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ON ORTHOGONAL POLYNOMIALS OF SOBOLEV TYPE: ALGEBRAIC PROPERTIES AND ZEROS*

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Abstract. In this paper the inner product $\langle f, g\rangle = \int_I fg\,d\mu + Mf(c)g(c) + Nf'(c)g'(c)$ is considered, where μ is a positive measure on the interval I, $c \in \mathbf{R}$ and $M, N \geqq 0$. General algebraic properties of the orthogonal polynomials associated with $\langle \cdot, \cdot \rangle$ as well as the zeros and their location are studied. In particular, the case of a symmetric measure μ is analyzed. Finally, a second-order linear differential equation and two applications are given.

Key words. orthogonal polynomials, inner product, kernels, zeros, differential equations

AMS(MOS) subject classification. 33C45

1. Introduction. Problems concerning the approximation of $C^{(k)}$ functions by polynomials, using the method of least squares, had been considered by Lewis [16], Gröbner [8], and Lesky [15]. In these papers, orthogonal polynomials associated to inner products involving derivatives appear in a natural way.

On the other hand, the study of the families of orthogonal polynomials related to inner products defined by

$$\langle f, g\rangle = \int_I fg\,d\mu + \lambda \int_I f'g'\,d\mu$$

and the properties of their zeros was begun by Althammer [1], Cohen [6], and Schäfke [22] in the case of Lebesgue measure with $I = (-1, 1)$, by Brenner [4] in the case $d\mu = e^{-x}\,dx$ with $I = (0, +\infty)$, and by Schäfke and Wolf [23] for the classical weights in the corresponding intervals I.

More recently, a group of Dutch mathematicians have considered similar problems for inner products

$$\langle f, g\rangle = \int_I fg\,d\mu + \sum_{k=0}^{n} \lambda_k f^{(k)}(0) g^{(k)}(0)$$

when $I = (0, +\infty)$ and μ is the Laguerre measure [11] or a q-discrete measure [12], as well as when μ is the Gegenbauer measure and $\langle \cdot, \cdot \rangle$ is given by

$$\langle f, g\rangle = \int_{-1}^{1} fg\,d\mu + M[f(-1)g(-1) + f(1)g(1)] + N[f'(-1)g'(-1) + f'(1)g'(1)]$$

with $I = (-1, 1)$, (see [2], [3]).

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Besides, Marcellán and Ronveaux [17] have studied the most general situation when the inner product is

$$\langle f, g\rangle = \int_I fg\, d\mu + \lambda f^{(r)}(c) g^{(r)}(c),$$

where $\lambda \in \mathbf{R}^+$ and $c \in \mathbf{R}$.

Finally, results relative to zeros have been the object of a very recent work by Meijer [20] and asymptotic properties have been obtained by Marcellán and van Assche [18].

The aim of this paper is to present the most general possible treatment of the families of orthogonal polynomials associated to an inner product of type

$$\langle f, g\rangle = \int_I fg\, d\mu + Mf(c)g(c) + Nf'(c)g'(c)$$

with $c \in \mathbf{R}$ and $M, N \geqq 0$.

In § 2, we study the algebraic properties of these orthogonal polynomials. An explicit representation in terms of the orthogonal polynomials associated to μ is given, as well as a five term recurrence relation, which is based on the self-adjoint character of a certain multiplication operator in the space of the polynomials. Moreover, a relation between the corresponding kernels and an analog of the Christoffel-Darboux formula is presented.

In § 3, we obtain results related to the distribution of the zeros, showing the dependence of this distribution from the position of the point c with respect to the support of the measure μ. Estimations about the position of the greatest zero are given.

In § 4, we consider a particularly simple situation corresponding to symmetric measures. In this case we can improve the results related to the zeros.

In § 5, we expose an application for semiclassical measures, deriving a second-order linear differential equation satisfied by the new orthogonal polynomials. Finally, two particularly interesting cases are considered: one of them deals with the case of Poisson's distribution, as an example of a discrete measure, and the other one corresponds to the case of Gegenbauer measure with $c=0$. In the latter, the mass is placed in an interior point of the support, unlike the usual location of masses in the ends of the support. This simplifies the calculations very much.

2. Algebraic properties.

2.1. Representation formulas. Let μ be a positive Borel measure on an interval (finite or infinite) $I \subset \mathbf{R}$ with infinite support such that all the moments $\int_I x^n\, d\mu$ exist. We define the following real inner product in the linear space of real polynomials $\mathcal{P}$:

$$\langle f, g\rangle = \int_I fg\, d\mu + Mf(c)g(c) + Nf'(c)g'(c),$$

where $c \in \mathbf{R}$ and $M, N \geqq 0$. This inner product cannot be associated to any positive measure on I in the standard sense [7], whenever $N > 0$.

Let $(P_n(x)) = (P_n)$ and $(Q_n(x)) = (Q_n)$ be the sequences of monic orthogonal polynomials (SMOP) with respect to μ and with the inner product $\langle \cdot, \cdot \rangle$, respectively. If we consider the representation of Q_n in terms of P_j,

$$Q_n(x) = P_n(x) + \sum_{j=0}^{n-1} \alpha_{nj} P_j(x)$$

from the orthogonality of Q_n with respect to P_j, $j=0, 1, \cdots, n-1$, it follows that

$$\alpha_{nj}=\frac{\int_I Q_n P_j \, d\mu}{\int_I P_j^2 \, d\mu}=-\frac{MQ_n(c)P_j(c)+NQ'_n(c)P'_j(c)}{\|P_j\|^2} \qquad 0 \leqq j \leqq n-1.$$

Then

$$Q_n(x)=P_n(x)-MQ_n(c)K_{n-1}(x, c)-NQ'_n(c)K_{n-1}^{(0,1)}(x, c), \tag{2.1}$$

where $(K_n(x, y))$ is the sequence of kernels associated to (P_n), and $K_n^{(r,s)}(x, y)$ denotes the generalized kernel

$$K_n^{(r,s)}(x, y)=\sum_{j=0}^{n} \frac{P_j^{(r)}(x)P_j^{(s)}(y)}{\|P_j\|^2}.$$

If we derive in (2.1) with respect to x and evaluating at $x=c$, the values $Q_n(c)$ and $Q'_n(c)$ can be expressed as the solutions of the system,

$$\begin{aligned} P_n(c)&=Q_n(c)[1+MK_{n-1}(c, c)]+Q'_n(c)NK_{n-1}^{(0,1)}(c, c), \\ P'_n(c)&=Q_n(c)MK_{n-1}^{(0,1)}(c, c)+Q'_n(c)[1+NK_{n-1}^{(1,1)}(c, c)], \end{aligned} \tag{2.2}$$

whose determinant:

$$\begin{aligned} D=1&+MK_{n-1}(c, c)+NK_{n-1}^{(1,1)}(c, c) \\ &+MN[K_{n-1}(c, c)K_{n-1}^{(1,1)}(c, c)-K_{n-1}^{(0,1)}(c, c)^2] \end{aligned}$$

is positive from the Cauchy-Schwartz inequality. Therefore,

$$\begin{aligned} Q_n(c)&=\frac{P_n(c)[1+NK_{n-1}^{(1,1)}(c, c)]-P'_n(c)NK_{n-1}^{(0,1)}(c, c)}{D}, \\ Q'_n(c)&=\frac{-P_n(c)MK_{n-1}^{(0,1)}(c, c)+P'_n(c)[1+MK_{n-1}(c, c)]}{D}. \end{aligned} \tag{2.3}$$

Then (2.1) becomes

$$\begin{aligned} Q_n(x)=P_n(x)&-M\frac{P_n(c)[1+NK_{n-1}^{(1,1)}(c, c)]-P'_n(c)NK_{n-1}^{(0,1)}(c, c)}{D}K_{n-1}(x, c) \\ &-N\frac{-P_n(c)MK_{n-1}^{(0,1)}(c, c)+P'_n(c)[1+MK_{n-1}(c, c)]}{D}K_{n-1}^{(0,1)}(x, c). \end{aligned} \tag{2.4}$$

We need establish some auxiliary results.

