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IN MEMORIAM

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Professor José Javier Guadalupe "Chicho" (1945–2000)

José Javier Guadalupe, professor at the University of La Rioja, Logroño, Spain, died on April 1, 2000 at the age of 54. Born in Santa Cruz de La Palma, Canary Islands, in 1945, Chicho studied at the University of Zaragoza, where he graduated in Mathematics in 1970. He was appointed Assistant Professor in the department of Mathematical Analysis of that university and in 1974 he moved to the University College at Logroño, a nice place to live, near Zaragoza. He wrote a dissertation on "The closure in $L^p(\mu)$ of analytic polynomials on the unit circle" under the advice of the late professor José Luis Rubio de Francia and received his Ph.D. at the University of Zaragoza in 1982.

He spent his professional activity in Logroño as an excellent teacher as well as a recognized and qualified mathematician. Unfortunately, despite the fact that he was not a driver, he died in a traffic accident on April 1, 2000.

A valuable service was provided by Chicho to the Spanish community on orthogonal polynomials. The group of mathematicians under the advice of Luis Vigil, the chairman of the department of Mathematical Analysis at the University of Zaragoza until his retirement in 1984, was structured under a solid activity not only from the scientific point of view but from the human links among their members.

In this sense, Chicho promoted the first meeting of the group of Zaragoza with others universities (Granada, Cantabria, Oviedo, Vigo, Madrid) in Logroño in 1983. This was the starting point of an important activity as a real group of mathematicians. In fact, an international congress on orthogonal polynomials was held in Segovia in 1986 under the organization of this team. Every two years, an international event with the participation of leading scientists in orthogonal polynomials and approximation theory is held in our country. Chicho attended all of them but we remark the value of his first contribution in 1983.

In 1995, he promoted a new activity in order to meet Spanish mathematicians working in classical analysis. His links with people from the universities of Zaragoza, Barcelona, Madrid and the Basque Country allowed the success of those biennal meetings. As a sign of the recognition of his enthusiasm, the meeting of the year 2001 will be held in his honour and in his birth place, Santa Cruz de La Palma.

Chicho was a very good friend of all people interested to share his vocation as a mathematician, musician, fine gourmet of wine and food, and sport supporter.

He is survived by his wife Mari Carmen and her daughter Zenaida.

As colleagues and friends of Chicho, the aim of this presentation is to summarize the basic contributions of his research activity. Probably, with his criticism, Chicho would point out many remarks to our work, but this is just our tribute to his memory.

We miss you, Chicho.

The scientific contributions of Chicho

Orthogonal polynomials. The study of the classical theory of orthogonal polynomials was a permanent attractor for the research activity of Chicho.

In a first contribution with Vinuesa [2] they analyzed several equivalent characterizations of sequences of polynomials orthogonal with respect to a positive measure supported on $\mathbb{R}$. In particular, they proved one of them, closely related with the Gaussian quadrature formula, which leads to the so-called typical extension of an inner product (in the discrete version) from $\mathbb{P}_{n-1}$ to $\mathbb{P}_n$.

Later on, and under the influence of the Szegő theory approach by Vigil, the connections between orthogonal polynomials on $\mathbb{T}$, reflection parameters (the values of these polynomials in 0), Schur and Carathéodory functions, and the absolutely continuous and singular components in the Lebesgue decomposition of the measure of orthogonality were analyzed in [10].

In those years Luis Vigil was the leader of a research group working on orthogonal polynomials on $\mathbb{T}$. José Luis Rubio participated and solved some questions related with the Szegő theory. The idea of connecting different aspects of the theory of orthogonal polynomials with modern problems of Fourier Analysis raised from this collaboration and Rubio proposed this subject to Chicho for his Ph.D. thesis. Under his adviser Rubio, Chicho studied general aspects of Szegő theory, Hardy spaces in the unit disc and in plane domains with weights and their relation with the closure of analytic polynomials on weighted Jordan curves, prediction theory, invariant subspaces in $L^p(\mathbb{T})$ and $L^p(w)$, boundedness of the conjugate function operator on plane curves with weights belonging to the A_p classes of Muckenhoupt, among others.

