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Laplacian coflow for warped G_2-structures

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We consider the Laplacian coflow of a G_2-structure on warped products of the form $M^7 = M^6 \times_f S^1$ with M^6 a compact 6-manifold endowed with an SU(3)-structure. We give an explicit reinterpretation of this flow as a set of evolution equations of the differential forms defining the SU(3)-structure on M^6 and the warping function f. Necessary and sufficient conditions for the existence of solution for this flow are given. Finally we describe new solutions for this flow where the SU(3)-structure on M^6 is nearly Kähler, symplectic half-flat or balanced.

0. Introduction

The first author to consider flows of G_2-structures was Bryant in 2006, [3]. Concretely he considered the Laplacian flow of a G_2-structure:

$$\frac{\partial}{\partial t}\varphi(t) = \Delta_7\varphi(t),$$

starting from a closed 3-form φ_0 defining the G_2-structure. Δ_7 is the corresponding Hodge Laplacian, given by the formula $\Delta_7 = *_7\, d_7 *_7 d_7 - d_7 *_7 d_7 *_7$.

In the last years there has been a lot of fundamental works on this issue. In [5] it was proved the short time existence and uniqueness of solution on compact manifolds. The first examples of long time solutions

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to this flow were described in [8]. These examples consist on non compact nilpotent Lie groups endowed with a one-parameter family of closed G_2-structures such that satisfies the Laplacian flow equation for all $t \in (a, +\infty)$ with $a < 0$.

Recent papers by Lotay and Wei [17–19] derived important properties of the Laplacian flow as long time existence or convergence results. Even more recently Fino and Raffero on [11] obtained sufficient conditions for the existence of solution of this flow on warped products of the form $M^6 \times_f S^1$ with M^6 a 6-dimensional manifold endowed with an SU(3)-structure. Recall that, if (B, g_B) and (F, g_F) are Riemannian manifolds and f is a non-vanishing differentiable function on B, then the warped product $W = B \times_f F$ consists on the product manifold $B \times F$ endowed with the metric $g = \pi_1^*(g_B) + f^2\pi_2^*(g_F)$ where π_1 and π_2 are the projections of W onto B and F respectively. They also reinterpret the flow as a set of evolution equations on M^6 involving the differential forms defining the SU(3)-structure and the warping function f. More details about the Laplacian flow of a closed G_2-structure can be found in the reviews [9,16] and the references therein. Another interesting result concerning this flow was due to Xu and Ye in [23], where they proved long time existence and uniqueness of solution for this flow starting near a torsion free G_2-structure.

In this work we consider the so-called Laplacian "coflow" of G_2-structures. This coflow was introduced by Karigiannis, McKay and Tsui in [15] and can be considered as the analogous of the Laplacian flow of a closed G_2-structure where the 3-form φ_0 is now considered to be coclosed instead of closed. Equivalently this flow can be stated as:

$$\frac{\partial}{\partial t} *_7 \varphi(t) = -\Delta_7 *_7 \varphi(t),$$

where the 4-form $*_7\varphi_0$ is closed and $*_7$ denotes the Hodge star operator. These authors considered more natural to define this flow with a minus sign in order to make it more likely to the heat equation. In order to obtain solutions they consider 7-dimensional manifolds $M^6 \times L^1$ with $L^1 = \mathbb{R}$ or S^1 where M^6 is endowed with a Calabi-Yau or a nearly Kähler structure. Grigorian in [13] introduced the modified Laplacian coflow, which consists on a modified version of the Laplacian coflow, proving short time existence and uniqueness of solution for this modified flow. He also derives the modified Laplacian coflow for warped G_2-structures of the form $M^6 \times_f L^1$ obtaining solution for M^6 being Calabi-Yau or nearly Kähler. Long time solutions for the Laplacian coflow on non compact nilpotent Lie groups were described in [1]. In this work we present solutions for the coflow on warped products where the base manifolds are Lie groups endowed with metrics belonging to the Gray-Hervella classes $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$.

The paper is structured as follows. In Section 1 we give an introduction to SU(3) and G_2-structures. Section 2 is devoted to G_2-structures of the form $M^6 \times_f S^1$ (M^6 being compact and endowed with an SU(3)-structure) whose induced metric describes a warped product. In particular in Theorem 2.4 we give an explicit description of the torsion forms of such a G_2-structure in terms of the torsion forms of the SU(3)-structure on the base manifold and the warping function. In Section 3 we reinterpret the Laplacian flow and coflow of a G_2-structure as a set of evolution equations of the SU(3)-structure and we describe the Laplacian coflow operator of the warped G_2-structure by means of the torsion forms of the SU(3)-structure and the warping function. In particular for the Laplacian flow we reobtain the equations due to Fino and Raffero in [11]. Finally the goal of Section 4 is to obtain new examples of solutions of the Laplacian coflow constructed as warped products where the base manifolds are 6-dimensional and they are endowed with nearly Kähler, symplectic half-flat or balanced SU(3)-structures.

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1. SU(3) and G$_2$-structures

In this section we review some preliminaries concerning SU(3) and G$_2$-structures. More concretely we present these structures, their corresponding SU(3) and G$_2$ type decomposition of the spaces of differential forms and finally their torsion forms.

1.1. SU(3)*-structures*

An SU(n)-structure on a differentiable manifold M^{2n} consists on a triple (g, J, Ψ) where (g, J) is an almost Hermitian structure on M^{2n} and Ψ is a complex $(n, 0)$ form, satisfying

$$(-1)^{n(n-1)/2}\left(\frac{\imath}{2}\right)^n \Psi \wedge \overline{\Psi} = \frac{1}{n!}\,\omega^n,$$

with $\overline{\Psi}$ the conjugated form of Ψ and ω the Kähler form of the almost Hermitian structure. An SU(n)-structure can equivalently be described by the triple (ω, ψ_+, ψ_-) where ψ_+ and ψ_- are, respectively the real and the imaginary part of the complex form Ψ. In what follows we will focus on SU(3)-structures on 6-dimensional manifolds. Note that in this case, the metric $g_{\omega,\psi_\pm}$ can be recovered from (ω, ψ_+, ψ_-) as

$$g_{\omega,\psi_\pm}(X, Y)vol_6 = -3\,(\iota_X)\omega \wedge (\iota_Y \psi_+) \wedge \psi_+,$$

where ι denotes the contraction operator, $vol_6 = \frac{1}{3!}\omega^3$ and $X, Y \in \mathfrak{X}(M^6)$.