LEMMA 2.1. *Let $(P_n^c(x))$ and $(P_n^{c,c}(x))$ be the* SMOP *with respect to the measures $(x-c)^2 \, d\mu$ and $(x-c)^4 \, d\mu$, respectively. Then*:

$$(x-c)P_{n-1}^c(x)=P_n(x)-\frac{P_n(c)}{K_{n-1}(c, c)}K_{n-1}(x, c), \tag{2.5}$$

$$P_{n-1}^c(c)=P'_n(c)-\frac{P_n(c)}{K_{n-1}(c, c)}K_{n-1}^{(0,1)}(c, c), \tag{2.6}$$

$$(x-c)P_{n-2}^{c,c}(x)=P_{n-1}^c(x)-\frac{P_{n-1}^c(c)}{K_{n-2}^c(c, c)}K_{n-2}^c(x, c), \tag{2.7}$$

$$(x-c)(y-c)K_{n-1}^c(x, y)=K_n(x, y)-\frac{K_n(x, c)K_n(c, y)}{K_n(c, c)}, \tag{2.8}$$

(2.9) $$(x-c)K^c_{n-1}(x, c)=K^{(0,1)}_n(x, c)-\frac{K^{(0,1)}_n(c, c)}{K_n(c, c)}K_n(x, c),$$

(2.9′) $$K^c_{n-1}(c, c)=K^{(1,1)}_n(c, c)-\frac{[K^{(0,1)}_n(c, c)]^2}{K_n(c, c)},$$

where $(K^c_n(x, y))$ *denotes the sequence of kernels associated to* (P^c_n).

Proof. Let us consider the representation of $(x-c)P^c_{n-1}(x)$ in terms of $P_j(x)$:

$$(x-c)P^c_{n-1}(x)=P_n(x)+\sum_{j=0}^{n-1}\alpha_{n-1,j}P_j(x).$$

By using the orthogonality of the sequence (P^c_{n-1}) with respect to the measure $(x-c)^2\,d\mu$ we get:

$$\alpha_{n-1,0}=\frac{P_0(c)}{\|P_0\|^2}\int_I(x-c)P^c_{n-1}(x)\,d\mu$$

and

$$\begin{aligned}\alpha_{n-1,j}&=\frac{1}{\|P_j\|^2}\int_I(x-c)P^c_{n-1}(x)P_j(x)\,d\mu\\&=\frac{1}{\|P_j\|^2}\left[\int_I P^c_{n-1}(x)\frac{P_j(x)-P_j(c)}{x-c}(x-c)^2\,d\mu+P_j(c)\int_I(x-c)P^c_{n-1}(x)\,d\mu\right]\\&=\frac{P_j(c)}{\|P_j\|^2}\int_I(x-c)P^c_{n-1}(x)\,d\mu\end{aligned}$$

if $j=1,\cdots,n-1$.

Then

$$(x-c)P^c_{n-1}(x)=P_n(x)+K_{n-1}(x, c)\int_I(t-c)P^c_{n-1}(t)\,d\mu(t).$$

Evaluating at $x=c$, it follows the value of the last integral and, therefore, (2.5).

In order to prove (2.7) it suffices to consider the representation of $(x-c)P^{c,c}_{n-2}(x)$ in terms of $P^c_j(x)$ and to repeat the above argument.

If we derive (2.5) with respect to x and evaluating at $x=c$, we deduce (2.6).

Formula (2.8) can be obtained from the representation

$$(x-c)(y-c)K^c_{n-1}(x, y)=\sum_{j=0}^{n}\beta_{n-1,j}(y)P_j(x).$$

By using the reproducing property of the kernels and the orthogonality of P_n we have:

$$\beta_{n-1,0}(y)=\frac{(y-c)}{\|P_0\|^2}P_0(c)\int_I(x-c)K^c_{n-1}(x, y)\,d\mu(x)$$

and

$$\begin{aligned}\beta_{n-1,j}(y)&=\frac{(y-c)}{\|P_j\|^2}\int_I(x-c)K^c_{n-1}(x, y)P_j(x)\,d\mu(x)\\&=\frac{(y-c)}{\|P_j\|^2}\left[\int_I K^c_{n-1}(x, y)\frac{P_j(x)-P_j(c)}{x-c}(x-c)^2\,d\mu(x)\right.\\&\qquad\left.+P_j(c)\int_I(x-c)K^c_{n-1}(x, y)\,d\mu(x)\right]\end{aligned}$$

$$=\frac{1}{\|P_j\|^2}\left[P_j(y)-P_j(c)+(y-c)P_j(c)\int_I (x-c)K^c_{n-1}(x,y)\,d\mu(x)\right]$$

for every $j=1,\cdots,n$.

Then,

$$(x-c)(y-c)K^c_{n-1}(x,y)=K_n(x,y)-K_n(x,c)$$
$$+(y-c)K_n(x,c)\int_I (t-c)K^c_{n-1}(x,t)\,d\mu(t).$$

Now, formula (2.8) can be derived directly from the last one.

By derivation in (2.8) with respect to y and evaluating at $y=c$ we get (2.9). In a similar way, we deduce (2.9′) from (2.9). □

The above lemma allows us to represent the kernels $K_{n-1}(x,c)$ and $K^{(0,1)}_{n-1}(x,c)$ in terms of the polynomials $P_n(x)$, $P^c_{n-1}(x)$, and $P^{c,c}_{n-2}(x)$. By substitution of these values in (2.4) we obtain the following.

PROPOSITION 2.2. *Let c be such that the condition $P_n(c)P^c_{n-1}(c)\neq 0$ is satisfied for every $n\in N$. Then, the formula*

$$Q_n(x)=(1-\alpha_n)P_n(x)+(\alpha_n-\beta_n)(x-c)P^c_{n-1}(x)+\beta_n(x-c)^2P^{c,c}_{n-2}(x), \tag{2.10}$$

where

$$\alpha_n=1-\frac{Q_n(c)}{P_n(c)}=1-\frac{[1+NK^{(1,1)}_{n-1}(c,c)]P_n(c)-NK^{(0,1)}_{n-1}(c,c)P'_n(c)}{DP_n(c)},$$

$$\beta_n=\frac{NQ'_n(c)K^c_{n-2}(c,c)}{P^c_{n-1}(c)}$$

holds.

Remarks. (1) Since all the zeros of the polynomials $P_n(x)$ and $P^c_{n-1}(x)$ are in the interior of the interval I, we conclude that if c is not an interior point of I, then the formula (2.10) is true.

(2) The polynomials P^c_n have been identified by Kautsky and Golub (see [10]). By using methods of linear algebra they prove the fact that, if J is the Jacobi matrix associated with (P_n), a single step of the QR algorithm with the (Wilkinson) shift c corresponds to finding the Jacobi matrix associated with (P^c_n). A proof of this result by an analytic technique can be found in [5].

Let us consider the Christoffel-Darboux formula for the kernel $K_{n-1}(x,y)$ ([7, Thm. 4.5, p. 23] or [24, Thm. 3.2.2, p. 43]). For the first consequence below we evaluate at $y=c$, and for the second we derive with respect to y and evaluate at $y=c$.

$$(x-c)K_{n-1}(x,c)=\frac{1}{\|P_{n-1}\|^2}[P_n(x)P_{n-1}(c)-P_{n-1}(x)P_n(c)],$$

$$(x-c)^2K^{(0,1)}_{n-1}(x,c)=\frac{1}{\|P_{n-1}\|^2}[P_n(x)\{P_{n-1}(c)+(x-c)P'_{n-1}(c)\}-P_{n-1}(x)\{P_n(c)+(x-c)P'_n(c)\}]. \tag{2.11}$$

Multiplying the formula (2.4) by $(x-c)^2$ and substituting (2.11) we obtain

$$(x-c)^2Q_n(x)=q_2(x,n)P_n(x)+q_1(x,n)P_{n-1}(x), \tag{2.12}$$

where

$$q_2(x, n) = (x-c)^2 - \frac{MQ_n(c)}{\|P_{n-1}\|^2} \sum_{k=0}^{1} P_{n-1}^{(k-1)}(c)(x-c)^k$$
$$- \frac{NQ'_n(c)}{\|P_{n-1}\|^2} \sum_{k=0}^{1} P_{n-1}^{(k)}(c)(x-c)^k,$$
$$q_1(x, n) = \frac{MQ_n(c)}{\|P_{n-1}\|^2} \sum_{k=0}^{1} P_n^{(k-1)}(c)(x-c)^k$$
$$+ \frac{NQ'_n(c)}{\|P_{n-1}\|^2} \sum_{k=0}^{1} P_n^{(k)}(c)(x-c)^k.$$

(We denote $P_n^{(-1)}(c) = 0$.)

From (2.12), it follows that the sequence (Q_n) is strictly quasi orthogonal of order 2 with respect to the measure $(x-c)^2 \, d\mu$ [19] and, therefore,

$$(x-c)^2 Q_n(x) = P_{n+2}(x) + \sum_{j=n-2}^{n+1} a_{nj} P_j(x), \tag{2.13}$$

where the numbers a_{nj} can be expressed in terms of the coefficients of the polynomials $q_2(x, n)$, $q_1(x, n)$, and the coefficients of the three term recurrence relation satisfied by the SMOP (P_n).