From Szegő theory to invariant subspaces. Let μ be a finite positive measure supported on the unit circle $\mathbb{T}$. A basic problem in the classic theory of orthogonal polynomials on $\mathbb{T}$ is the study of the closure of the analytic polynomials, $H^2(\mathbb{T},\mu) = H^2(\mu)$, in the Hilbert space $L^2(\mathbb{T},\mu) = L^2(\mu)$. Two important questions arise: when is $H^2(\mu) \subsetneq L^2(\mu)$? (case C in Akhiezer's notation) and in this case, how to describe the functions in $H^2(\mu)$.

The answer to the first problem is given by Szegő, Kolmogorov and Krein: $H^2(\mu) \subsetneq L^2(\mu)$ is equivalent to $\log w \in L^1(\mathbb{T})$, where $\mu = wd\theta + \mu_s$ is the Lebesgue decomposition. The answer to the second question is the Szegő theorem:

$$H^2(\mu) = KH^2(\mathbb{T}) \oplus L^2(\mu_s)$$

where $H^2(\mathbb{T})$ is the classic Hardy space and K is the projection of the function 1 on the orthogonal linear subespace of $e^{i\theta}H^2(\mu)$, i.e. $1-K$ is the best approximation of 1 in the subspace $e^{i\theta}H^2(\mu)$.

In [4], this result was extended to $L^p(\mu)$ $(1 \leq p < \infty)$ with techniques of best approximation and James orthogonality. For $1 < p < \infty$ we have

$$H^p(\mu) \subsetneq L^p(\mu) \Leftrightarrow \log w \in L^1(\mathbb{T}) \Leftrightarrow e^{i\theta}H^p(\mu) \subsetneq H^p(\mu).$$

Also,

$$H^p(\mu) = K_p H^p(\mathbb{T}) \oplus L^p(\mu_s),$$

where $1 - K_p$ is the best approximation of 1 in $e^{i\theta}H^p(\mu)$. This representation is also true for $p = 1$, where K_1 is obtained as the limit of K_p in $L^1(\mu)$.

Moreover, as some geometric and functional consequences, we get

$$L^p(\mu_s) = \bigcap_{m \geq 0} e^{im\theta}H^p(\mu), \quad H^p(\mu) = e^{i\theta}H^p(\mu) + L[K_p],$$

as well as the fact that the spaces $H^p(\mathbb{T})$ and $H^p(w)$ are isometric.

Furthermore, the main properties of $H^p(\mathbb{T})$ with respect to bases, duality, factorization and Fourier coefficients were obtained for $H^p(\mu)$ $(1 < p < \infty)$.

The space $H^p(\mu)$ can also be considered as an invariant subspace of $L^p(\mu)$ under the shift operator S defined by $S(f)(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$. As in the classical theory a subspace $M \subseteq L^p(\mu)$ is said to be invariant if $S(M) \subseteq M$, doubly invariant if $S(M) = M$ and simply invariant if it is invariant but not doubly invariant. For $1 < p < \infty$, M is doubly invariant under $S \Leftrightarrow M = \chi_E L^p(\mu_s)$, and M is simply invariant $\Leftrightarrow M = qH^p(\mathbb{T}) \bigoplus \chi_E L^p(\mu_s)$ where E is a measurable subset of $\mathrm{supp}(\mu_s)$ and $q \in L^p(w)$ verifies $|q|^p w = 1$ and it is determined by the subspace up to constant factors of modulus one. Thus, the following consequences are deduced for $0 < p < \infty$ and an arbitrary measure μ satisfying the Szegő condition ($\log w \in L^1(\mathbb{T})$), see [1, 3, 4, 5]:

i) Szegő's theorem

$$H^p(\mu) = K_p H^p(\mathbb{T}) \oplus L^p(\mu_s)$$

where $1 - K_p$ is the best approximation of the function 1 in $e^{i\theta}H^p(\mu)$ and can be expressed in terms of $\exp \int \log w d\theta$ (the geometric mean of the weight).

ii) The simply invariant subspaces of $L^p(w)$ (respectively of $H^p(w)$) are of the form $M = uH^p(w)$ where $u \in L^p(w)$ with $|u| = 1$ a.e. (respectively u is an inner function) and it is unique up to constant factors of modulus one. These and other results extend those obtained by Beurling in $H^p(\mathbb{T})$.