The presence of such structure on a manifold M^6 can also be characterized by the existence of a local basis of 1-forms $\{e^1, \ldots, e^6\}$ such that (ω, ψ_+, ψ_-) can be described as:

$$\begin{gathered} \omega = e^{12} + e^{34} + e^{56}, \\ \psi_+ = e^{135} - e^{146} - e^{236} - e^{245}, \quad \psi_- = -e^{246} + e^{235} + e^{145} + e^{136}, \end{gathered} \tag{1}$$

where we denote, as usual in the related literature, e^{ij} the wedge product $e^i \wedge e^j$ and e^{ijk} the wedge product $e^i \wedge e^j \wedge e^k$. In the following, a basis in which the SU(3)-structure has the expression (1) will be called an *adapted basis*.

In [6] it is described how the intrinsic torsion of an SU(3)-structure, namely τ, lies in a space of the form

$$\tau \in \mathcal{W}_1^\pm \oplus \mathcal{W}_2^\pm \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5,$$

where $\mathcal{W}_i$ denote the irreducible components under the action of the group SU(3). This torsion can be described by the exterior derivatives of ω, ψ_+ and ψ_- and also in terms of the so called torsion forms. This latter description is given in [4] where the authors consider the natural action of the group SU(3) on $\Omega^k(M^6)$, the space of k-forms on M^6. Thus, the different spaces of forms $\Omega^k(M^6)$ can be splitted into SU(3) irreducible subspaces as follows:

$$\Omega^1(M^6) \text{ is irreducible,}$$
$$\Omega^2(M^6) = \Omega^2_1(M^6) \oplus \Omega^2_6(M^6) \oplus \Omega^2_8(M^6),$$

with

$\Omega^2_1(M^6) = \{f\omega | f \in \mathcal{C}^\infty(M^6)\}$,

$\Omega^2_6(M^6) = \{*_6 J(\eta \wedge \psi_+) | \eta \in \Omega^1(M^6)\} = \{\sigma \in \Omega^2(M^6) | J\sigma = \sigma\}$,

$\Omega^2_8(M^6) = \{\sigma \in \Omega^2(M^6) | \sigma \wedge \psi_+ = 0, *_6 J\sigma = -\sigma \wedge \omega\} = \{\sigma \in \Omega^2(M^6) | J\sigma = -\sigma, \sigma \wedge \omega^2 = 0\}$;
and

$$\Omega^3(M^6) = \Omega^3_+(M^6) \oplus \Omega^3_-(M^6) \oplus \Omega^3_6(M^6) \oplus \Omega^3_{12}(M^6),$$

with

$$\Omega^3_+(M^6) = \{f\psi_+ | \, f \in \mathcal{C}^\infty(M^6)\}, \quad \Omega^3_6(M^6) = \{\eta \wedge \omega | \, \eta \in \Omega^1(M^6)\} = \{\gamma \in \Omega^3(M^6) | *_6 J\gamma = \gamma\},$$
$$\Omega^3_-(M^6) = \{f\psi_- | \, f \in \mathcal{C}^\infty(M^6)\}, \quad \Omega^3_{12}(M^6) = \{\gamma \in \Omega^3(M^6) | \, \gamma \wedge \omega = 0, \gamma \wedge \psi_\pm = 0\},$$

where $*_6$ denotes the Hodge star operator associated to the induced metric $g_{\omega,\psi_\pm}$ and the volume form vol_6. Notice that $\Omega^k_d(M^6)$ denotes the SU(3)-irreducible space of k-forms having dimension d. Decompositions of the spaces of k-forms for $k = 4, 5$ and 6 need not to be detailed since they can be achieved via the Hodge star operator, $*_6\Omega^k_d(M^6) = \Omega^{6-k}_d(M^6)$.

With all these previous descriptions the derivatives of ω, ψ_+ and ψ_- can be decomposed into summands belonging to the SU(3)-invariant spaces as follows (see [4] for details):

$$\begin{aligned} d\omega &= \tfrac{-3}{2}\sigma_0\psi_+ + \tfrac{3}{2}\pi_0\psi_- + \nu_1 \wedge \omega + \nu_3, \\ d\psi_+ &= \pi_0\,\omega^2 + \pi_1 \wedge \psi_+ - \pi_2 \wedge \omega, \\ d\psi_- &= \sigma_0\,\omega^2 + \pi_1 \wedge \psi_- - \sigma_2 \wedge \omega, \end{aligned} \tag{2}$$

where $\sigma_0, \pi_0 \in \mathcal{C}^\infty(M^6), \pi_1, \nu_1 \in \Omega^1(M^6), \pi_2, \sigma_2 \in \Omega^2_8(M^6)$ and $\nu_3 \in \Omega^3_{12}(M^6)$ are the *torsion forms* of the SU(3)-structure.

Some classes of SU(3)-structures that are useful for our purposes are given in Table 1.

Table 1
Some classes of SU(3)-structures.

Class	Non-vanishing torsion forms	Structure
$\{0\}$	–	Calabi-Yau
$\mathcal{W}_1^-$	σ_0	Nearly Kähler
$\mathcal{W}_2^-$	σ_2	Symplectic half-flat
$\mathcal{W}_3$	ν_3	Balanced

1.2. G$_2$-*structures*

A G$_2$-structure on a 7-dimensional differentiable manifold consists on a three form φ defining a metric, namely g_φ, a volume form vol_7 and a 2-fold vector cross product, see [7,14]. The metric g_φ can be recovered from φ as

$$g_\varphi(X,Y)vol_7 = \frac{1}{6}\,(\iota_X\varphi) \wedge (\iota_Y\varphi) \wedge \varphi,$$

with $X, Y \in \mathfrak{X}(M^7)$. The presence of such structure on a manifold M^7 can be characterized by the existence of an adapted basis, i.e. a local basis of 1-forms $\{e^1, \dots, e^7\}$ such that φ can be described as:

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

Concerning the intrinsic torsion of a G$_2$-structure, namely $\mathcal{T}$, in [7] it is described how this torsion lies in a space of the form

$$\mathcal{T} \in \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4,$$

where $\mathcal{X}_i$ denotes the irreducible components under the action of the group G_2. Thus, we can distinguish between 16 different classes of G_2-structures, the so-called *Fernández-Gray classes*, which can be characterized by the behavior of the exterior derivative of φ and $*_7\varphi$ where $*_7$ is the Hodge star operator induced by the G_2-structure. In [3] it is given a description of the derivatives of φ and $*_7\varphi$ as summands belonging to the different G_2-invariant spaces $\mathcal{X}_i$.