Now, we can obtain a recurrence relation for the orthogonal polynomials Q_n.

PROPOSITION 2.3. *The polynomials Q_n satisfy a five term recurrence relation*:

$$(x-c)^2 Q_n(x) = Q_{n+2}(x) + \sum_{j=n-2}^{n+1} \gamma_{nj} Q_j(x) \qquad n \geqq 0, \tag{2.14}$$

where $\gamma_{n,n-2} > 0$ $(n \geqq 2)$ and the convention $Q_{-1} = Q_{-2} = 0$.

Proof. Let

$$(x-c)^2 Q_n(x) = \sum_{j=0}^{n+2} \gamma_{nj} Q_j(x)$$

be the expansion of the polynomial $(x-c)^2 Q_n(x)$ with respect to the sequence (Q_n).

Obviously, $\gamma_{n,n+2} = 1$ because Q_n is monic. On the other hand, if $0 \leqq j < n-2$, $\gamma_{n,j} = 0$ from the orthogonality of the sequence (Q_n).

The remaining coefficients $\gamma_{n,j}$ can be found as follows: from the definition of the inner product, if $n-2 \leqq j \leqq n+1$

$$\gamma_{n,j} = \frac{\langle (x-c)^2 Q_n(x), Q_j(x) \rangle}{\langle Q_j, Q_j \rangle}$$
$$= \frac{1}{\langle Q_j, Q_j \rangle} \int_I (x-c)^2 Q_n(x) Q_j(x) \, d\mu(x).$$

But from (2.13),

$$(x-c)^2 Q_j(x) = \sum_{h=j-2}^{j+2} a_{jh} P_h(x) \tag{2.15}$$

with $a_{j,j+2} = 1$, and from (2.1)

$$Q_n(x) = \sum_{h=0}^{n} \beta_{nh} P_h(x),$$

where $\beta_{nn}=1$, and if $h<n$

$$\beta_{nh}=\frac{-1}{\|P_h\|^2}[MQ_n(c)P_h(c)+NQ'_n(c)P'_h(c)],$$

then,

$$\int_I (x-c)^2 Q_n(x)Q_j(x)\,d\mu(x)$$

$$=a_{jn}\|P_n\|^2+\sum_{h=j-2}^{n-1}\beta_{nh}a_{jh}\|P_h\|^2$$

$$=a_{jn}\|P_n\|^2-\sum_{h=j-2}^{n-1}a_{jh}[MQ_n(c)P_h(c)+NQ'_n(c)P'_h(c)].$$

Also, from (2.15)

$$\sum_{h=j-2}^{j+2}a_{jh}P_h(c)=\sum_{h=j-2}^{j+2}a_{jh}P'_h(c)=0,$$

and hence

$$(2.16)\qquad \gamma_{nj}=\langle Q_j, Q_j\rangle^{-1}\left[a_{jn}\|P_n\|^2+MQ_n(c)\sum_{h=n}^{j+2}a_{jh}P_h(c)+NQ'_n(c)\sum_{h=n}^{j+2}a_{jh}P'_h(c)\right]$$

holds. Finally, from the definition of the inner product $\langle\,,\rangle$

$$\langle Q_n, Q_n\rangle=\|P_n\|^2+MQ_n(c)P_n(c)+NQ'_n(c)P'_n(c).$$

So, if $j=n-2$ we get

$$\gamma_{n,n-2}=\frac{\langle Q_n, Q_n\rangle}{\langle Q_{n-2}, Q_{n-2}\rangle}>0. \qquad \square$$

Remark. In the above proposition we have pointed out that

$$\langle Q_n, Q_n\rangle=\|P_n\|^2+MQ_n(c)P_n(c)+NQ'_n(c)P'_n(c).$$

An explicit expression of $\langle Q_n, Q_n\rangle$ in terms of M, N, and the polynomials P_n can be derived by using (2.3). We find, by straightforward calculations,

$$(2.17)\qquad \langle Q_n, Q_n\rangle=\|P_n\|^2\frac{\gamma_{n-1}\lambda_n+\gamma_n\lambda_{n-1}-2MNK_n^{(0,1)}(c,c)K_{n-1}^{(0,1)}(c,c)-D}{D},$$

where $\gamma_n=1+MK_n(c,c)$ and $\lambda_n=1+NK_n^{(1,1)}(c,c)$.

2.2. Kernels. We are going to derive a formula relating the kernel associated to the new polynomials Q_n with the kernels $K_n(x,c)$ and $K_{n-1}^{(0,1)}(x,c)$.

Let

$$L_n(x,y)=\sum_{h=0}^{n}\frac{Q_h(x)Q_h(y)}{\langle Q_h, Q_h\rangle}.$$

If we consider its expansion in terms of the polynomials $P_j(x)$,

$$L_n(x,y)=\sum_{j=0}^{n}\alpha_{nj}(y)P_j(x),$$

we have:

$$\alpha_{nj}(y) = \int_I L_n(x, y) \frac{P_j(x)}{\|P_j\|^2} d\mu$$
$$= \left\langle L_n(x, y), \frac{P_j(x)}{\|P_j\|^2} \right\rangle - ML_n(c, y) \frac{P_j(c)}{\|P_j\|^2} - NL_n^{(1,0)}(c, y) \frac{P_j'(c)}{\|P_j\|^2}$$
$$= \frac{1}{\|P_j\|^2} [P_j(y) - ML_n(c, y) P_j(c) - NL_n^{(1,0)}(c, y) P_j'(c)].$$

This proves that the kernels $L_n(x, y)$ and $K_n(x, y)$ satisfy the following formula:

$$(2.18) \qquad \begin{aligned} L_n(x, y) = K_n(x, y) - ML_n(c, y) K_n(x, c) \\ - NL_n^{(1,0)}(c, y) K_n^{(0,1)}(x, c). \end{aligned}$$

Explicit expressions for $L_n(c, y)$ and $L_n^{(1,0)}(c, y)$ can be obtained as solutions of the system

$$(2.19) \qquad \begin{aligned} K_n(c, y) &= L_n(c, y)[1 + MK_n(c, c)] + L_n^{(1,0)}(c, y) NK_n^{(1,0)}(c, c), \\ K_n^{(1,0)}(c, y) &= L_n(c, y) MK_n^{(1,0)}(c, c) \\ &\quad + L_n^{(1,0)}(c, y)[1 + NK_n^{(1,1)}(c, c)]. \end{aligned}$$

Now, we obtain an analog of the Christoffel-Darboux formula for the new polynomials.

PROPOSITION 2.4. *The relation*

$$(2.20) \qquad \begin{aligned} &(x + y - 2c)(x - y) L_n(x, y) \\ &= \frac{1}{\langle Q_n, Q_n \rangle} [Q_{n+2}(x) Q_n(y) - Q_{n+2}(y) Q_n(x)] \\ &+ \frac{\gamma_{n,n+1}}{\langle Q_n, Q_n \rangle} [Q_{n+1}(x) Q_n(y) - Q_{n+1}(y) Q_n(x)] \\ &+ \frac{1}{\langle Q_{n-1}, Q_{n-1} \rangle} [Q_{n+1}(x) Q_{n-1}(y) - Q_{n+1}(y) Q_{n-1}(x)] \end{aligned}$$

and its confluent form

$$(2.21) \qquad \begin{aligned} 2(x - c) L_n(x, x) &= \frac{1}{\langle Q_n, Q_n \rangle} [Q_{n+2}'(x) Q_n(x) - Q_{n+2}(x) Q_n'(x)] \\ &+ \frac{\gamma_{n,n+1}}{\langle Q_n, Q_n \rangle} [Q_{n+1}'(x) Q_n(x) - Q_{n+1}(x) Q_n'(x)] \\ &+ \frac{1}{\langle Q_{n-1}, Q_{n-1} \rangle} [Q_{n+1}'(x) Q_{n-1}(x) - Q_{n+1}(x) Q_{n-1}'(x)] \end{aligned}$$

hold.

Proof. Multiplying in the relation (2.14) by $Q_n(y)$ and multiplying the same relation evaluated at $x = y$ by $Q_n(x)$, we obtain after subtraction:

$$(2.22) \quad (x + y - 2c)(x - y) Q_n(x) Q_n(y) = \sum_{0 \neq k = -2}^{2} \gamma_{n,n+k} [Q_{n+k}(x) Q_n(y) - Q_{n+k}(y) Q_n(x)],$$

where $\gamma_{n,n+2} = 1$.