Weighted Hardy spaces in plane domains. Assume now the measure μ on $\mathbb{T}$ is absolutely continuous with respect to the Lebesgue measure, $d\mu = wd\theta$. $H^p(\mathbb{D}, w)$ denotes the space of analytic functions in the unit disc $\mathbb{D}$ such that $\sup_{0 \leq r < 1} \int |f(re^{i\theta})|^p w(\theta) d\theta < \infty$ and $H^p(\mathbb{T}, w)$ is the closure of the analytic polynomials in $L^p(\mathbb{T}, w)$.

If $\log w \in L^1(\mathbb{T})$ and $0 < p < \infty$ then $H^p(\mathbb{T}, w) = K_p H^p(\mathbb{T})$ where $\frac{1}{K_p}$ is the outer function (unique up to constant factors of modulus one) such that $|K_p|^p w = 1$ a.e. Is there an analogous result for $H^p(\mathbb{D}, w)$?, i.e.

a) does $H^p(\mathbb{D}, w) = K_p H^p(\mathbb{D})$ hold? Furthermore, it is known that the spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$ are isometrically isomorphic.
b) What is the situation for $H^p(\mathbb{D}, w)$ and $H^p(\mathbb{T}, w)$?

A sufficient condition on the weight w in order to get the same results as before, see [K], is $w \in A_\infty$ (A_∞ is the union of the A_p classes of Muckenhoupt).

Let Γ be a rectifiable Jordan curve. For $0 < p < \infty$ and a measure $d\mu = w ds$ on Γ, where ds is the arc length measure on Γ, let $H^p(\Gamma, \mu)$ be the closed subspace of $L^p(\Gamma, \mu)$ generated by the analytic polynomials $P(z) = \sum_{k=0}^n a_k z^k$.

Then,

$$f \in L^p(\Gamma, \mu) \Leftrightarrow f \circ \varphi \in L^p(\mathbb{T}, \nu),$$

and

$$f \in H^p(\Gamma, \mu) \Leftrightarrow f \circ \varphi \in H^p(\mathbb{T}, \nu),$$

where $d\nu = (w \circ \varphi)|\varphi'| d\theta$.

A necessary and sufficient condition for $H^p(\Gamma, \mu) \subsetneq L^p(\Gamma, \mu)$ is $\log(w \circ \varphi) \in L^1(\mathbb{T})$. However, there is not an equivalent condition on w with respect to the arc length ds and the class of curves has to be restricted.

Now, let Γ be a chord-arc curve, i.e., a rectifiable Jordan curve such that for some constant $C > 0$ the length of the shorter arc along Γ with endpoints z_1 and z_2 is less than or equal to $C|z_1 - z_2|$, for every $z_1, z_2 \in \Gamma$.

Denote by Ω the region inside Γ, and by φ a conformal mapping from $|z| \le 1$ onto $\overline{\Omega}$ that, without loss of generality, will be normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$.

Chord-arc curves are characterized by the condition $\log \varphi' \in BMOA$, so that Ω is a Smirnov domain. These are the most general domains where a consistent theory of Hardy spaces can be developed.

The two questions a) and b) with $\mathbb{D}$ replaced by Ω have an affirmative solution in [9] if the weight w belongs to the class $A_\infty(\Gamma)$ (the natural extension of the class A_∞) and the curve Γ is chord-arc. For these curves a Szegő theorem was obtained in [11]: $H^p(\Gamma, w) = K_p H^p(\Gamma, ds)$, where

$$K_p = w^{-\frac{1}{p}} exp\left(-\frac{i}{p}\left(\log w\right)^{\sim}\right)$$

and $\sim$ denotes the conjugate function on Γ defined by $\widetilde{f} = \widetilde{(f \circ \varphi)} \circ \varphi^{-1}$. Also, analogous results to the classical case ($\Gamma = \mathbb{T}$) were given concerning duality, bases, Fourier coefficients, etc.

In [4], an analog of the Riesz theorem ($L^p(\mathbb{T}) = H^p(\mathbb{T}) \bigoplus e^{-i\theta}\widetilde{H}^p(\mathbb{T})$) was proved. In fact,

$$L^p(w) = H^p(w) \oplus e^{-i\theta}\widetilde{H}^p(w) \Leftrightarrow w \in A_p, \quad 1 < p < \infty,$$

where $\widetilde{H}^p(\mu)$ is the closure in $L^p(\mu)$ of the conjugate analytic polynomials. Likewise, a sharp condition ($w^{-p'/p} \in L^1(\mathbb{T})$) was given which implies $H^p(w) \bigcap \widetilde{H}^p(w) = \{\text{constant functions}\}$. In the framework of prediction theory, the fact $H^p(w) \bigcap \widetilde{H}^p(w)$

= {constant functions} means that the intersection between the past and the future is the present (where the past and the future refer to the stochastic process spectrally represented by $wd\theta$). This result and similar ones can be seen in [6] in a more general context than the L^p-spaces.

