gamma_2 + (s_3 - \beta_2)\left(\frac{s_3\gamma_2}{t_3} - \beta_3 - s_3 + s_4\right),$$

and using the definition of $\widetilde{\beta}_n$ (see (3.2)), and the fact already proved $A_3 = a$ we obtain

$$B = \frac{\gamma_2 - t_3}{t_3}a_3 + (t_3 - \gamma_2) + (r_3 - \widetilde{\beta}_2 + s_2 - r_2)(a + \widetilde{\beta}_2 - r_3 + r_2 - s_2).$$

Hence, taking into account the expression of b in terms of λ, a and c given by (3.30), we have to prove

$$B = -\widetilde{\gamma}_1 + \lambda(\beta_0 - c) - \widetilde{\beta}_0(\widetilde{\beta}_0 + a).$$

Observe that by the definition of $\widetilde{\beta}_n$ and $\widetilde{\gamma}_n$ (see (3.2) and (3.3)) we can write

$$t_3 - \gamma_2 = t_2 - r_2(s_1 - r_1) - \widetilde{\gamma}_2 + (s_2 - r_2)(r_3 - \widetilde{\beta}_2 + \beta_1) \tag{3.37}$$

and then by Eq. (3.33) we get

$$\frac{\gamma_2 - t_3}{t_3}a_3 = -\gamma_1 + (s_1 - r_1)\frac{r_3\widetilde{\gamma}_2}{t_3} + a_3\frac{\widetilde{\gamma}_2}{t_3} - \frac{a_3(s_2 - r_2)(r_3 - \widetilde{\beta}_2 + \beta_1)}{t_3}. \tag{3.38}$$

Next, we analyze the last term in the expression of B, that is $(r_3 - \widetilde{\beta}_2 + s_2 - r_2)(a + \widetilde{\beta}_2 - r_3 + r_2 - s_2)$.

In the sequel of the proof, we will use the following identity

$$\beta_0 - \widetilde{\beta}_0 - \widetilde{\beta}_1 + \beta_1 + r_2 - s_2 = 0,$$

which is a direct consequence of the definition of $\widetilde{\beta}_1$ and $\widetilde{\beta}_0$; see (3.2). This relation together with the first equality in (3.36) and the initial condition (3.5) leads to

$$a + \widetilde{\beta}_2 - r_3 + r_2 - s_2 = \frac{r_3\widetilde{\gamma}_2}{t_3} + \frac{a_3(s_2 - r_2)}{t_3} - \beta_1.$$

Then

$$\begin{aligned} (r_3 - \widetilde{\beta}_2 + s_2 - r_2)(a + \widetilde{\beta}_2 - r_3 + r_2 - s_2) &= (r_3 - \widetilde{\beta}_2 + \beta_1 + \beta_0 - \widetilde{\beta}_0 - \widetilde{\beta}_1)(a + \widetilde{\beta}_2 - r_3 + r_2 - s_2) \\ &= (r_3 - \widetilde{\beta}_2 + \beta_1)\left(\frac{r_3\widetilde{\gamma}_2}{t_3} + \frac{a_3(s_2 - r_2)}{t_3} - \beta_1\right) \end{aligned}$$

$$+(\beta_0-\widetilde{\beta}_0-\widetilde{\beta}_1)(a+\widetilde{\beta}_2-r_3+\widetilde{\beta}_0+\widetilde{\beta}_1-\beta_1-\beta_0)$$
$$=(r_3-\widetilde{\beta}_2+\beta_1)\left(\frac{r_3\widetilde{\gamma}_2}{t_3}+\frac{a_3(s_2-r_2)}{t_3}-\beta_1+\widetilde{\beta}_1-\beta_0+\widetilde{\beta}_0\right)$$
$$+(\beta_0-\widetilde{\beta}_0-\widetilde{\beta}_1)(a+\widetilde{\beta}_0+\widetilde{\beta}_1-\beta_0). \tag{3.39}$$