On the other hand, the inner product $\langle\,,\rangle$ is such that

$$\langle (x-c)^2 Q_n(x), Q_m(x)\rangle = \langle Q_n(x), (x-c)^2 Q_m(x)\rangle$$

for all $n, m \in N$. Hence, as

$$(x-c)^2 Q_{n-i}(x) = \sum_{k=i-2}^{i+2} \gamma_{n-i,n-k} Q_{n-k}(x)$$

we get

(2.23) $$\gamma_{n-i,n-k}\langle Q_{n-k}, Q_{n-k}\rangle = \gamma_{n-k,n-i}\langle Q_{n-i}, Q_{n-i}\rangle, \qquad k-2 \leqq i \leqq k+2.$$

From (2.22) and (2.23), by straightforward calculations, we get (2.20).

The result in (2.21) follows immediately from (2.20). □

3. Zeros of Q_n. In this section, we always consider $N>0$. It is well known that the zeros of P_n are real, simple, and belong to $\mathring{I}$ ($\mathring{I}$ denotes the interior of the true interval of orthogonality I). But this result may be false for polynomials Q_n. In fact, the general result we can prove is the following.

PROPOSITION 3.1. *If $n \geqq 3$, the polynomial Q_n has at least $n-2$ different zeros with odd multiplicity in $\mathring{I}$.*

Proof. Let $\xi_{n1}, \cdots, \xi_{nk}$ denote all the distinct zeros of Q_n of odd multiplicity which are in $\mathring{I}$. Define $p(x) = (x-\xi_{n1}) \cdots (x-\xi_{nk})$. The polynomial $(x-c)^2 p(x) Q_n(x)$ does not change sign in the interval I; hence,

$$\langle (x-c)^2 p(x) Q_n(x), 1\rangle = \int_I (x-c)^2 p(x) Q_n(x)\, d\mu(x) \neq 0.$$

Since (Q_n) is a quasi-orthogonal sequence of order 2 with respect to $(x-c)^2\, d\mu$, it follows that $\deg p(x) \geqq n-2$. □

PROPOSITION 3.2. *The zeros of the polynomial Q_n are real, simple, and at least $n-1$ of them belong to $\mathring{I}$, whenever either $c = \inf I$ or $c = \sup I$.*

Proof. Suppose $c = \sup I$. Let $\xi_{n1}, \cdots, \xi_{nk}$ denote all the zeros of Q_n in $\mathring{I}$. From Proposition 3.1, it follows that $k \geqq n-2$. Set $p(x) = (x-\xi_{n1}) \cdots (x-\xi_{nk})$; then, the polynomials $p(x)Q_n(x)$ and $(x-c)p(x)Q_n(x)$ have constant but opposite signs in $\mathring{I}$.

If $Q_n'(c) = 0$, we have

$$\langle (x-c)p(x), Q_n(x)\rangle = \int_I (x-c)p(x)Q_n(x)\, d\mu(x) \neq 0,$$

and hence, $\deg p(x) \geqq n-1$.

Let $Q_n'(c) \neq 0$. If we suppose $k = n-2$, the following formulas hold:

$$0 = \langle (x-c)p(x), Q_n(x)\rangle = \int_I (x-c)p(x)Q_n(x)\, d\mu + Np(c)(Q_n'(c)$$

$$0 = \langle (p(x), Q_n(x)\rangle = \int_I p(x)Q_n(x)\, d\mu + Mp(c)Q_n(c) + Np'(c)Q_n'(c).$$

Hence, $p(c)Q_n'(c)$ and $p'(c)Q_n'(c)$ have opposite signs, which is a contradiction. Thus $k \geqq n-1$. As a consequence, all the zeros of $Q_n(x)$ are real and simple.

If $c = \inf I$, the proof is similar. □

Remark. We want to note that if we consider the inner product

$$\langle f, g\rangle = \int_I f(x)g(x)\, d\mu(x) + Mf(c)g(c) + Nf^{(r)}(c)g^{(r)}(c)$$

with $r \in N$, by using the above arguments, we can deduce that the polynomial Q_n associated to the new inner product has at least $n-(r+1)$ different zeros in $\mathring{I}$. Whenever either $c=\sup I$ or $c=\inf I$, then Q_n has at least $n-1$ zeros in $\mathring{I}$ and, therefore, all the zeros are real and simple (see [11]).

Note that if $c=\sup I$ and all the roots of $Q_n(x)$ are located in the interior of I, then both conditions

$$Q_n(c)>0 \quad \text{and} \quad Q_n'(c)>0 \tag{3.1}$$

hold. In the similar way, if $c=\inf I$ and all the roots of $Q_n(x)$ belong to $\mathring{I}$, then

$$\operatorname{sgn} Q_n(c)=(-1)^n \quad \text{and} \quad \operatorname{sgn} Q_n'(c)=(-1)^{n-1} \tag{3.2}$$

hold.

This remark allows us to easily deduce sufficient conditions to assure a zero of $Q_n(x)$ is not in $\mathring{I}$, and besides we can give some results about its location.

From now on, if $c=\sup I$ or $c=\inf I$, we shall denote the zeros of $Q_n(x)$ being ordered by increasing size: $\xi_{n1}<\cdots<\xi_{nn}$.

PROPOSITION 3.3. *The following statements hold*:

(a) *Let $c=\sup I$. If the property* (3.1) *is not true then the greatest zero of $Q_n(x)$ satisfies*

$$c \leqq \xi_{nn} < c+\frac{c-\xi_{n1}}{n-1} \quad \textit{and} \quad |\xi_{nn}-c|<|\xi_{n,n-1}-c|.$$

Moreover, if $M \neq 0$, $\xi_{nn}-c<\frac{1}{2}\sqrt{N/M}$.

(b) *Let $c=\inf I$. If the property* (3.2) *is not true then the lowest zero of $Q_n(x)$ satisfies*

$$c-\frac{\xi_{nn}-c}{n-1}<\xi_{n1} \leqq c \quad \textit{and} \quad |\xi_{n1}-c|<|\xi_{n2}-c|.$$

Moreover, if $M \neq 0$, $c-\xi_{n1}<\frac{1}{2}\sqrt{N/M}$.

Proof. It suffices to prove (a). It is easy to deduce that if (3.1) is not true, we have $c \leqq \xi_{nn}$.

Assume $c<\xi_{nn}$, then $Q_n(c)=(c-\xi_{n1})\cdots(c-\xi_{nn})<0$. In this situation,

$$K_{n-1}^{(0,1)}(c, c)=\sum_{h=0}^{n-1} \frac{P_h(c) P_h'(c)}{\|P_h\|^2}>0,$$

from (2.2) it follows $Q_n'(c)>0$. Since

$$\frac{Q_n'(c)}{Q_n(c)}=\sum_{j=1}^{n-1} \frac{1}{c-\xi_{nj}}-\frac{1}{\xi_{nn}-c},$$

we get

$$\frac{1}{\xi_{nn}-c}>\sum_{j=1}^{n-1} \frac{1}{c-\xi_{nj}}>\frac{n-1}{c-\xi_{n1}}.$$

Hence,

$$\xi_{nn}<c+\frac{c-\xi_{n1}}{n-1} \quad \text{and} \quad |\xi_{nn}-c|<|\xi_{n,n-1}-c|.$$

Now, let us set $Q_n(x)=(\xi_{nn}-x)\varphi(x)$. Then,

$$\langle Q_n, \varphi\rangle=\int_I Q_n \varphi \, d\mu+M Q_n(c) \varphi(c)+N Q_n'(c) \varphi'(c)=0.$$

As $Q_n(x)\varphi(x)>0$ in I, in the above formula the integral is positive and so

$$MQ_n(c)\varphi(c)+NQ'_n(c)\varphi'(c)=(\xi_{nn}-c)[M\varphi(c)^2+N\varphi'(c)^2]-N\varphi(c)\varphi'(c)<0.$$

Whenever $M>0$, taking into account that $\varphi(c)<0$ and $\varphi'(c)<0$ and by using the Cauchy-Schwarz inequality, we obtain

$$\xi_{nn}-c<\frac{1}{2}\sqrt{\frac{N}{M}}. \qquad \square$$

Remark. The same results for $c=0$ and the Laguerre weight have been obtained in [13], and for some generalizations of the Laguerre weight see [20].

4. Analysis of the symmetric case. If I is a symmetric interval and the measure μ is symmetric on I (i.e., $\mu(A)=\mu(-A)$ for every $A\subset I$ measurable), it is well known [7, Thm. 4.3] that the SMOP (P_n) associated to μ satisfies $P_n(-x)=(-1)^nP_n(x)$ for all $n\in N$. As examples of this situation we have Hermite polynomials and Gegenbauer polynomials. We want to emphasize that the condition $P_n(-x)=(-1)^nP_n(x)$ for all $n\in N$ is equivalent to $K_n^{(0,1)}(0,0)=0$ for all $n\in N$.