One of the main problems in Fourier analysis at that time was to characterize the families of plane curves for which the Cauchy operator T is bounded in $L^p(\Gamma)$. For $\Gamma = \mathbb{T}$, the boundedness of this operator is equivalent to the boundedness of the conjugate function operator, but in general this is not true. Indeed, T is bounded on $L^2(\Gamma)$ iff Γ is regular (see [D]), while Γ is regular if $\sim$ is bounded (see [Z]), but the converse is not true (see [JZ])

The condition $|\varphi'| \in A_p$ is equivalent to the boundedness of the operator $\sim$ in $L^p(\Gamma)$, $1 < p < \infty$ (this is the natural extension for curves Γ of the classical theorem of M. Riesz). In [13], this characterization was extended to $L^p(\Gamma, w)$ when Γ is a quasiregular curve (all quasiregular curves are chord-arc) obtaining that the operator $\sim$ is bounded in $L^p(\Gamma, w)$ if and only if $w \in A_p(\Gamma)$.

Fourier series. The convergence of the trigonometric Fourier series

$$\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{ikt}, \qquad f \in L^1(\mathbb{T}),$$

means the convergence of the sequence $S_n f$, where $S_n f(t) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikt}$. The convergence in norm, almost everywhere and weak convergence of these series have been widely studied.

One of the first classical results about mean convergence is due to M. Riesz, who proved the convergence in $L^p(\mathbb{T})$ with $1 < p < \infty$ of trigonometric Fourier series. This is equivalent to the corresponding uniform boundedness of the operators S_n, and it is also equivalent to the boundedness of the conjugate function operator $\sim: L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T})$.

The famous Carleson result, generalized by R. Hunt, ensures the almost everywhere convergence for every $f \in L^p(\mathbb{T}), 1 < p < \infty$ (see [C], [Hu]).

In the seventies, many results where obtained in this framework, more specifically on mean convergence for weighted Fourier series on $\mathbb{T}$. A remarkable result was given in [HMW]: in the case of weighted L^p spaces ($L^p(w)$), $1 < p < \infty$, the operators S_n are uniformly bounded if and only if the weight w belongs to the A_p class of Muckenhoupt.

Taking this situation into account, Rubio suggested to Chicho to study the convergence of Fourier series with respect to families of classical orthogonal polynomials and also for some perturbations of them.

Questions like the ones above quoted can be set up for orthonormal systems in a function space. For instance, assume that μ is a positive Borel measure on

$\mathbb{R}$ with finite moments and let $(P_n)_{n\geq 0}$ be the sequence of polynomials orthonormal with respect to μ. For every $f \in L^p(\mu)$, $1 < p < \infty$, its Fourier series is $\sum_{k=0}^{\infty} c_k(f)P_k$, $c_k(f) = \int f P_k d\mu$ and then the operators S_n are given by

$$S_n f(x) = \sum_{k=0}^{n} c_k(f)P_k(x) = \int_{\mathbb{R}} K_n(x,y)\, f(y)\, d\mu(y)$$

where $K_n(x,y)$ is the usual reproducing kernel related with the orthonormal polynomials.

The goal was to find conditions which imply the convergence of the Fourier series in $L^p(\mu)$, and to analyze the weak convergence of the series for extremal values of p. The almost everywhere convergence should be studied, also. In general, the techniques used with nontrigonometric Fourier series consist of obtaining explicit expressions in terms of the weighted Hilbert transform or finding results either of equiconvergence or transplantation with the trigonometric Fourier series. In both cases it is also necessary to have estimates of the corresponding orthogonal polynomials.

The first answers to these problems were given for specific families of orthogonal polynomials (classical or generalized classical orthogonal polynomials) or related special functions (see [S]).