Now, we add the formulas (3.37)–(3.39). Thus, by the relation $\widetilde{\beta}_0-\beta_0+s_1-r_1=0$ and the definition of $\widetilde{\gamma}_1$ (see (3.3)) we deduce

$$B=-\widetilde{\gamma}_1-\widetilde{\beta}_0(a+\widetilde{\beta}_0)+(s_1-r_1)\frac{r_3\widetilde{\gamma}_2}{t_3}+\frac{a_3\widetilde{\gamma}_2}{t_3}-\widetilde{\gamma}_2+(r_3-\widetilde{\beta}_2+\beta_1)\frac{r_3\widetilde{\gamma}_2}{t_3}+(\beta_0-\widetilde{\beta}_1)(a+\widetilde{\beta}_0+\widetilde{\beta}_1).$$

Therefore, since $C=c$ that is $\widetilde{\beta}_2-r_3-\frac{a_3}{r_3}=c$ we get

$$B=-\widetilde{\gamma}_1-\widetilde{\beta}_0(a+\widetilde{\beta}_0)-\widetilde{\gamma}_2+\frac{r_3\widetilde{\gamma}_2}{t_3}(\beta_1-c+s_1-r_1)+(\beta_0-\widetilde{\beta}_1)(a+\widetilde{\beta}_0+\widetilde{\beta}_1)$$
$$=-\widetilde{\gamma}_1-\widetilde{\beta}_0(a+\widetilde{\beta}_0)-\widetilde{\gamma}_2+\frac{\lambda\gamma_1}{\widetilde{\gamma}_1}(\beta_1-c+s_1-r_1+\beta_0-\widetilde{\beta}_1)+\frac{\lambda}{\widetilde{\gamma}_1}(\beta_0-c)(\beta_0-\widetilde{\beta}_1)(s_1-r_1),$$

where in the second equality we have used $\lambda=\frac{r_3}{t_3}\frac{\widetilde{\gamma}_1\widetilde{\gamma}_2}{\gamma_1}$ and the relation (3.31). Finally, it is sufficient to observe (3.32) and the definition of $\widetilde{\gamma}_1$ to conclude that $B=b$. □

4. Example

A new example of a non-degenerate 2–3 type relation (1.1) is presented.

From some rational transformations of the Jacobi weight function we construct two regular functionals **u** and **v** such that $(1+x)^2\mathbf{v}=(1-x)\mathbf{u}$, in the distributional sense. Moreover we analyze when their corresponding MOPSs $(P_n)_n$ and $(Q_n)_n$ satisfy a non-degenerate 2–3 type relation (1.1), and in this case we give explicitly the parameters involved in this relation. The construction will be done in several steps.

Let $\mathbf{w}=\mathbf{w}^{(\alpha,\beta)}$ be the positive definite linear functional defined by the weight function $(1-x)^\alpha\,(1+x)^\beta\chi_{(-1,1)}$ where $\alpha>-1$ and $\beta>-1$, and χ_E represents the characteristic function of a set E. Denote by $(W_n)_n$ its corresponding MOPS (we have chosen this notation instead of the classical one $(P_n^{(\alpha,\beta)})_n$ to avoid confusions with the notation $(P_n)_n$ used along all this paper). It is well known (see for instance [6]) that the recurrence coefficients (β_n,γ_n) of $(W_n)_n$ are given by

$$\beta_n=\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},\quad n\ge 0,$$
$$\gamma_n=\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},\quad n\ge 1.$$

Besides

$$w_0=\langle\mathbf{w},1\rangle=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$

and for $n\ge 1$

$$\langle\mathbf{w},W_n^2\rangle=\frac{2^{2n+\alpha+\beta+1}\Gamma(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(2n+\alpha+\beta+2)}. \tag{4.1}$$

First, we consider a functional $\widetilde{\mathbf{w}}$ such that $(1-x)\widetilde{\mathbf{w}}=\mathbf{w}$, that is

$$\widetilde{\mathbf{w}}=(1-x)^{-1}\mathbf{w}+\widetilde{w}_0\delta_1.$$

If the functional $\widetilde{\mathbf{w}}$ is regular and $(\widetilde{W}_n)_n$ is the corresponding MOPS, then there exists a sequence of complex numbers $(a_n)_n$ with $a_n\ne 0$ for every $n\ge 1$, such that

$$\widetilde{W}_n(x)=W_n(x)+a_nW_{n-1}(x),\quad n\ge 1. \tag{4.2}$$

The regularity of the functional $\widetilde{\mathbf{w}}$ is equivalent to the parameters a_n in (4.2) being the solution of the nonlinear difference equation

$$\beta_n-a_{n+1}-\frac{\gamma_n}{a_n}=1,\quad n\ge 1, \tag{4.3}$$

(see Theorem 2 in [11] and its proof). Observe that there is only a free parameter, namely a_1.