Let us consider the condition

$$K_n^{(0,1)}(c,c)=0 \quad \text{for every } n\in N \tag{4.1}$$

is satisfied. Let us remark that

(i) $P_n(c)P'_n(c)=0$ for every $n\in N$;

(ii) $P_n(c)P_{n-1}(c)=0$ and $P'_n(c)P'_{n-1}(c)=0$ for every $n\in N$

are separately equivalent to (4.1). From (i) or (ii), it follows that c must belong to $\mathring{I}$. Furthermore there exists at most one c, which is determined by $P_1(c)=0$. Then, in general, we have

$$P_{2n-1}(c)=0 \quad \text{and} \quad P'_{2n}(c)=0 \quad \text{for every } n\in N.$$

We point out that no number c satisfies (4.1) for Jacobi polynomials with $\alpha\neq\beta$ or for Laguerre polynomials.

Now, it is not difficult to prove the polynomials P_n are symmetric with respect to the point c is equivalent to the condition (4.1). Since translation of the centre of symmetry is trivial, in the sequel, we assume (with absolutely no loss of generality) that $c=0$.

Since the determinant D is

$$D=[1+MK_{n-1}(0,0)][1+NK_{n-1}^{(1,1)}(0,0)],$$

we achieve

$$\begin{aligned}Q_n(0)&=\frac{P_n(0)}{1+MK_{n-1}(0,0)},\\ Q'_n(0)&=\frac{P'_n(0)}{1+NK_{n-1}^{(1,1)}(0,0)}.\end{aligned} \tag{4.2}$$

Then (2.4) becomes

$$\begin{aligned}Q_{2n}(x)&=P_{2n}(x)-\frac{MP_{2n}(0)}{1+MK_{2n-1}(0,0)}K_{2n-1}(x,0),\\ Q_{2n+1}(x)&=P_{2n+1}(x)-\frac{NP'_{2n+1}(0)}{1+NK_{2n}^{(1,1)}(0,0)}K_{2n}^{(0,1)}(x,0).\end{aligned} \tag{4.3}$$

Some properties about the quantities $Q_n(0)$ and $Q'_n(0)$ can be derived directly from (4.1) and (4.2). For instance:

(a) $Q_{2n}(0)\neq 0$ and $Q_{2n-1}(0)=0$ for every $n\in N$;
(b) $Q'_{2n}(0)=0$ and $Q'_{2n-1}(0)\neq 0$ for every $n\in N$;
(c) sign $Q_n(0)=$ sign $P_n(0)$ and sign $Q'_n(0)=$ sign $P'_n(0)$ for every $n\in N$.

In order to obtain Proposition 2.2 we might impose $P_n(0)P'_n(0)\neq 0$ for every $n\in N$. This restriction is not necessary now. Indeed, from (2.5)

$$P_{2n}(0)K_{2n-1}(x,0)=K_{2n-1}(0,0)[P_{2n}(x)-xP^c_{2n-1}(x)],$$

and from (2.9), (2.6), (2.7), (2.5), and (2.9′)

$$P'_{2n+1}(0)K^{(0,1)}_{2n}(x,0)=K^{(1,1)}_{2n}(0,0)[P_{2n+1}(x)-x^2P^{c,c}_{2n-1}(x)].$$

By substituting these values in (4.3) we obtain the following.

PROPOSITION 4.1. *The decomposition*:

$$Q_n(x)=(1-\alpha_n)P_n(x)+(\alpha_n-\beta_n)xP^c_{n-1}(x)+\beta_n x^2P^{c,c}_{n-2}(x) \tag{4.4}$$

where

$$\alpha_{2n}=\frac{MK_{2n-1}(0,0)}{1+MK_{2n-1}(0,0)}\qquad \beta_{2n}=0$$

$$\alpha_{2n+1}=\frac{NK^{(1,1)}_{2n}(0,0)}{1+NK^{(1,1)}_{2n}(0,0)}\qquad \beta_{2n+1}=\alpha_{2n+1}$$

holds.

Remark. It is interesting to point out that α_n and β_n are nonnegative and bounded by 1. Consequently, all the coefficients in (4.4) are nonnegative and bounded.

By substituting the values of $Q_n(0)$ and $Q'_n(0)$ (see (4.2)) in (2.12) and simplifying, we obtain:

$$x^2Q_n(x)=[x^2-a_n]P_n(x)+b_nxP_{n-1}(x), \tag{4.5}$$

where

$$a_n=\frac{1}{\|P_{n-1}\|^2}\frac{NP'_n(0)P_{n-1}(0)}{1+NK^{(1,1)}_{n-1}(0,0)},$$

$$b_n=\frac{1}{\|P_{n-1}\|^2}\left[\frac{MP_n(0)^2}{1+MK_{n-1}(0,0)}+\frac{NP'_n(0)^2}{1+NK^{(1,1)}_{n-1}(0,0)}\right].$$

Note that from the above formula it follows that the polynomials Q_n are also symmetric.

To deduce the recurrence relation we shall employ the expansion of $x^2Q_n(x)$ in terms of the polynomials P_n. Using the three term recurrence formula verified by the SMOP (P_n),

$$xP_n(x)=P_{n+1}(x)+B_{n+1}P_{n-1}(x)$$

and

$$x^2Q_n(x)=P_{n+2}(x)+\sum_{j=n-2}^{n+1}a_{n,j}P_j(x),$$

we find:

$$\begin{aligned} a_{n,n+1}&=0,\\ a_{n,n}&=B_{n+2}+B_{n+1}-a_n+b_n,\\ a_{n,n-1}&=0,\\ a_{n,n-2}&=B_n(B_{n+1}+b_n).\end{aligned} \tag{4.6}$$

Substituting these values (2.16) we obtain the coefficients of the five term recurrence relation verified by the SMOP (Q_n). To do this, it suffices to note that if $|m-n|$ is odd,

$$P_m(0)P_n(0)=P'_m(0)P'_n(0)=0$$

holds. Thus, by using the notations $\gamma_n=1+MK_n(0,0)$ and $\lambda_n=1+NK_n^{(1,1)}(0,0)$ we can give the following.

PROPOSITION 4.2. *The SMOP (Q_n) satisfies the formula*

$$x^2Q_n(x)=Q_{n+2}(x)+\sum_{j=n-2}^{n+1}\gamma_{n,j}Q_j(x),$$

where

$$\gamma_{n,n+1}=0,$$

$$\gamma_{n,n}=a_{n,n}+\frac{1}{\langle Q_n,Q_n\rangle}\left[\frac{MP_n(0)P_{n+2}(0)}{1+MK_{n-1}(0,0)}+\frac{NP'_n(0)P'_{n+2}(0)}{1+NK_{n-1}^{(1,1)}(0,0)}\right],$$

(4.7)
$$\gamma_{n,n-1}=0,$$

$$\gamma_{n,n-2}=\frac{\langle Q_n,Q_n\rangle}{\langle Q_{n-2},Q_{n-2}\rangle},$$

$$\langle Q_n,Q_n\rangle=\|P_n\|^2\left[\frac{\gamma_n}{\gamma_{n-1}}+\frac{\lambda_n}{\lambda_{n-1}}-1\right].$$

Note that, in the symmetric case, the recurrence formula satisfied by the polynomials Q_n is

$$x^2Q_n(x)=Q_{n+2}(x)+\gamma_{n,n}Q_n(x)+\gamma_{n,n-2}Q_{n-2}(x). \tag{4.8}$$

Moreover, the explicit expression concerning the kernels $L_n(0,y)$ and $L_n^{(1,0)}(0,y)$ is very simple. Then (2.18) becomes

(4.9)
$$L_n(x,y)=K_n(x,y)-\frac{MK_n(0,y)}{1+MK_n(0,0)}K_n(x,0)$$
$$-\frac{NK_n^{(1,0)}(0,y)}{1+NK_n^{(1,1)}(0,0)}K_n^{(0,1)}(x,0).$$

PROPOSITION 4.3. *The kernel $L_n(x,y)$ associated to the the polynomials Q_n can be expressed, in terms of the kernels associated to the polynomials P_n, P_n^c, and $P_n^{c,c}$:*

$$L_n(x,y)=r_nK_n(x,y)+s_nxyK_{n-1}^c(x,y)+t_nx^2y^2K_{n-2}^{c,c}(x,y), \tag{4.10}$$

where

$$r_n=\frac{1}{1+MK_n(0,0)},$$

$$s_n=\frac{MK_n(0,0)}{1+MK_n(0,0)}-\frac{NK_n^{(1,1)}(0,0)}{1+NK_n^{(1,1)}(0,0)},$$

$$t_n=\frac{NK_n^{(1,1)}(0,0)}{1+NK_n^{(1,1)}(0,0)}.$$

Proof. Using the formulas (2.9), the analog of (2.8) for $K_{n-2}^{c,c}(x,y)$, and (2.9′) we obtain

$$K_n^{(0,1)}(x,0)K_n^{(1,0)}(0,y)=K_n^{(1,1)}(0,0)[xyK_{n-1}^c(x,y)-x^2y^2K_{n-2}^{c,c}(x,y)].$$

Then the decomposition (4.10) holds and the explicit expression of coefficients r_n, s_n, t_n is obtained. □

Remarks.