In a seminal paper by H.L. Krall [Kr] a classification of the set of polynomial eigenfunctions of a fourth order linear differential operator with polynomial coefficients was given. Together with the classical orthogonal polynomials (Jacobi, Laguerre, and Hermite) three new families of orthogonal polynomials related to non classical weight functions appear:

(1) $d\mu = e^{-x}\,\chi_{(0,+\infty)}(x)dx + M\delta(x)$, $M \geq 0$ (Laguerre-type polynomials)
(2) $d\mu = (1-x)^{\alpha}\,\chi_{[0,1]}(x)dx + M\delta(x)$, $M \geq 0$, $\alpha > -1$ (Jacobi-type polynomials)
(3) $d\mu = \frac{\alpha}{2}\chi_{[-1,1]}(x)\,dx + \frac{1}{2}\delta(x-1) + \frac{1}{2}\delta(x+1)$, $\alpha > 0$ (Legendre-type polynomials)

These orthogonal polynomials attracted the attention of many researchers in the seventies and eighties.

In particular, T.H. Koornwinder [Ko] introduced an extension of (3) namely
$d\mu = (1-x)^{\alpha}(1+x)^{\beta}\chi_{[-1,1]}(x)\,dx + M\delta(x+1) + N\delta(x-1), \quad \alpha,\beta > -1.$

The main results of Koornwinder's paper focused on the hypergeometric character of such polynomials as well as the fact that they are solutions of a second order linear differential equation $a(x;n)y'' + b(x;n)y' + c(x;n)y = 0$ where a, b, c are polynomials of fixed degree independent of n.

Nevertheless the fact that no more analytic properties of such polynomials were considered by Koornwinder (coefficients of the three-term recurrence relation, distribution of their zeros, asymptotic properties and so on), this paper was the starting point of a very active research about the perturbations of positive measures by the addition of mass points. The location of such points with respect to the support of the measure was important from the point of view of the relative asymptotic

[N] but other people focused their attention on the spectral theory associated with these polynomials. The Dutch school (H. Bavinck, H.G. Meijer, M.G. de Bruin, R. Koekoek) has developed the analysis of differential operators whose eigenfunctions are those orthogonal polynomials. Another approach in the framework of the semiclassical families of orthogonal polynomials was given by F. Marcellán and P. Maroni ([M-Ma]) among others, when a quasi–definite linear functional is perturbed by mass points located in the complex plane.

About the middle of the eighties, Chicho had two doctoral students: J.L. Varona and M. Pérez Riera. He started to work with them in this context and they opened the study of Fourier series for such non–classical measures.

In [15], Chicho and his team obtained an estimate for the Koornwinder polynomials in $[-1,1]$ and proved the convergence of the Fourier projection operator in $L^p(\mu)$, $(1<p<\infty)$, when

$$\frac{\alpha+1}{4(2\alpha+3)}<p<\frac{\alpha+1}{4(2\alpha+1)},\quad \frac{\beta+1}{4(2\beta+3)}<p<\frac{\beta+1}{4(2\beta+1)}$$

and μ is the Jacobi weight plus the discrete measure $M\delta(x+1)+N\delta(x-1)$. Later on, in [26] they allow the discrete measure perturbation to be supported in a point $c\in(-1,1)$. Instead of the analysis of the kernel polynomials and the Christoffel-Darboux representation, they used a very simple relation between the polynomials orthogonal with respect to a measure ν supported on $\mathbb{R}$, those associated with $(x-c)^2d\nu$ and the corresponding sequence with respect to $d\nu+M\delta(x-c)$.

As a very nice application they consider the absolutely continuous measure

$$d\nu = h(x)(1-x)^{\alpha}(1+x)^{\beta}\prod_{i=1}^{N}|x-t_i|^{\gamma_i}\chi_{[-1,1]}(x)dx,$$

where

(i) α, β, $\gamma_i>-1$, $t_i\in(-1,1)$, $t_i\neq t_j$, $i\neq j$.
(ii) h is a positive continuous function on $[-1,1]$ and $w(h,\delta)\delta^{-1}\in L^1(0,2)$.

Such a measure is said to be a generalized Jacobi weight. They proved that the upper bound estimates in $[-1,1]$ for the generalized Jacobi polynomials and the corresponding kernel polynomials remain valid when the perturbation is introduced. The case $c=\pm1$ is also analyzed.