In order to obtain this description it is considered the natural action of the group G_2 on $\Omega^k(M^7)$. Thus, the different spaces of forms $\Omega^k(M^7)$ can be splitted into G_2-irreducible subspaces as follows:

$$\Omega^1(M^7) \text{ is irreducible,}$$
$$\Omega^2(M^7) = \Omega^2_7(M^7) \oplus \Omega^2_{14}(M^7),$$

with

$\Omega^2_7(M^7) = \{*_7(\eta \wedge *_7\varphi) | \, \eta \in \Omega^1(M^7)\} = \{\sigma \in \Omega^2(M^7) | \sigma \wedge \varphi = 2 *_7 \sigma\}$,
$\Omega^2_{14}(M^7) = \{\sigma \in \Omega^2(M^7) | \, \sigma \wedge \varphi = - *_7 \sigma\}$;

and

$$\Omega^3(M^7) = \Omega^3_1(M^7) \oplus \Omega^3_7(M^7) \oplus \Omega^3_{27}(M^7),$$

with

$$\Omega^3_1(M^7) = \{f\varphi | \, f \in \mathcal{C}^\infty(M^7)\}, \qquad \Omega^3_{27}(M^7) = \{\gamma \in \Omega^3(M^7) | \, \gamma \wedge \varphi = \gamma \wedge *_7\varphi = 0\}.$$
$$\Omega^3_7(M^7) = \{*_7(\eta \wedge \varphi) | \, \eta \in \Omega^1(M^7)\},$$

Similarly to the previous case, $\Omega^k_d(M^7)$ denotes the G_2-irreducible space of k-forms which has dimension d. For the rest of dimensions ($k = 4, 5, 6$ and 7) use the relation: $*_7\Omega^k_d(M^7) = \Omega^{7-k}_d(M^7)$.

Thus, the derivatives of φ and $*_7\varphi$ can be decomposed into summands belonging to the G_2-invariant spaces as follows (see [3]):

$$d\,\varphi = \tau_0 *_7 \varphi + 3\tau_1 \wedge \varphi + *_7\tau_3, \qquad d(*_7\varphi) = 4\tau_1 *_7 \varphi + \tau_2 \wedge \varphi, \tag{3}$$

where $\tau_0 \in \mathcal{C}^\infty(M^7), \tau_1 \in \Omega^1(M^7), \tau_2 \in \Omega^2_{14}(M^7)$ and $\tau_3 \in \Omega^3_{27}(M^7)$ are the torsion forms.

In particular:

$$\begin{aligned} \tau_0 &= \frac{1}{7} *_7 (d\varphi \wedge \varphi), & \tau_2 &= - *_7 d *_7 \varphi + 4 *_7 (\tau_1 \wedge *_7\varphi), \\ \tau_1 &= \frac{-1}{12} *_7 (*_7 d\varphi \wedge \varphi), & \tau_3 &= *_7 d\varphi - \tau_0\varphi - 3 *_7 (\tau_1 \wedge \varphi). \end{aligned} \tag{4}$$

The principal Fernández-Gray classes are given in Table 2.

Table 2
Some classes of G_2-structures.

Class	Non-vanishing torsion forms	Structure
$\mathcal{P}$	$-$	Parallel
$\mathcal{X}_1$	τ_0	Nearly Parallel
$\mathcal{X}_2$	τ_2	Closed
$\mathcal{X}_3$	τ_3	Coclosed of pure type
$\mathcal{X}_4$	τ_1	Locally conformal parallel
$\mathcal{X}_1 \oplus \mathcal{X}_3$	τ_0, τ_3	Coclosed

2. Warped G_2-structures

Consider two Riemannian manifolds, namely (F, g_F) and (B, g_B), and f a non-vanishing real differentiable function on B. The warped product, denoted as $B \times_f F$, consists on the product manifold

$$W = B \times F$$

endowed with the metric $g_f = \pi_1^*(g_B) + f^2\pi_2^*(g_F)$ with π_1 and π_2 being the projections of W onto B and F respectively.

Starting from an SU(3)-structure $(\omega, \psi_\pm)$ over M^6, and considering a function $f \in \mathcal{C}^\infty(M^6)$ it is possible to construct a G_2-structure φ over $M^7 = M^6 \times S^1$ such that:

$$\varphi = f\,\omega \wedge ds + (\alpha\,\psi_+ - \beta\,\psi_-), \tag{5}$$

with s the coordinate on S^1 and α, $\beta \in \mathbb{R}$ satisfying $\alpha^2 + \beta^2 = 1$. Thus, the metric and the volume form of this G_2-structure are given in terms of the SU(3)-structure by:

$$g_\varphi = g_{\omega,\psi_\pm} + f^2 ds^2, \qquad vol_7 = f vol_6 \wedge ds.$$

Observe that $g_\varphi = g_f$, so M^7 is in fact a warped product. In what follows we will call *warped* G_2*-structure* to this G_2-structure (5).

Remark 2.1. If we consider the pair $(\alpha, \beta) = (1, 0)$, this definition of warped G_2-structure is exactly the one already given in [11].