Besides since

$$\widetilde{w}_0(1-\beta_0+a_1)=w_0$$

it follows that $1-\beta_0+a_1\ne 0$ and using the value of β_0 we obtain

$$2(\alpha+1)+a_1(\alpha+\beta+2)\ne 0. \tag{4.4}$$

Now, the main goal is to characterize under which conditions the functional $\widetilde{\mathbf{w}}$ is regular and to obtain the expression of the parameters a_n.

Through clever calculations, we can deduce that if we take $a_1 \neq 0$ satisfying the above condition (4.4), then for $\alpha \neq 0$ the functional $\widetilde{\mathbf{w}}$ is regular if and only if

$$A_n := \Gamma(\alpha+1)\Gamma(\alpha+\beta+2)\Gamma(n)\Gamma(n+\beta) + M\Gamma(\beta+1)\Gamma(n+\alpha)\Gamma(n+\alpha+\beta) \neq 0, \quad n \geq 2, \tag{4.5}$$

where

$$M := \frac{2\alpha\widetilde{w}_0}{w_0} - (\alpha+\beta+1) = -\frac{2(\beta+1)+a_1(\alpha+\beta+1)(\alpha+\beta+2)}{2(\alpha+1)+a_1(\alpha+\beta+2)},$$

and moreover, we can deduce by induction that the parameters

$$a_n = \frac{-2}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}\frac{A_{n+1}}{A_n}, \quad n \geq 2, \tag{4.6}$$

are the solution of Eq. (4.3). Note that when $\alpha+\beta > -1$, (4.6) is valid for $n \geq 1$.

Whenever $\alpha = 0$, the functional $\widetilde{\mathbf{w}}$ is regular if and only if the condition (4.4) holds and

$$\widetilde{A}_n := \frac{2\widetilde{w}_0}{w_0} - (\beta+1)\sum_{i=1}^{n-1}\left(\frac{1}{i}+\frac{1}{\beta+i}\right) \neq 0, \quad n \geq 2. \tag{4.7}$$

Besides it can be proved by induction (empty sum equals zero) that

$$a_n = \frac{-2n(n+\beta)}{(2n+\beta)(2n+\beta-1)}\frac{\widetilde{A}_{n+1}}{\widetilde{A}_n}, \quad n \geq 1. \tag{4.8}$$

From (4.2) and the relation $(1-x)\widetilde{\mathbf{w}} = \mathbf{w}$, we obtain

$$\langle \widetilde{\mathbf{w}}, \widetilde{W}_n^2 \rangle = -a_n \langle \mathbf{w}, W_{n-1}^2 \rangle, \quad n \geq 1, \tag{4.9}$$

and so we have an explicit expression for $\langle \widetilde{\mathbf{w}}, \widetilde{W}_n^2 \rangle$ in terms of the parameter a_1.

In a second step, we consider the functional $\mathbf{u}$ verifying $(1+x)\widetilde{\mathbf{w}} = \mathbf{u}$. Then,

$$\mathbf{u} = (1-x)^{-1}\mathbf{w}^{(\alpha,\beta+1)} + u_0\,\delta_1,$$

where $u_0 = 2\widetilde{w}_0 - w_0 = \frac{1+\beta_0-a_1}{1-\beta_0+a_1} w_0$. Indeed, for any polynomial p

$$\begin{aligned}
\langle (1+x)\widetilde{\mathbf{w}}, p(x) \rangle &= \langle \widetilde{\mathbf{w}}, (1+x)p(x) \rangle \\
&= \left\langle \mathbf{w}, \frac{(1+x)p(x) - 2p(1)}{1-x} \right\rangle + 2\widetilde{w}_0 p(1) \\
&= \left\langle \mathbf{w}, (1+x)\frac{p(x)-p(1)}{1-x} \right\rangle + (2\widetilde{w}_0 - w_0)p(1) \\
&= \langle (1-x)^{-1}\,(1+x)\mathbf{w}^{(\alpha,\beta)} + (2\widetilde{w}_0 - w_0)\delta_1, p(x) \rangle.
\end{aligned}$$