Notice that the addition of a finite number of mass points $c_i\in(-1,1)$ does not change essentially the results obtained for a single mass point.

The natural continuation of this paper was [27]. In our opinion this is one of the most remarkable contributions by Chicho and his team. Here they proved some weighted norm inequalities for the Fourier projection operators S_n, their maximal operator and the commutator $[M_b, S_n]$ where M_b is the operator of pointwise multiplication by $b \in BMO$. They checked the main results for the generalized Jacobi weights with mass points on $[-1, 1]$ and extended some previous results by Badkov.

On the other hand, it was known that for Laguerre and Hermite polynomials, the uniform boundedness $\|S_n f\|_{p,w} \leq C\|f\|_{p,w}$ holds if and only if $4/3 < p < 4$. However, for $p \leq 4/3$ or $p \geq 4$, inequalities of the above form remain true if we consider two different weights u, v instead of w. This fact suggests the study of partial sum operators $S_n \colon L^p(u) \longrightarrow L^p(v)$.

Because of Christoffel-Darboux formula, Pollard's decomposition of the kernel $K_n(x, y)$ is valid; assuming that the pair of weights $(u, v) \in A_p$ and using estimates of the orthogonal polynomials it can be derived that the Hilbert transform is bounded. As a consequence, the boundedness of partial sum operators is characterized for families of orthogonal polynomials including the generalized Jacobi polynomials (see [15, 16, 17, 20, 27]) and for Bessel series ([24]). The case when the measure is a generalized Jacobi weight plus a finite sum of Dirac deltas is analyzed in [27] as mentioned above.

The weak convergence of the Fourier expansion means that there exists a constant C, independent of n, y and f such that

$$\int_{|S_n f(x)|>y} d\mu(x) \leq C y^{-p} \int |f(x)|^p d\mu(x), \quad y > 0$$

which is equivalent to the uniform boundedness of S_n from $L^p(\mu)$ into $L^p_*(\mu)$, $1 < p < \infty$.

The set of values of p for which the Fourier series converges in norm is an interval and the above weak inequality only can be true in the closure of this interval. In its endpoints the weak convergence fails for generalized Jacobi polynomials and Bessel functions. It has sense to ask about the restricted weak convergence: in both cases the answer is affirmative (see [17,19, 23-25, 27]).

With respect to the a.e. convergence, in [22, 27] the maximal operator of the Fourier series is proved to be bounded for generalized Jacobi weights with a finite sum of Dirac deltas. Therefore, the series converges a.e. The main tool is a refinement of a result of Gilbert about transplantation.

The results above concern to specific classes of measures. There exist necessary conditions for the convergence which, in general, are not sufficient. If $S_n(\mu)$ denotes the nth partial sum operator of the Fourier series associated with the measure μ, the analysis of the mapping

$$b \longmapsto S_n(e^b \, d\mu)$$

can give results about the convergence of the Fourier series associated with measures of the form $e^b \, d\mu$, where b is any *perturbation*, with small norm in some adequate function space. This problem is closely related with the study of the commutator $[b, S_n]f = bS_n(f) - S_n(bf)$. For this commutator, as well as for some perturbations, information is obtained in [28, 29, 31] in the cases of Jacobi polynomials and Bessel functions whenever $b \in BMO$.

Stieltjes polynomials. Some of his last papers ([30, 34, 35]) were devoted to the Stieltjes polynomials. Given a measure μ as above, the Stieltjes polynomials with respect to μ can be introduced as polynomials S_n with deg$S_n = n$ such that

$$\int x^k S_n(x) P_{n-1}(x)\, d\mu(x) = 0, \quad k = 0, 1, \ldots, n-1.$$

In these papers the contribution to this topic is related with Gauss-Kronrod quadrature formulas and Padé-type approximation. In the first topic, some results about analytic functions in a neighbourhood of the interval $[-1, 1]$ are improved by studying the properties of the zeros of S_n. Concerning Padé-type approximation, new results on rational approximation, with some prefixed poles, to the Markov functions are deduced from the asymptotic behaviour of the Stieltjes polynomials.