The metrics $g_{\omega,\psi_\pm}$ and g_φ on the base manifold M^6 and the warped product $M^6 \times_f S^1$ respectively define two star operators $*_6$ and $*_7$ related by the following:

Lemma 2.2 *(Lemma 3.2, [11]). Let $\eta \in \Omega^k(M^6)$ be a differential k-form on M^6, and let $*_6$ and $*_7$ be the Hodge star operator determined by the* SU(3)*-structure and the warped* G_2*-structure, respectively. Then*

$$*_7\eta = f *_6 \eta \wedge ds,$$

$$*_7(\eta \wedge ds) = (-1)^k f^{-1} *_6 \eta.$$

Hence from (5) and the previous lemma it can be checked that

$$*_7\varphi = \frac{1}{2}\omega^2 + f\,(\alpha\,\psi_- + \beta\,\psi_+) \wedge ds. \tag{6}$$

Remark 2.3. The key idea of this section is to study how the G_2-geometry of the warped product $M^6 \times_f S^1$ forces conditions on the SU(3)-geometry of the base M^6. Having this idea in mind, we are going to describe the torsion forms (4) of the warped G_2-structure in terms of the torsion forms of the SU(3)-structure and the warping function.

In the spirit of [20, Theorem 3.4] we can prove:

Theorem 2.4. *Let $(M^6, \omega, \psi_\pm)$ be an* SU(3)*-manifold with torsion forms $\pi_0, \sigma_0, \pi_1, \nu_1, \pi_2, \sigma_2$ and ν_3. Then, the torsion forms (4) of a warped* G_2*-manifold $(M^7 = M^6 \times_f S^1, \varphi)$ are given by*

$$\begin{aligned}
\tau_0 &= \tfrac{12}{7}(\alpha\pi_0 - \beta\sigma_0),\\
\tau_1 &= \tfrac{1}{2}(\alpha\sigma_0 + \beta\pi_0) f ds + \tfrac{1}{6}\eta_1,\\
\tau_2 &= -\alpha\sigma_2 - \beta\pi_2 + \tfrac{f}{3} *_6 \left(\eta_2 \wedge \omega^2\right) \wedge ds - \tfrac{1}{3} *_6 \left(\eta_2 \wedge (\alpha\psi_- + \beta\psi_+)\right),\\
\tau_3 &= \left[\tfrac{2}{7}(\alpha\pi_0 - \beta\sigma_0) f\omega - \tfrac{f}{2} *_6 \left(\eta_3 \wedge (\alpha\psi_+ - \beta\psi_-)\right) + f(\alpha\pi_2 - \beta\sigma_2)\right] \wedge ds - \tfrac{1}{2} *_6 (\eta_3 \wedge \omega) -\\
&\quad \tfrac{3}{14}(\alpha\pi_0 - \beta\sigma_0)(\alpha\psi_+ - \beta\psi_-) - *_6\nu_3,
\end{aligned} \tag{7}$$

where η_i are the following 1-forms:

$$\eta_1 = \frac{1}{f} d_6 f + \pi_1 + \nu_1, \quad \eta_2 = \frac{1}{f} d_6 f + \pi_1 - 2\nu_1, \quad \eta_3 = \frac{1}{f} d_6 f - \pi_1 + \nu_1.$$

Proof. The result holds after long computations where the definition of the spaces $\Omega^k_d(M^6)$ are used. As hint, let us write down the expressions for $d\varphi$, $*_7(d\varphi)$ and $d(*_7\varphi)$. From (5) and (6) one gets:

$$\begin{aligned}
d\varphi &= \left(df \wedge \omega - \frac{3}{2} f\sigma_0\psi_+ + \frac{3}{2} f\pi_0\psi_- + f\,\nu_1 \wedge \omega + f\,\nu_3\right) \wedge ds\\
&\quad + (\alpha\,\pi_0 - \beta\,\sigma_0)\omega^2 + \pi_1 \wedge (\alpha\psi_+ - \beta\psi_-) - (\alpha\,\pi_2 - \beta\,\sigma_2) \wedge \omega,\\
*_7(d\varphi) &= -f^{-1} *_6 (df \wedge \omega) + \frac{3}{2}\sigma_0\psi_- + \frac{3}{2}\pi_0\psi_+ - *_6(\nu_1 \wedge \omega) - *_6\,\nu_3\\
&\quad + [2f(\alpha\,\pi_0 - \beta\,\sigma_0)\omega + f *_6 (\pi_1 \wedge (\alpha\psi_+ - \beta\psi_-)) + \alpha\, f\,\pi_2 - \beta\, f\,\sigma_2] \wedge ds,\\
d(*_7\varphi) &= \nu_1 \wedge \omega^2 + \left[-f(\alpha\sigma_2 + \beta\pi_2) \wedge \omega + f(\alpha\sigma_0 + \beta\pi_0)\omega^2 + (df + f\pi_1) \wedge (\alpha\psi_- + \beta\psi_+)\right] \wedge ds.
\end{aligned}$$

Finally, from (4) and using Lemma 2.2 the result is achieved after long and standard computations. □

Most of the Fernández-Gray classes of G_2-structures are characterized in terms of the cancellation of some of their torsion forms (see Table 2). Using expressions (7), the cancellations of τ_0, τ_1, τ_2 and τ_3 are expressed by using the SU(3)-torsion forms of the base M^6 and the warping function f.

Corollary 2.5. *Let $(M^6, \omega, \psi_\pm)$ be an* SU(3)*-manifold. Thus, the torsion forms of the warped* G_2*-structure satisfy:*

$$\tau_0 = 0 \iff \left\{ i) \quad \alpha\pi_0 - \beta\sigma_0 = 0. \right.$$

$$\tau_1 = 0 \iff \begin{cases} ii) & \alpha\sigma_0 + \beta\pi_0 = 0,\\ iii) & \eta_1 = 0. \end{cases}$$

$$\tau_2 = 0 \iff \begin{cases} iv) & \eta_2 = 0,\\ v) & \alpha\sigma_2 + \beta\pi_2 = 0. \end{cases}$$

$$\tau_3 = 0 \iff \begin{cases} vi) & \alpha\pi_0 - \beta\sigma_0 = 0,\\ vii) & \eta_3 = 0,\\ viii) & \alpha\pi_2 - \beta\sigma_2 = 0,\\ ix) & \nu_3 = 0. \end{cases}$$

In Table 3 we show how the G_2-geometry of the warped product $M^6 \times_f S^1$ forces conditions on the SU(3)-geometry of the base M^6.