(a) The coefficients in (4.10) are bounded and besides r_n, s_n are nonnegative.

(b) If $N=0$ there is always a decomposition as (4.10). But, if $N\neq 0$ there is such a decomposition if and only if $K_n^{(0,1)}(0,0)=0$ for all $n\in N$.

Next, we shall work in the symmetric case to obtain some strong results about zeros.

PROPOSITION 4.4. *All the zeros of Q_n are real, simple and belong to $\mathring{I}$.*

Proof. By Proposition 3.1, Q_n has at least $n-2$ different zeros in $\mathring{I}$, and all of them have odd multiplicity. As, $Q_n(-x)=(-1)^nQ_n(x)$ for every $x\in I$ and $Q'_{2n-1}(0)\neq 0$ for every $n\in N$, all the zeros of Q_n are simple. Suppose ξ is a complex zero of Q_n; then $\bar{\xi}$ is also a zero of Q_n and hence $-\xi=\bar{\xi}$. Thus $\xi=ir$ with $r\in\mathbf{R}$. Let us denote ξ_{nj}, $j=1,\cdots,n-2$, the remaining zeros of Q_n. Setting $p(x)=(x-\xi_{n1})\cdots(x-\xi_{n,n-2})$ we can write $Q_n(x)=p(x)(x^2+r^2)$. Then

$$\langle p, Q_n\rangle=\int_I p^2(x)(x^2+r^2)\,d\mu(x)+Mr^2p(0)^2+Nr^2p'(0)^2>0,$$

which is a contradiction; hence all the zeros are real.

Finally, we are going to show that ξ and $-\xi$ belong to $\mathring{I}$. Indeed, as $Q_n(x)=p(x)(x^2-\xi^2)$, it follows that $\langle p, Q_n\rangle=0$. But if we suppose $\xi\notin\mathring{I}$, then

$$\langle p, Q_n\rangle=\int_I p^2(x)(x^2-\xi^2)\,d\mu(x)-Mp(0)^2\xi^2-N[p'(0)]^2\xi^2<0.$$

Therefore, $\xi\in\mathring{I}$ holds. □

It is possible as well to deduce a separation property of the zeros. In order to prove it we will use the following.

LEMMA 4.5. *Between two consecutive zeros of $P_n(x)$ there is exactly one zero of $P_{n-1}^c(x)$.* (see [20, Lemma 6.1] or [9, Prop. 1.4.9]).

Since P_n and Q_n have symmetric zeros it suffices to consider the positive zeros. Let M, N be positive, real numbers.

PROPOSITION 4.6. *The positive zeros of P_n and Q_n mutually separate each other and the greatest positive zero of Q_n is less than the greatest positive zero of P_n. Moreover, the positive zeros of Q_{2n} alternate with the positive zeros of P_{2n-1}^c and the positive zeros of Q_{2n+1} alternate with the positive zeros of $P_{2n-1}^{c,c}$.*

Proof. Let us consider $n=2m$. As in (4.4) $\beta_{2m}=0$, we may write

$$Q_{2m}(x)=(1-\alpha_{2m})P_{2m}(x)+\alpha_{2m}xP_{2m-1}^c(x). \tag{4.11}$$

We denote $(x_{2m-1,j})_1^{m-1}$, $(x_{2m,j})_1^m$, $(\xi_{2m,j})_1^m$ the systems of the positive zeros of polynomials P_{2m-1}^c, P_{2m}, and Q_{2m}, respectively, each system arranged by increasing order.

From (4.11) and Lemma 4.5 it follows that whenever $x\geqq x_{2m,m}$, $Q_{2m}(x)>0$, and so $\xi_{2m,m}<x_{2m,m}$. On the other hand, as by Lemma 4.5 $P_{2m}(x_{2m-1,m-1}^c)\leqq 0$, we have $Q_{2m}(x_{2m-1,m-1}^c)\leqq 0$ and so $x_{2m-1,m-1}^c\leqq\xi_{2m,m}$. Since the roots of P_{2m} and P_{2m-1}^c are real and simple using, once more, Lemma 4.5 we have that the sign of $P_{2m-1}^c(x)$ changes in every $x_{2m,j}$ $(j=1,\cdots,m)$ and by (4.11) the sign of $Q_{2m}(x)$ changes in $x_{2m,j}$. Therefore, in each interval $(x_{2m,j-1}, x_{2m,j})$ there exists only one root of Q_{2m}.

In a similar way, the sign of $P_{2m}(x)$ changes in the roots of $P_{2m-1}^c(x)$, and, consequently, the sign of $Q_{2m}(x)$. Hence the positive roots of Q_{2m} and P_{2m-1}^c are interlaced.

If we suppose $n=2m+1$, then $\beta_{2m+1}=\alpha_{2m+1}$ and $P_{2m+1}(x)=xP_{2m}^{c}(x)$. Thus

$$Q_{2m+1}(x)=(1-\alpha_{2m+1})xP_{2m}^{c}(x)+\alpha_{2m+1}x^2P_{2m-1}^{c,c}(x).$$

Using the above argument and taking into account that the positive zeros of P_{2m+1} coincide with the positive zeros of P_{2m}^{c}, the result follows. □

Remark. Note that if $M=0$, $Q_{2m}(x)=P_{2m}(x)$, and if $N=0$, $Q_{2m+1}(x)=P_{2m+1}(x)$.

5. Differential properties.

5.1. Differential equation. Let us consider the case of (P_n) being a sequence of semiclassical orthogonal polynomials (see [19]). This means that the linear functional $\mathscr{L}$ defined by

$$\int_I P\,d\mu=\langle\mathscr{L},P\rangle,\qquad P\in\mathscr{P} \tag{5.1}$$

is characterized by polynomials ϕ and ψ such that a functional equation for $\mathscr{L}$

$$D(\phi\mathscr{L})+\psi\mathscr{L}=0 \tag{5.2}$$

holds with

$$\begin{aligned}\langle\psi\mathscr{L},P\rangle&=\langle\mathscr{L},\psi P\rangle,\\ \langle D(\phi\mathscr{L}),P\rangle&=-\langle\phi\mathscr{L},P'\rangle\end{aligned} \tag{5.3}$$

for every $P\in\mathscr{P}$.

It is easy to construct a second-order linear differential equation for the SMOP (Q_n) using the representation (2.12), where the polynomials q_2 and q_1 are known explicitly in terms of P_n.

Let us use the structure relation (see [19]) for semiclassical orthogonal polynomials P_n of class s ($s=\max\{(\deg\psi)-1,(\deg\phi)-2\}$).

$$\phi P'_{n+1}=\sum_{k=n-s}^{n+t}\theta_{nk}P_k, \tag{5.4}$$

where $t=\deg\phi$ and θ_{nk} are constants. This relation can be writen

$$\phi P'_{n+1}=C_nP_n+D_nP_{n-1}, \tag{5.5}$$

where the polynomials $C_n=C(x,n)$ and $D_n=D(x,n)$ are computed from the three term recurrence relation for the SMOP (P_n).

The usual 3 step procedure (see [21]) now give the relations

$$(x-c)^2Q_n=q_2P_n+q_1P_{n-1}, \tag{5.6}$$

$$\begin{aligned}\phi[(x-c)^2Q_n]'&=\phi(q_2'P_n+q_1'P_{n-1})+q_2(C_nP_n+D_nP_{n-1})\\ &\quad+q_1(C_{n-1}P_{n-1}+D_{n-1}P_{n-2})\\ &=q_{2,1}P_n+q_{1,1}P_{n-1},\end{aligned} \tag{5.7}$$

$$\phi[\phi[(x-c)^2Q_n]']'=q_{2,2}P_n+q_{1,2}P_{n-1}. \tag{5.8}$$

In the computation of the polynomials $q_{i,j}$ $(i,j=1,2)$, we need again the recurrence relation of the P_n in order to eliminate P_{n-2} in terms of P_n and P_{n-1}.