References

1. *Subespacios invariantes en $L^p(\mu)$ sobre el toro*, Actas de las VII Jornadas Hispano-lusas de Matemáticas, Publ. Mat. **21** (1980), 191–194.
2. *Una condición adicional al teorema de equivalencia de las fórmulas fundamentales para sucesiones reales de polinomios ortogonales* (with J. Vinuesa), Rev. Acad. Cienc. Zaragoza (2) **36** (1981), 53–56.
3. *Invariant subspaces and H^p spaces with respect to arbitrary measures*, Boll. Un. Mat. Ital. B (7) **1** (1982), 1067–1077.
4. *Clausura en $L^p(\mu)$ de los polinomios analíticos sobre la circunferencia unidad*, Doctoral Dissertation, Publicaciones de la Universidad de Zaragoza, 1982.
5. *Invariant subspaces in $L^p(w)$, ($0 < p < \infty$)*, Rev. Acad. Cienc. Zaragoza (2) **37** (1982), 11–14.
6. *Some problems arising from prediction theory and a theorem of Kolmogorov* (with J. L. Rubio de Francia), Collect. Math. **33** (1982), 249–257.
7. *Clausura de polinomios y subespacios invariantes en $L^p(\mu)$* (with M. L. Rezola), In Contribuciones Matemáticas en honor de Luis Vigil, Zaragoza, 1984, 203–212.
8. *Simply invariant subspaces in $L^p(\Gamma, \mu)$* (with M. L. Rezola), II Simposium de Polinomios Ortogonales y Aplicaciones, Zaragoza, 1984, 81–87.
9. *Valores frontera de $H^p(\mu)$ en dominios planos con pesos* (with M. L. Rezola), III Simposium sobre Polinomios Ortogonales y Aplicaciones, Madrid, 1985, 117–122.
10. *Correspondance entre suites de polynômes ortogonaux et fonctions de la boule unité de $H_0^\infty(D)$* (with M. Alfaro, M. P. Alfaro and L. Vigil), Lecture Notes in Math. **1171** (1985), 158–163.
11. *Closure of analytic polynomials in weighted Jordan curves* (with M. L. Rezola), Lecture Notes in Math. **1171** (1985), 204–210.
12. *Productos infinitos de tipo Cardano* (with M. A. Hernández), Actas de las XII Jornadas Luso-españolas de Matemáticas, Braga, 1987, vol. II, 359–364.
13. *The conjugate function in plane curves* (with M. L. Rezola), Canad. Math. Bull. **31** (1988), 147–152.
14. *Cardano type entire functions* (with M. A. Hernández), Extracta Math. **3** (1988), 98–101.
15. *Convergencia en media de la serie de Fourier respecto de polinomios asociados a la medida* $(1-x)^\alpha(1+x)^\beta\, dx + M\delta_{-1} + N\delta_1$ (with M. Pérez, F. J. Ruiz and J. L. Varona), V Simposium sobre Polinomios Ortogonales y Aplicaciones, Vigo, 1988, 91–100.