Table 3
Relation between torsion forms of the warped G_2-structure and the SU(3)-structure.

Class	G_2-torsion forms	SU(3)-torsion forms	Class
$\mathcal{P}$	$\tau_0 = \tau_1 = \tau_2 = \tau_3 = 0$	$\sigma_i = \pi_i = \nu_i = 0$ $d_6 f = 0$	0
$\mathcal{X}_2$	$\tau_0 = \tau_1 = \tau_3 = 0$	$\pi_0 = \sigma_0 = \pi_1 = \nu_3 = 0$ $\alpha\,\pi_2 - \beta\sigma_2 = 0$ $\frac{1}{f} d_6 f = -\nu_1$	$\mathcal{W}_2^{\pm} \oplus \mathcal{W}_4$
$\mathcal{X}_3$	$\tau_0 = \tau_1 = \tau_2 = 0$	$\pi_0 = \sigma_0 = \nu_1 = 0$ $\alpha\sigma_2 + \beta\pi_2 = 0$ $\frac{1}{f} d_6 f = -\pi_1$	$\mathcal{W}_2^{\pm} \oplus \mathcal{W}_3 \oplus \mathcal{W}_5$
$\mathcal{X}_4$	$\tau_0 = \tau_2 = \tau_3 = 0$	$\sigma_2 = \pi_2 = \nu_3 = 0$ $\frac{1}{f} d_6 f = \frac{1}{2}\nu_1 = \frac{1}{3}\pi_1$	$\mathcal{W}_1^{\pm} \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$
$\mathcal{X}_1 \oplus \mathcal{X}_3$	$\tau_1 = \tau_2 = 0$	$\alpha\sigma_0 + \beta\pi_0 = 0$ $\alpha\sigma_2 + \beta\pi_2 = 0$ $\nu_1 = 0,\ \frac{1}{f} d_6 f = -\pi_1$	$\mathcal{W}_1^{\pm} \oplus \mathcal{W}_2^{\pm} \oplus \mathcal{W}_3 \oplus \mathcal{W}_5$

Remark 2.6. From Corollary 2.5, $\tau_3 = 0$ implies $\tau_0 = 0$, therefore nearly Parallel structures can not be achieved as warped G_2-structures of the form (5).

3. The Laplacian flow and coflow of warped G_2-structure of the form $M^6 \times_f S^1$

Recall the definitions of the Laplacian flow and coflow, that are respectively:

$$(LF)\ \begin{cases} \dfrac{\partial}{\partial t}\varphi(t) = \Delta_t \varphi(t), \\ d_7\, \varphi(t) = 0, \end{cases} \qquad (LcF)\ \begin{cases} \dfrac{\partial}{\partial t}(*_t\varphi(t)) = -\Delta_t(*_t\varphi(t)), \\ d_7\,(*_t\varphi(t)) = 0, \end{cases}$$

where $\varphi(t)$ is a one-parameter family of G_2-structures and Δ_t, $*_t$ denote the Laplacian and the Hodge star operator induced by $\varphi(t)$ for every t.

Our objective in this section is to particularize the Laplacian flow and coflow considering one-parameter families of G_2-structures obtained as warped products, i.e.

$$\varphi(t) = f(t)\omega(t) \wedge ds + (\alpha\psi_+(t) - \beta\psi_-(t)). \tag{8}$$

From the previous expression, we derive the following:

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(t) &= \left(\frac{\partial}{\partial t} f(t)\,\omega(t) + f(t)\frac{\partial}{\partial t}\omega(t)\right) \wedge ds + \alpha\frac{\partial}{\partial t}\psi_+(t) - \beta\frac{\partial}{\partial t}\psi_-(t), \\ \frac{\partial}{\partial t}(*_7\varphi(t)) &= \left[\frac{\partial}{\partial t} f(t)\,(\beta\psi_+(t) + \alpha\psi_-(t)) + f(t)\left(\beta\frac{\partial}{\partial t}\psi_+(t) + \alpha\frac{\partial}{\partial t}\psi_-(t)\right)\right] \wedge ds + \frac{1}{2}\frac{\partial}{\partial t}\omega^2(t). \end{aligned} \tag{9}$$

Now we focus on the 3-form $\Delta_7\varphi$, resp. the 4-form $\Delta_7 *_7 \varphi$. For a generic G_2-structure, considering the formulas given in (3) of the exterior derivatives of φ and $*_7\varphi$, a description of the Laplacian in terms of the torsion forms can be given as

$$\Delta_7\varphi = d_7(\tau_2 - 4 *_7 (\tau_1 \wedge *_7\varphi)) + *_7 d_7(\tau_0\varphi + 3 *_7 (\tau_1 \wedge \varphi) + \tau_3). \tag{10}$$

Since the Laplacian commutes with the Hodge star operator, $\Delta_7 *_7 = *_7\Delta_7$, combining (7) and (10) it is also possible to describe $\Delta_7 *_7 \varphi$ of a warped G_2-structure in terms of the torsion forms of the SU(3)-structure and the warping function f for particular classes of G_2-structures.

Provided that we are interested in the Laplacian flow, resp. coflow, we consider the 3-form $\Delta_7\varphi$, resp. the 4-form $\Delta_7 *_7 \varphi$, when φ is closed, resp. coclosed. Let us start with the closed ones:

Proposition 3.1. *Let φ be a warped closed* G_2*-structure* (5) *on* $M^6 \times_f S^1$ *where* $(\omega, \psi_\pm)$ *is an* SU(3)*-structure on* M^6*. Then* $\Delta_7\varphi$ *has the following expression:*

$$\Delta_7\varphi = -d_6(\alpha\sigma_2 + \beta\pi_2) + d_6 *_6 \left(\nu_1 \wedge (\alpha\psi_- + \beta\psi_+)\right) + f\left[\nu_1 \wedge *_6(\nu_1 \wedge \omega^2) - d_6 *_6 (\nu_1 \wedge \omega^2)\right] \wedge ds,$$

where $\alpha\pi_2 - \beta\sigma_2 = 0$.