Observe that the value of u_0 yields $1+\beta_0-a_1 \neq 0$, that is

$$2(\beta+1) - a_1(\alpha+\beta+2) \neq 0. \tag{4.10}$$

Since the expression of the functional $\mathbf{u}$ is similar to the one of $\widetilde{\mathbf{w}}$, taking in mind the previous study of the regularity of the functional $\widetilde{\mathbf{w}}$ and exchanging β for $\beta+1$ and M for $\widetilde{M}$ where

$$\widetilde{M} := \frac{2\alpha\, u_0}{w_0^{(\alpha,\beta+1)}} - (\alpha+\beta+2) = \frac{\alpha+\beta+2}{\beta+1} M,$$

we can ensure that for $\alpha \neq 0$ the functional $\mathbf{u}$ is a regular functional whenever we also impose that the conditions

$$B_n := \Gamma(\alpha+1)\,\Gamma(\alpha+\beta+3)\,\Gamma(n)\,\Gamma(n+\beta+1) + \widetilde{M}\,\Gamma(\beta+2)\,\Gamma(n+\alpha)\,\Gamma(n+\alpha+\beta+1) \neq 0, \quad n \geq 1, \tag{4.11}$$

hold. For $\alpha = 0$, these conditions should be replaced by

$$\widetilde{B}_n := \frac{2\,u_0}{w_0^{(0,\beta+1)}} - (\beta+2)\sum_{i=1}^{n-1}\left(\frac{1}{i}+\frac{1}{\beta+1+i}\right) \neq 0, \quad n \geq 1, \tag{4.12}$$

(empty sum equals zero).

Denoting by $(P_n)_n$ the MOPS associated with this regular functional $\mathbf{u}$, then the following linear relation

$$\widetilde{W}_n(x) = P_n(x) + b_n P_{n-1}(x), \quad n \geq 1, \tag{4.13}$$

holds, where $b_n = \langle \widetilde{\mathbf{w}}, \widetilde{W}_n^2 \rangle / \langle \mathbf{u}, P_{n-1}^2 \rangle$.

Furthermore, from the value of $\langle \widetilde{\mathbf{w}}, \widetilde{W}_n^2 \rangle$ given in (4.9) and exchanging β for $\beta+1$ and $\widetilde{w}_0$ for u_0, we can obtain the value of $\langle \mathbf{u}, P_n^2 \rangle$. Thus, from (4.1), (4.6) and (4.8) and straightforward computations we obtain

$$b_1 = -a_1 \frac{w_0}{u_0}$$

$$b_n = -a_n(n-1)(n+\alpha-1)\frac{B_{n-1}}{B_n}, \quad n \geq 2,\ \alpha \neq 0$$

$$b_n = -a_n \frac{n-1}{n+\beta} \frac{\widetilde{B}_{n-1}}{\widetilde{B}_n}, \quad n \geq 2,\ \alpha = 0.$$

Notice that we have obtained the following 2–2 relation

$$W_n(x) + a_n W_{n-1}(x) = P_n(x) + b_n P_{n-1}(x), \quad n \geq 1, \tag{4.14}$$

and the corresponding functionals satisfy

$$(1+x)\mathbf{w} = (1-x)\mathbf{u}.$$

In the last step, we consider a new functional $\mathbf{v}$ defined by $(1+x)\mathbf{v} = \mathbf{w}$, that is

$$\mathbf{v} = (1+x)^{-1}\mathbf{w} + v_0 \delta_{-1}.$$

Again, if the functional $\mathbf{v}$ is regular and $(Q_n)_n$ is the corresponding MOPS, there exists a sequence of complex numbers $(c_n)_n$ with $c_n \neq 0,\ n \geq 1$, such that