The following determinant gives the expected differential equation for the sequence (Q_n):

$$\begin{vmatrix}(x-c)^2Q_n & q_2 & q_1\\ \phi[(x-c)^2Q_n]' & q_{2,1} & q_{1,1}\\ \phi\{\phi[(x-c)^2Q_n]'\}' & q_{2,2} & q_{1,2}\end{vmatrix}=0. \tag{5.9}$$

This differential equation becomes particularly simple in the symmetric case with $c=0$. The Hermite case was already treated in [17], so we study here the Gegenbauer case. Bavinck and Meijer also analyze this situation (Gegenbauer case), but with two mass points located at the endpoints of the interval (see [2]).

5.2. Applications. As a first example, we consider the inner product of Sobolev type

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)^{\lambda-1/2}\,dx + Mf(0)g(0) + Nf'(0)g'(0) \tag{5.10}$$

with $\lambda > -\frac{1}{2}$. In this case, the point c $(c=0)$ is in the support of the measure, and the symmetric character is preserved.

It is well known that the monic Gegenbauer polynomials verify a three term recurrence relation.

$$xP_{n+1}^{(\lambda)}(x) = P_{n+2}^{(\lambda)}(x) + \frac{(n-1)(n+2\lambda)}{4(n+\lambda)(n+\lambda+1)} P_n^{(\lambda)}(x) \qquad n \geqq 0,$$

$$P_0^{(\lambda)}(x) = 1 \qquad P_1^{(\lambda)}(x) = x$$

and

$$P_{2n}^{(\lambda)}(0) = \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{n!} \frac{\Gamma(n+\lambda)}{\Gamma(2n+\lambda)},$$

$$P_{2n}^{(\lambda)\prime}(0) = P_{2n}^{(\lambda)\prime\prime\prime}(0) = 0,$$

$$P_{2n+1}^{(\lambda)}(0) = P_{2n+1}^{(\lambda)\prime\prime}(0) = 0,$$

$$P_{2n+1}^{(\lambda)\prime}(0) = \frac{(2n+1)(n+\lambda)}{2n+\lambda} P_{2n}^{(\lambda)}(0),$$

$$P_{2n}^{(\lambda)\prime\prime}(0) = -4n(n+\lambda)P_{2n}^{(\lambda)}(0),$$

$$P_{2n+1}^{(\lambda)\prime\prime\prime}(0) = -\frac{4n(2n+1)(n+\lambda)(n+\lambda+1)}{2n+\lambda} P_{2n}^{(\lambda)}(0),$$

$$\|P_n^{(\lambda)}\|^2 = 2^{1-2(\lambda+n)}\pi \frac{n!\Gamma(n+2\lambda)}{(n+\lambda)[\Gamma(n+\lambda)]^2}.$$

Moreover, they satisfy a structure relation

$$(x^2-1)P_{n+1}^{(\lambda)\prime}(x) = (n+1)xP_{n+1}^{(\lambda)}(x) - \frac{(n+1)(n+2\lambda)}{2(n+\lambda)} P_n^{(\lambda)}(x) \tag{5.11}$$

(see [24, formula 4.7.27, p. 83]). Thus (4.3) becomes

$$Q_{2n}(x) = P_{2n}^{(\lambda)}(x) + M_n \frac{P_{2n-1}^{(\lambda)}(x)}{x}, \tag{5.12}$$

$$Q_{2n+1}(x) = P_{2n+1}^{(\lambda)}(x) - N_n \frac{P_{2n+1}^{(\lambda)}(x) - \dfrac{P_{2n+1}^{(\lambda)\prime}(0)}{P_{2n}^{(\lambda)}(0)} xP_{2n}^{(\lambda)}(x)}{x^2}, \tag{5.13}$$

where

$$M_n = M \frac{[P_{2n}^{(\lambda)}(0)]^2}{\|P_{2n-1}^{(\lambda)}\|^2[1+MK_{2n-1}(0,0)]},$$

$$N_n = N \frac{P_{2n+1}^{(\lambda)\prime}(0)P_{2n}^{(\lambda)}(0)}{\|P_{2n}^{(\lambda)}\|^2[1+NK_{2n}^{(1,1)}(0,0)]},$$

but,

$$P_{2n}^{(\lambda)}(x)=S_n(x^2); \qquad P_{2n+1}^{(\lambda)}(x)=xS_n^*(x^2)$$

and

$$K_{2n}^{(0,1)}(x,0)=-\frac{P_{2n}(0)}{\|P_{2n}^{(\lambda)}\|^2}\frac{n(2n-1+2\lambda)}{2n+\lambda}xS_{n-1}^{**}(x^2)$$

(see [7, Chap. 1, § 8]). Then

$$Q_{2n}(x)=S_n(x^2)+M_nS_{n-1}^*(x^2),$$

$$Q_{2n+1}(x)=x\left[S_n^*(x^2)+N_n\frac{n(2n-1+2\lambda)}{2n+\lambda}S_{n-1}^{**}(x^2)\right].$$

The following proposition can easily be derived from the above comments and from (4.7) and (4.8).

PROPOSITION 5.1. *For the* SMOP (Q_n) *corresponding to the inner product defined by* (5.10), $Q_n(-x)=(-1)^nQ_n(x)$. *If*

$$Q_{2n}(x)=U_n(x^2) \quad \textit{and} \quad Q_{2n+1}(x)=xV_n(x^2),$$

then

(5.14) $$U_n(x)=S_n(x)+M_nS_{n-1}^*(x),$$

(5.15) $$V_n(x)=S_n^*(x)+N_n\frac{n(2n-1+2\lambda)}{2n+\lambda}S_{n-1}^{**}(x),$$

and U_n, V_n *satisfy a three term recurrence relation in the standard sense.*

Remark. In general, for a symmetric SMOP associated to a Sobolev type inner product, we can define two SMOP in the standard sense. They satisfy a decomposition in terms of (5.14) and (5.15).

PROPOSITION 5.2. *The* SMOP (Q_n) *verifies a second-order linear differential equation*

$$A(x;n)Q_n''(x)+B(x;n)Q_n'(x)+C(x;n)Q_n(x)=0,$$

where A, B, C are polynomials of degree independent of n. More precisely, $\deg B(x;n)\leqq \deg A(x;n)-1$; $\deg C(x;n)\leqq \deg A(x;n)-2$; $\deg A(x;2n)=6$ *and* $\deg A(x;2n+1)=8$.

Proof. From (5.11) and (5.12)

$$\begin{aligned}xQ_{2n}(x)&=xP_{2n}^{(\lambda)}(x)+M_n\frac{2n-1+\lambda}{(2n-1+2\lambda)n}\\&\quad\cdot[2nxP_{2n}^{(\lambda)}(x)-(x^2-1)P_{2n}^{(\lambda)\prime}(x)]\\&=\left(1+2M_n\frac{2n-1+\lambda}{2n-1+2\lambda}\right)xP_{2n}^{(\lambda)}(x)\\&\quad-M_n\frac{2n-1+\lambda}{n(2n-1+2\lambda)}(x^2-1)P_{2n}^{(\lambda)\prime}(x).\end{aligned}$$

On the other hand, from (5.11) and (5.13)

$$\begin{aligned}x^2Q_{2n+1}(x)&=(x^2-N_n)P_{2n+1}^{(\lambda)}(x)-N_n\left[\frac{-P_{2n+1}^{(\lambda)\prime}(0)}{P_{2n}^{(\lambda)}(0)}xP_{2n}^{(\lambda)}(x)\right]\\&=(x^2-N_n)P_{2n+1}^{(\lambda)}(x)-N_n\\&\quad\cdot[x(x^2-1)P_{2n+1}^{(\lambda)\prime}(x)-(2n+1)x^2P_{2n+1}^{(\lambda)}(x)]\\&=([1+(2n+1)N_n]x^2-N_n)P_{2n+1}^{(\lambda)}(x)-N_nx(x^2-1)P_{2n+1}^{(\lambda)\prime}(x).\end{aligned}$$

Then

$$Q_n(x)=\tilde{M}_n(x)P_n^{(\lambda)}(x)+\tilde{N}_n(x)P_n^{(\lambda)\prime}(x), \tag{5.16}$$

where

$$\tilde{M}_{2n}(x)=1+2M_n\frac{2n-1+\lambda}{2n-1+2\lambda},$$

$$\tilde{M}_{2n+1}(x)=1+(2n+1)N_n-\frac{N_n}{x^2},$$

$$\tilde{N}_{2n}(x)=M_n\frac{2n-1+\lambda}{n(2n-1+2\lambda)}\,\frac{1-x^2}{x},$$

$$\tilde{N}_{2n+1}(x)=N_n\frac{1-x^2}{x}.$$

Using derivatives in (5.16),

$$\begin{aligned}Q_n'(x)=\tilde{M}_n'(x)P_n^{(\lambda)}(x)+[\tilde{M}_n(x)+\tilde{N}_n'(x)]P_n^{(\lambda)\prime}(x)\\+\tilde{N}_n(x)P_n^{(\lambda)\prime\prime}(x).\end{aligned} \tag{5.17}$$