In the particular case that the warping function f *is constant* $(d_6 f = 0)$*, then*

$$\Delta_7\varphi = -d_6(\alpha\sigma_2 + \beta\pi_2).$$

Proof. Since φ is closed, $\tau_0 = \tau_1 = \tau_3 = 0$ and by (10)

$$\Delta_7\varphi = d_7\tau_2,$$

where in view of (7)

$$\tau_2 = -\alpha\sigma_2 - \beta\pi_2 + *_6(\nu_1 \wedge (\alpha\psi_- + \beta\psi_+)) - f *_6 (\nu_1 \wedge \omega^2) \wedge ds.$$

For the case f constant, since $\frac{1}{f} d_6 f = -\nu_1$ (see Table 3) then $\nu_1 = 0$ and the result holds. □

Consider now coclosed G_2-structures:

Proposition 3.2. *Let φ be a warped coclosed* G_2*-structure* (5) *on* $M^6 \times_f S^1$ *where* $(\omega, \psi_\pm)$ *is an* SU(3)*-structure on* M^6*. Then* $\Delta_7 *_7 \varphi$ *has the following expression:*

$$\begin{aligned}\Delta_7 *_7 \varphi = {} & \tfrac{3}{2}(\alpha\pi_0 - \beta\sigma_0)\left[(\alpha\pi_0 - \beta\sigma_0)\omega^2 + \pi_1 \wedge (\alpha\psi_+ - \beta\psi_-) - (\alpha\pi_2 - \beta\sigma_2) \wedge \omega\right] + d_6 *_6 (\pi_1 \wedge \omega) \\ & - d_6(*_6\nu_3) + \tfrac{3}{2} d_6(\alpha\pi_0 - \beta\sigma_0) \wedge (\alpha\psi_+ - \beta\psi_-) \\ & + f\Big[2d_6(\alpha\pi_0 - \beta\sigma_0) \wedge \omega + (\alpha\pi_0 - \beta\sigma_0)\left(-2\pi_1 \wedge \omega - 3\sigma_0\psi_+ + 3\pi_0\psi_- + 2\nu_3\right) + d_6(\alpha\pi_2 - \beta\sigma_2) \\ & - \pi_1 \wedge *_6\left(\pi_1 \wedge (\alpha\psi_+ - \beta\psi_-)\right) + d_6 *_6 \left(\pi_1 \wedge (\alpha\psi_+ - \beta\psi_-)\right) - \pi_1 \wedge (\alpha\pi_2 - \beta\sigma_2)\Big] \wedge ds,\end{aligned}$$

where $\alpha\sigma_i + \beta\pi_i = 0$ *for* $i = 0, 2$.

Moreover, if f *is constant, then*

$$\begin{aligned}\Delta_7 *_7 \varphi = {} & \tfrac{3}{2}(\alpha\pi_0 - \beta\sigma_0)\left((\alpha\pi_0 - \beta\sigma_0)\omega^2 - (\alpha\pi_2 - \beta\sigma_2) \wedge \omega\right) - d_6(*_6\nu_3) \\ & + \tfrac{3}{2} d_6(\alpha\pi_0 - \beta\sigma_0) \wedge (\alpha\psi_+ - \beta\psi_-) \\ & + f\Big[2d_6(\alpha\pi_0 - \beta\sigma_0) \wedge \omega + (\alpha\pi_0 - \beta\sigma_0)\left(-3\sigma_0\psi_+ + 3\pi_0\psi_- + 2\nu_3\right) + d_6(\alpha\pi_2 - \beta\sigma_2)\Big] \wedge ds.\end{aligned} \tag{11}$$

Proof. The condition φ being coclosed is equivalent to $\tau_1 = \tau_2 = 0$ and as a consequence of (10):

$$\Delta_7 *_7 \varphi = *_7\Delta_7\varphi = d_7(\tau_0\varphi + \tau_3).$$

Now, using (7):

$$\begin{aligned}\Delta_7 *_7 \varphi = d_7\Big[& f\Big(2(\alpha\pi_0 - \beta\sigma_0)\omega + *_6\left(\pi_1 \wedge (\alpha\psi_+ - \beta\psi_-)\right) + (\alpha\pi_2 - \beta\sigma_2)\Big) \wedge ds \\ & + \tfrac{3}{2}(\alpha\pi_0 - \beta\sigma_0)(\alpha\psi_+ - \beta\psi_-) + *_6(\pi_1 \wedge \omega) - *_6\nu_3\Big],\end{aligned}$$

and the result follows. In order to prove (11), observe that $\pi_1 = 0$ according to Table 3. □

Remark 3.3. In what follows, and similarly as in [11], we restrict our attention to the case of the warping function f is constant over the base manifold M^6.

In order to obtain solutions of the Laplacian flow of a warped closed G_2-structure, combining the expressions (9) and Proposition 3.1, we can set the system of equations that must be satisfied:

Proposition 3.4. *For a closed warped* G_2*-structure* (5)*, the equation of the Laplacian flow (LF) is equivalent to:*

$$\begin{cases} f'(t)\,\omega(t) + f(t)\dfrac{\partial}{\partial t}\omega(t) = 0, \\ \alpha\dfrac{\partial}{\partial t}\psi_+(t) - \beta\dfrac{\partial}{\partial t}\psi_-(t) = -d_6(\alpha\sigma_2(t) + \beta\pi_2(t)), \end{cases}$$

where $\alpha\pi_2(t) - \beta\sigma_2(t) = 0$*.*

Remark 3.5. For the particular case of $(\alpha, \beta) = (1, 0)$, we recover the system already studied by Fino and Raffero in [11, Prop. 5.2].