$$Q_n(x) = W_n(x) + c_n W_{n-1}(x), \quad n \geq 1. \tag{4.15}$$

The functional $\mathbf{v}$ is regular if and only if the parameters c_n satisfy

$$\beta_n - c_{n+1} - \frac{\gamma_n}{c_n} = -1, \quad n \geq 1, \tag{4.16}$$

(see Theorem 2 in [11]). Moreover,

$$v_0(1 + \beta_0 - c_1) = w_0$$

hence $1 + \beta_0 - c_1 \neq 0$, and using the value of β_0 we obtain

$$2(\beta+1) - c_1(\alpha+\beta+2) \neq 0. \tag{4.17}$$

Now, working in the same way as we have done before with the functional $\widetilde{\mathbf{w}}$, we can prove (by induction) that for $\beta \neq 0$ if we take $c_1 \neq 0$ satisfying the above condition (4.17) and

$$C_n := \Gamma(\beta+1)\Gamma(\alpha+\beta+2)\Gamma(n)\Gamma(n+\alpha) + N\Gamma(\alpha+1)\Gamma(n+\beta)\Gamma(n+\alpha+\beta) \neq 0, \quad n \geq 2, \tag{4.18}$$

where

$$N := \frac{2\beta\, v_0}{w_0} - (\alpha+\beta+1) = -\frac{2(\alpha+1) - c_1(\alpha+\beta+1)(\alpha+\beta+2)}{2(\beta+1) - c_1(\alpha+\beta+2)},$$

then the functional $\mathbf{v}$ is regular and the parameters c_n in the relation (4.16) are given by

$$c_n = \frac{2}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)} \frac{C_{n+1}}{C_n}, \quad n \geq 2. \tag{4.19}$$

For $\beta = 0$, the functional $\widetilde{\mathbf{v}}$ is regular if and only if the condition (4.17) holds and

$$\widetilde{C}_n := \frac{2v_0}{w_0} - (\alpha+1)\sum_{i=1}^{n-1}\left(\frac{1}{i} + \frac{1}{\alpha+i}\right) \neq 0, \quad n \geq 2, \tag{4.20}$$

and besides it can be proved by induction that

$$c_n = \frac{-2n(n+\alpha)}{(2n+\alpha)(2n+\alpha-1)} \frac{\widetilde{C}_{n+1}}{\widetilde{C}_n}, \quad n \geq 2.$$

Summarizing: if we take $a_1 \neq 0$ and $c_1 \neq 0$ satisfying the conditions (4.4), (4.5) or (4.7) if $\alpha = 0$, (4.10), (4.11) or (4.12) if $\alpha = 0$, (4.17), and (4.18) or (4.20) if $\beta = 0$, then the functionals

$$\mathbf{u} = (1-x)^{-1}\mathbf{w}^{(\alpha,\beta+1)} + \frac{1+\beta_0-a_1}{1-\beta_0+a_1} w_0\, \delta_1,$$

and

$$\mathbf{v} = (1+x)^{-1}\mathbf{w}^{(\alpha,\beta)} + \frac{1}{1+\beta_0-c_1} w_0 \delta_{-1},$$

are regular and they are related by

$$(1-x)\mathbf{u} = (1+x)^2\mathbf{v}.$$

In general, the above relation does not imply the existence of a non-degenerate 2–3 type relation (1.1) between the sequences $(P_n)_n$ and $(Q_n)_n$ associated with the functionals $\mathbf{u}$ and $\mathbf{v}$, respectively. More precisely we can ensure that a necessary and sufficient condition to get this type of relation is

$$a_n \neq c_n, \quad n \geq 2. \tag{4.21}$$

Indeed, if there exists a non-degenerate 2–3 type relation (1.1), since the functional $(1-x)\mathbf{u}$ is regular, from Proposition 2.2 we have $t_n \neq r_n(s_{n-1} - r_{n-1})$ for all $n \geq 2$. Now, Theorem 5.1 in [14] yields

$$\langle \mathbf{v}, Q_n^2 \rangle \neq -\langle \mathbf{u}, Q_n P_{n-1} \rangle, \quad n \geq 2.$$