16. *Mean and weak convergence of some orthogonal Fourier expansions by using A_p theory* (with M. Pérez and J. L. Varona), Lecture Notes in Pure and Appl. Math. **117** (1989), 161–169.
17. *L^p-boundedness of the kernels relative to generalized Jacobi weights* (with M. Pérez, F. J. Ruiz and J. L. Varona), VI Simposium sobre Polinomios Ortogonales y Aplicaciones, Gijón, 1989, 168–178.
18. *Convergencia de los núcleos para polinomios ortogonales sobre un recinto plano* (with J. Vinuesa), Actas de las XIV Jornadas Hispano-lusas de Matemáticas, La Laguna, 1989, vol. I, 389–393.
19. *Weak behaviour of Fourier-Jacobi series* (with M. Pérez and J. L. Varona), J. Approx. Theory **61** (1990), 222–238.
20. *Weighted L^p-boundedness of Fourier series with respect to generalized Jacobi weights* (with M. Pérez, F. J. Ruiz and J. L. Varona), Publ. Mat. **35** (1991), 449–459.
21. *Convergencia y sumabilidad de desarrollos ortogonales*, Zubía **10** (1992), 185–189.
22. *Two notes on convergence and divergence a.e. of Fourier series with respect to some orthogonal systems* (with M. Pérez, F. J. Ruiz and J. L. Varona), Proc. Amer. Math. Soc. **116** (1992), 457–464.
23. *Weighted weak behavior of Fourier-Jacobi series* (with M. Pérez and J. L. Varona), Math. Nachr. **158** (1992), 161–174.
24. *Mean and weak convergence of Fourier-Bessel series* (with M. Pérez, F. J. Ruiz and J. L. Varona), J. Math. Anal. Appl. **173** (1993), 370–389.
25. *Endpoint weak boundedness of some polynomial expansions* (with M. Pérez, F. J. Ruiz and J. L. Varona), J. Comput. Appl. Math. **49** (1993), 93–102.
26. *Asymptotic behaviour of orthogonal polynomials relative to measures with mass points* (with M. Pérez, F. J. Ruiz and J. L. Varona), Mathematika **40** (1993), 331-344.
27. *Weighted norm inequalities for polynomial expansions associated to some measures with mass points* (with M. Pérez, F. J. Ruiz and J. L. Varona), Constr. Approx. **12** (1996), 341–360.
28. *Estimates for commutators of orthogonal Fourier series* (with M. Pérez and F. J. Ruiz), J. London Math. Soc. **54** (1996), 311–322.
29. *Perturbation of orthogonal Fourier expansions* (with M. Pérez), J. Approx. Theory **92** (1998), 294–307.
30. *Convergence rate of Padé-type approximants for Stieltjes functions* (with M. Bello, F. Cala and G. López), J. Comput. Appl. Math. **99** (1998), 47–53.
31. *Commutators and analytic dependence of Fourier-Bessel series on* $(0,\infty)$ (with M. Pérez and J. L. Varona), Canad. Math. Bull. **42** (1999), 198–208.
32. *Mean and almost everywhere convergence of Fourier-Neumann series* (with Ó. Ciaurri, M. Pérez and J. L. Varona), J. Math. Anal. Appl. **236** (1999), 125–147.
33. *Solving dual integral equations on Lebesgue spaces* (with Ó. Ciaurri, M. Pérez and J. L. Varona), Studia Math. **142** (2000), 253–267.
34. *Asymptotics for Stieltjes polynomials, Padé-type approximants and Gauss-Kronrod quadrature* (with M. Bello, B. de la Calle and G. López), (to appear in J. Anal. Math.).
35. *The zero distribution of orthogonal polynomials with respect to varying measures on the unit circle* (with M. Bello and J. L. Varona), submitted for publication.
36. *Dibujando mediante iteraciones* (with M. Benito), Números **42** (2000), 15–28.

Auxiliary References

C. L. Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157.
D. G. David, *Operateurs intégraux singuliers sur certaines courbes du plan complexe*, Ann. Sci. École Norm. Sup. (4), Ser. 17 (1984), 157–189.

Hu. R. Hunt, *On the convergence of Fourier series*, Proc. Conf. Orthogonal Expansions and Continuous Analogues, 235–255. Southern Illinois Univ. Press, Carbondale, III., 1968.

HMW. R. Hunt; B. Muckenhoupt; R. Wheeden, *Weightd norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.

JZ. P. Jones; M. Zinsmeister, *Sur la transformation conforme des domaines de Laurentiev*, C.R. Acad. Sci. Paris, **295** (1982), 563–556.

K. C.E. Kenig, *Weighted H^p spaces on Lipschitz domains*, Amer. J. Math. **102** (1980), 129–163.

Ko. T.H. Koornwinder, *Orthogonal Polynomials with weight function* $(1-x)^\alpha(1+x)^\beta + M\delta(x+1) + N\delta(x-1)$, Canad. Math. Bull **27** (1984), 205–214.

Kr. H.L. Krall, *On orthogonal polynomials satisfying fourth order differential equation* , The Pensylvania State College Studies 6, State College, PA 1940.

KK. J. Koekoek; R. Koekoek, *Differential equations for generalized Jacobi polynomials*, J. Comput. Appl. Math. **126** (2000), 1–31.

M-Ma. F. Marcellán; P. Maroni, *Sur l'adjonction d'une masse de Dirac à une forme regulière et semiclassique*, Ann. Mat. Pura Appl. (4) **162** (1992), 1–22.

N. P. Nevai, *Orthogonal Polynomials*, Mem. Amer. Math. Soc. 213, Providence, R.I., 1979.

S. G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, R.I., 1975 (4th edition).

Z. M. Zinsmeister, *Représentation conforme et courbes presque lispchitziennes*, Ann. Inst. Fourier **34** (1984), 29–44.

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