Similarly, for the coflow, we get the following system of equations:

Proposition 3.6. *For a coclosed warped* G_2*-structure* (5)*, the equation of the Laplacian coflow (LcF) is equivalent to:*

$$\begin{cases} \dfrac{\partial\omega^2(t)}{\partial t} = -3(\alpha\pi_0(t) - \beta\sigma_0(t))^2\omega^2(t) + 3(\alpha\pi_0(t) - \beta\sigma_0(t))(\alpha\pi_2(t) - \beta\sigma_2(t)) \wedge \omega(t) \\ \qquad\qquad +2d_6(*_6\nu_3(t)) - 3d_6(\alpha\pi_0(t) - \beta\sigma_0(t)) \wedge (\alpha\psi_+(t) - \beta\psi_-(t)), \\ \dfrac{f'(t)}{f(t)}\,(\beta\psi_+(t) + \alpha\psi_-(t)) + \left(\beta\dfrac{\partial\psi_+(t)}{\partial t} + \alpha\dfrac{\partial\psi_-(t)}{\partial t}\right) = \\ \qquad\qquad -(\alpha\pi_0(t) - \beta\sigma_0(t))\,[-3\sigma_0(t)\psi_+(t) + 3\pi_0(t)\psi_-(t) + 2\nu_3(t)] \\ \qquad\qquad -d_6(\alpha\pi_2(t) - \beta\sigma_2(t)) - 2d_6(\alpha\pi_0(t) - \beta\sigma_0(t)) \wedge \omega(t), \end{cases}$$

where $\alpha\sigma_i(t) + \beta\pi_i(t) = 0$ *for* $i = 0, 2$*.*

Corollary 3.7. *For the particular case of* $(\alpha, \beta) = (0, 1)$*, the Laplacian coflow becomes:*

$$\begin{cases} \dfrac{\partial\omega^2(t)}{\partial t} = -3\sigma_0(t)^2\omega^2(t) + 3\sigma_0(t)\sigma_2(t) \wedge \omega(t) + 2d_6(*_6\nu_3(t)) - 3d_6\sigma_0(t) \wedge \psi_-(t), \\ \dfrac{f'(t)}{f(t)}\psi_+(t) + \dfrac{\partial\psi_+(t)}{\partial t} = -3\sigma_0(t)^2\psi_+(t) + 2\sigma_0(t)\nu_3(t) + d_6\sigma_2(t) + 2d_6\sigma_0(t) \wedge \omega(t). \end{cases} \tag{12}$$

Remark 3.8. For the Laplacian coflow we chose the parameters (α, β) to be $(0, 1)$ in order to obtain equations depending on the torsion forms σ_0, σ_2 and ν_3 (see (2)) which are the ones that appear in the canonical definitions of the SU(3)-structures, nearly Kähler, symplectic half-flat and balanced, respectively (see equations (19), (22) and (28) in the next sections).

4. New solutions to the Laplacian coflow

Our main objective is to provide new solutions $\varphi(t)$ for the Laplacian coflow (12). In what follows we will consider one parameter families of warped G_2-structures (8) on $G \times S^1$, being G a Lie group. The underlying SU(3)-structures $(\omega(t), \psi_+(t), \psi_-(t))$ are left-invariant and can be locally described as

$$\begin{aligned} \omega(t) &= x^{12} + x^{34} + x^{56}, \\ \psi_+(t) = x^{135} - x^{146} - x^{236} - x^{245}, &\quad \psi_-(t) = -x^{246} + x^{235} + x^{145} + x^{136}, \end{aligned} \tag{13}$$

where $\{x^i(t)\}$ denotes for every t a local adapted basis, x^{ij} stands for $x^i(t) \wedge x^j(t)$ and x^{ijk} stands for $x^i(t) \wedge x^j(t) \wedge x^k(t)$. Our ansatz consists on stating that

$$x^i(t) = f_i(t)h^i, \tag{14}$$

where $f_i(t)$ are differentiable non-vanishing real functions satisfying $f_i(0) = 1$ and $\{h^1, \dots, h^6\}$ is an adapted basis for the SU(3)-structure for $t = 0$. Notice that (14) defines in fact a global basis since we are considering parallelizable manifolds.

Observe that the volume induced by $\varphi(t)$ is given by $vol_7(t) = f(t)vol_6(t) \wedge ds$ where

$$vol_6(t) = x^{123456}(t) = \prod_{i=1}^{6} f_i(t)h^{123456} = \prod_{i=1}^{6} f_i(t)vol_6,$$

that is

$$vol_7(t) = \left(\prod_{i=1}^{6} f_i(t)\right) f(t)\, vol_6 \wedge ds. \tag{15}$$

Direct computations show:

$$\frac{\partial \omega(t)}{\partial t} = \sum_{k=1}^{3} \left(\frac{f'_{2k-1}(t)}{f_{2k-1}(t)} + \frac{f'_{2k}(t)}{f_{2k}(t)}\right) x^{2k-1}(t) \wedge x^{2k}(t). \tag{16}$$

$$\frac{\partial \omega^2(t)}{\partial t} = 2\textstyle\sum_{(i,j,k,l)\in\mathcal{J}} \left(\frac{f'_i(t)}{f_i(t)} + \frac{f'_j(t)}{f_j(t)} + \frac{f'_k(t)}{f_k(t)} + \frac{f'_l(t)}{f_l(t)}\right) x^{ijkl}, \tag{17}$$

with $\mathcal{J} = \{(1,2,3,4), (1,2,5,6), (3,4,5,6)\}$.

$$\begin{aligned} \frac{f'(t)}{f(t)}\psi_+(t) + \frac{\partial \psi_+(t)}{\partial t} &= \left(\frac{f'(t)}{f(t)} + \frac{f'_1(t)}{f_1(t)} + \frac{f'_3(t)}{f_3(t)} + \frac{f'_5(t)}{f_5(t)}\right) x^{135} \\ &\quad - \textstyle\sum_{(i,j,k)\in\mathcal{I}} \left(\frac{f'(t)}{f(t)} + \frac{f'_i(t)}{f_i(t)} + \frac{f'_j(t)}{f_j(t)} + \frac{f'_k(t)}{f_k(t)}\right) x^{ijk}, \end{aligned} \tag{18}$$

with $\mathcal{I} = \{(1,4,6), (2,3,6), (2,4,5)\}$.

As we mentioned before, the G_2-geometry of the warped product imposes conditions on the SU(3)-geometry of the base M^6. Concretely, the G_2-structure is coclosed if and only if the corresponding SU(3)-structure lies on the space $\mathcal{W}_1^{\pm} \oplus \mathcal{W}_2^{\pm} \oplus \mathcal{W}_3 \oplus \mathcal{W}_5$ (see Table 3). Notice that if we consider a one-parameter family of SU(3)-structures $(\omega(t), \psi_{\pm}(t))$ belonging to the previous space for any t, then the corresponding warped G_2-structure will remain coclosed for any t. Moreover, in what follows we will impose that $(\omega(t), \psi_{\pm}(t))$ belongs to $\mathcal{W}_1^-$, $\mathcal{W}_2^-$ or $\mathcal{W}_3$ for any t. Now we particularize (12) for some interesting cases of SU(3)-structures lying on these particular subspaces.