From (4.13)–(4.15) we have

$$\langle \mathbf{u}, Q_n P_{n-1} \rangle = (b_n + c_n - a_n)\langle \mathbf{u}, P_{n-1}^2 \rangle,$$

and

$$\langle \mathbf{v}, Q_n^2 \rangle = c_n \langle \mathbf{w}, W_{n-1}^2 \rangle = -\frac{c_n}{a_n}\langle \widetilde{\mathbf{w}}, \widetilde{W}_n^2 \rangle = -\frac{c_n}{a_n} b_n \langle \mathbf{u}, P_{n-1}^2 \rangle.$$

Therefore, we get

$$b_n + c_n - a_n \neq \frac{c_n}{a_n} b_n, \quad n \geq 2,$$

and then (4.21) holds.

Conversely, from (4.14) and (4.15) we obtain

$$\begin{aligned} Q_1(x) &= P_1(x) + b_1 + c_1 - a_1, \\ Q_2(x) + (a_2 - c_2)Q_1(x) &= P_2(x) + b_2 P_1(x) + (a_2 - c_2)c_1 \end{aligned} \tag{4.22}$$

and straightforward computations lead us to the following explicit non-degenerate 2–3 type relation

$$Q_n(x) + r_n Q_{n-1}(x) = P_n(x) + s_n P_{n-1}(x) + t_n P_{n-2}(x),$$

where

$$r_n = a_{n-1}\frac{a_n - c_n}{a_{n-1} - c_{n-1}} \neq 0, \quad n \geq 3,$$

$$s_n = b_n + c_{n-1}\frac{a_n - c_n}{a_{n-1} - c_{n-1}}, \quad n \geq 3,$$

$$t_n = b_{n-1}c_{n-1}\frac{a_n - c_n}{a_{n-1} - c_{n-1}} \neq 0, \quad n \geq 3.$$

Observe that the condition $t_2 \neq r_2(s_1 - r_1)$ is satisfied because $a_1 \neq b_1$.

We want to remark that in the case $a_1 \neq c_1$, the above relation for $n = 2$ is equivalent to the relation (4.22).

Finally, by the sake of completeness, we show that there are a wide spectrum of free parameters a_1 and c_1 which allows us to build these examples.

For instance, taking $\alpha = \beta = 1/2$, $a_1 \notin (-1/2, 0] \cup \{\pm 1\}$, and $c_1 = -a_1$, the conditions (4.4), (4.10) and (4.17) are trivially satisfied, and besides, it is not difficult to verify that $A_n \neq 0$, $B_n \neq 0$, $C_n = A_n \neq 0$, for every $n \geq 1$. So the functionals

$$\mathbf{u} = \mathbf{w}^{(-1/2,3/2)} - \pi \frac{1 + 2a_1}{1 + a_1}\delta_1,$$

and

$$\mathbf{v} = \mathbf{w}^{(1/2,-1/2)} - \frac{\pi}{2}\frac{1 + 2a_1}{1 + a_1}\delta_{-1},$$

are regular and satisfy

$$(1 - x)\mathbf{u} = (1 + x)^2 \mathbf{v}.$$

Furthermore since $c_1 = -a_1$, from (4.6) and (4.19) we deduce $c_n = -a_n$, $n \geq 1$. In this case the values of the parameters of the 2–3 type relation are given by

$$r_n = a_n, \quad n \geq 2,$$

$$s_n = b_n - a_n, \quad n \geq 2,$$

$$t_n = -b_{n-1}a_n, \quad n \geq 2,$$

with

$$a_n = -\frac{1}{2}\,\frac{1 - (1 + 2a_1)n}{1 - (1 + 2a_1)(n - 1)}, \quad n \geq 2,$$

and

$$b_n = -a_n \frac{(2n - 1)(1 + a_1) - (1 + 2a_1)(n - 1)n}{(2n + 1)(1 + a_1) - (1 + 2a_1)n(n + 1)}, \quad n \geq 1.$$

Acknowledgments

This work has been supported by the research project MTM2012-36732-C03-02 (MEC, Spain). J. Petronilho was also partially supported by CMUC and FCT (Portugal), through European program COMPETE/FEDER, and the research project PTDC/MAT/098060/2008 (FCT). The other three authors were also partially supported by DGA project E-64 (Spain).

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