But, from the second-order linear differential equation satisfied by Gegenbauer polynomials (see [24, formula 4.7.5, p. 80]),

$$(x^2-1)P_n^{(\lambda)\prime\prime}(x)+(2\lambda+1)xP_n^{(\lambda)\prime}(x)-n(n+2\lambda)P_n^{(\lambda)}(x)=0$$

formula (5.17) becomes

$$Q_n'(x)=\hat{M}_n(x)P_n^{(\lambda)}(x)+\hat{N}_n(x)P_n^{(\lambda)\prime}(x), \tag{5.18}$$

where

$$\hat{M}_n(x)=\tilde{M}_n'(x)+\frac{\tilde{N}_n(x)}{x^2-1}n(n+2\lambda),$$

$$\hat{N}_n(x)=\tilde{M}_n(x)+\tilde{N}_n'(x)-(2\lambda+1)x\frac{\tilde{N}_n(x)}{x^2-1}.$$

From (5.16) and (5.18)

$$P_n^{(\lambda)}(x)=\frac{\begin{vmatrix}Q_n(x) & \tilde{N}_n(x)\\ Q_n'(x) & \hat{N}_n(x)\end{vmatrix}}{\Delta_n}, \tag{5.19}$$

$$P_n^{(\lambda)\prime}(x)=\frac{\begin{vmatrix}\tilde{M}_n(x) & Q_n(x)\\ \hat{M}_n(x) & Q_n'(x)\end{vmatrix}}{\Delta_n}, \tag{5.20}$$

where $\Delta_n=\tilde{M}_n(x)\hat{N}_n(x)-\hat{M}_n(x)\tilde{N}_n(x)$ is a rational function.

From derivation in (5.19) and taking into account (5.20), the result follows. □

Remark. The above result should be compared with Proposition 6.1 in [14].

We consider, as a second example, an inner product of Sobolev type when μ is a discrete positive measure. More precisely, μ is a step function whose jumps are

$$d\mu(x)=\frac{e^{-a}a^x}{x!}\quad \text{at } x=0,1,2,\cdots \quad\text{and}\quad a\in \boldsymbol{R}^+.$$

This corresponds to Poisson distribution in Probability Theory. The corresponding sequence $(C_n^{(a)})$ of monic orthogonal polynomials is called Charlier polynomials in the literature (see [7, p. 170]).

They can be expressed in terms of Laguerre polynomials as $C_n^{(a)}(x)=n!L_n^{(x-n)}(a)$ and satisfy a three term recurrence relation

$$C_{n+1}^{(a)}(x)=(x-n-a)C_n^{(a)}(x)-anC_{n-1}^{(a)}(x) \qquad n\geqq 0,$$
$$C_{-1}^{(a)}(x)=0 \qquad C_0^{(a)}(x)=1.$$

Moreover, Charlier polynomials can be characterized as the only SMOP belonging to Δ-Appell class, i.e.,

$$\Delta C_n^{(a)}(x)=nC_{n-1}^{(a)}(x) \qquad n\geqq 1,$$

where

$$\Delta p(x)=p(x+1)-p(x).$$

In this case, (2.12) becomes

$$x^2Q_n(x)=q_2(x;\,n)C_n^{(a)}(x)+q_1(x;\,n)C_{n-1}^{(a)}(x), \tag{5.21}$$

where

$$q_2(x;\,n)=x^2-a_nx-b_n,$$
$$q_1(x;\,n)=c_nx+d_n$$

and

$$a_n=\frac{MQ_n(0)C_{n-1}^{(a)}(0)+NQ_n'(0)C_{n-1}^{(a)\prime}(0)}{\|C_{n-1}^{(a)}\|^2},$$

$$b_n=\frac{NQ_n'(0)C_{n-1}^{(a)}(0)}{\|C_{n-1}^{(a)}\|^2},$$

$$c_n=\frac{MQ_n(0)C_n^{(a)}(0)+NQ_n'(0)C_n^{(a)\prime}(0)}{\|C_{n-1}^{(a)}\|^2},$$

$$d_n=\frac{NQ_n'(0)C_n^{(a)}(0)}{\|C_{n-1}^{(a)}\|^2}=-ab_n.$$

If in (5.21) we apply the Δ-operator and the recurrence relation for $C_n^{(a)}$, we get

$$\begin{aligned}
&(x+1)^2\Delta Q_n(x)+(2x+1)Q_n(x)\\
&\quad=q_2(x+1;\,n)nC_{n-1}^{(a)}(x)+\Delta q_2(x;\,n)C_n^{(a)}(x)\\
&\qquad+q_1(x+1;\,n)(n-1)C_{n-2}^{(a)}(x)+\Delta q_1(x;\,n)C_{n-1}^{(a)}(x)\\
&\quad=\left[\Delta q_2(x;\,n)-\frac{1}{a}\,q_1(x+1;\,n)\right]C_n^{(a)}(x)\\
&\qquad+\left[nq_2(x+1;\,n)+\Delta q_1(x;\,n)+\frac{1}{a}(x+1-n-a)q_1(x+1;\,n)\right]C_{n-1}^{(a)}(x).
\end{aligned}$$

Thus,

$$(x+1)^2\Delta Q_n(x)+(2x+1)Q_n(x)=A(x;\,n)C_n^{(a)}(x)+B(x;\,n)C_{n-1}^{(a)}(x) \tag{5.22}$$

with

$$A(x;\, n)=\left(2-\frac{c_n}{a}\right)x+1-a_n-\frac{1}{a}(c_n+d_n),$$

$$B(x;\, n)=\left(n+\frac{c_n}{a}\right)(x+1)^2$$
$$+\left[\frac{1}{a}\{d_n-(n+a)c_n\}-na_n\right](x+1)+c_n-d_n.$$

Then, from (5.21) and (5.22), Cramer's rule gives

$$C_n^{(a)}(x)=\frac{E_n(x)}{S_n(x)}Q_n(x)+\frac{F_n(x)}{S_n(x)}\Delta Q_n(x),$$

$$C_{n-1}^{(a)}(x)=\frac{G_n(x)}{S_n(x)}Q_n(x)+\frac{H_n(x)}{S_n(x)}\Delta Q_n(x),$$

where

$$E_n(x)=x^2B(x;\, n)-(2x+1)q_1(x;\, n),$$
$$F_n(x)=-(x+1)^2q_1(x;\, n),$$
$$S_n(x)=q_2(x;\, n)B(x;\, n)-q_1(x;\, n)A(x;\, n),$$
$$G_n(x)=(2x+1)q_2(x;\, n)-x^2A(x;\, n),$$
$$H_n(x)=(x+1)^2q_2(x;\, n).$$

Finally, using $\Delta C_n^{(a)}(x)=nC_{n-1}^{(a)}(x)$

$$\frac{E_n(x+1)}{S_n(x+1)}\Delta Q_n(x)+\left(\Delta\frac{E_n(x)}{S_n(x)}\right)Q_n(x)$$
$$+\frac{F_n(x+1)}{S_n(x+1)}\Delta^2Q_n(x)+\left(\Delta\frac{F_n(x)}{S_n(x)}\right)\Delta Q_n(x)$$
$$=n\left(\frac{G_n(x)}{S_n(x)}Q_n(x)+\frac{H_n(x)}{S_n(x)}\Delta Q_n(x)\right).$$

Therefore,

$$F_n(x+1)S_n(x)\Delta^2Q_n(x)$$
$$+([E_n(x+1)+F_n(x+1)]S_n(x)-[nH_n(x)+F_n(x)]S_n(x+1))\Delta Q_n(x)$$
$$+(E_n(x+1)S_n(x)-[E_n(x)+nG_n(x)]S_n(x+1))Q_n(x)=0.$$

In conclusion, we present the following.

PROPOSITION 5.3. *The* SMOP (Q_n) *satisfies a second-order linear difference equation*

$$U_n(x;\, n)\Delta^2Q_n(x)+V_n(x;\, n)\Delta Q_n(x)+W_n(x;\, n)Q_n=0,$$

where U, V and W are polynomials with degree independent of n. More precisely, $\deg U=7$, $\deg V\leqq 8$ *and* $\deg W\leqq 7$.

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