4.1. The nearly Kähler case ($\mathcal{W}_1^-$)

Recall that a nearly Kähler SU(3)-structure satisfies

$$d\omega = -\frac{3}{2}\,\sigma_0\,\psi_+, \quad d\psi_+ = 0, \quad d\psi_- = \sigma_0\omega^2. \tag{19}$$

In particular, $\sigma_2 = \nu_3 = 0$. Particularizing (12) for $\sigma_2(t) = \nu_3(t) = 0$, we get

$$\begin{cases} \dfrac{\partial\omega^2(t)}{\partial t} = -3\sigma_0(t)^2\omega^2(t) - 3d_6\sigma_0(t)\wedge\psi_-(t), \\ \dfrac{f'(t)}{f(t)}\psi_+(t) + \dfrac{\partial\psi_+(t)}{\partial t} = -3\sigma_0(t)^2\psi_+(t) + 2d_6\sigma_0(t)\wedge\omega(t). \end{cases}$$

Observe that with this particular ansatz, the left-hand side of the first equation above is a combination of the 4-forms x^{1234}, x^{1256} and x^{3456} (see (17)); however, it can be easily proved that if η is a one-form, then $\eta\wedge\psi_-(t)$ never belongs to the space generated by x^{1234}, x^{1256} and x^{3456}, unless $\eta = 0$. Therefore, we need $d_6\sigma_0(t) = 0$, which means that $\sigma_0(t)$ is constant as a differentiable function on M^6.

Now, the previous system simplifies as:

$$\begin{cases} \dfrac{\partial\omega^2(t)}{\partial t} = -3\sigma_0(t)^2\omega^2(t), \\ \frac{f'(t)}{f(t)}\psi_+(t) + \dfrac{\partial\psi_+(t)}{\partial t} = -3\sigma_0(t)^2\psi_+(t). \end{cases} \tag{20}$$

Let us solve this system (as before, we denote $f_i(t)f_j(t)$ simply as f_{ij}).

Lemma 4.1. *If* $\dfrac{\partial\omega^2(t)}{\partial t} = -3\sigma_0(t)^2\,\omega^2(t)$, *then,* $f_{12} = f_{34} = f_{56}$, *where* $f_i(t)$ *are the functions in* (14).

Proof. Using the symplectic operator $L : \Omega^q(M) \to \Omega^{q+2}(M)$ defined by $L(\eta) = \eta\wedge\omega$, the previous equation can be expressed as:

$$\frac{\partial\omega^2(t)}{\partial t} + 3\sigma_0(t)^2\,\omega^2(t) = 0 \Longleftrightarrow L_t\left(2\,\frac{\partial\omega(t)}{\partial t} + 3\sigma_0(t)^2\,\omega(t)\right) = 0.$$

It happens that L is injective for $q \le n-1$, being $\dim M = 2n$ [2]. Since in our case $n = 3$, we have that

$$L_t\left(2\,\frac{\partial\omega(t)}{\partial t} + 3\sigma_0(t)^2\,\omega(t)\right) = 0 \Longleftrightarrow \frac{\partial\omega(t)}{\partial t} = -\frac{3}{2}\sigma_0(t)^2\,\omega(t).$$

Using (16), $\dfrac{\partial\omega(t)}{\partial t} = -\frac{3}{2}\sigma_0(t)^2\,\omega(t)$ if and only if

$$\left(\frac{f_1'(t)}{f_1(t)} + \frac{f_2'(t)}{f_2(t)}\right) = \left(\frac{f_3'(t)}{f_3(t)} + \frac{f_4'(t)}{f_4(t)}\right) = \left(\frac{f_5'(t)}{f_5(t)} + \frac{f_6'(t)}{f_6(t)}\right) = -\frac{3}{2}\sigma_0(t)^2,$$

which is equivalent to say

$$\frac{d}{dt}(\ln f_{12}) = \frac{d}{dt}(\ln f_{34}) = \frac{d}{dt}(\ln f_{56}) = -\frac{3}{2}\sigma_0(t)^2.$$

In particular,

$$\frac{f_{12}}{f_{34}} = c_1, \quad \frac{f_{12}}{f_{56}} = c_2, \quad \frac{f_{34}}{f_{56}} = c_3,$$

where c_i are constants. Since $f_i(0) = 1$, we obtain that $f_{12} = f_{34} = f_{56}$. □

For the second equation we get:

Lemma 4.2. *If* $\frac{f'(t)}{f(t)}\psi_+(t) + \dfrac{\partial\psi_+(t)}{\partial t} = -3\sigma_0(t)^2\,\psi_+(t)$*, then,* $f_1(t) = f_2(t)$*,* $f_3(t) = f_4(t)$*,* $f_5(t) = f_6(t)$*, where* $f_i(t)$ *are the functions in* (14)*.*

Proof. Arguing as before, if $\frac{f'(t)}{f(t)}\psi_+(t) + \dfrac{\partial\psi_+(t)}{\partial t} = -3\sigma_0(t)^2\,\psi_+(t)$, then:

$$\frac{d}{dt}(\ln(f(t)f_{135})) = \frac{d}{dt}(\ln(f(t)f_{146})) = \frac{d}{dt}(\ln(f(t)f_{236})) = \frac{d}{dt}(\ln(f(t)f_{245})) = -3\sigma_0(t)^2.$$

In particular, observe that:

$$\frac{d}{dt}(\ln(f(t)f_{ijk})) = \frac{d}{dt}(\ln(f(t)f_{ipq})) \iff \frac{d}{dt}\left(\ln\frac{f(t)f_{ijk}}{f(t)f_{ipq}}\right) = 0 \iff \ln\frac{f_{jk}}{f_{pq}} = c \iff \frac{f_{jk}}{f_{pq}} = 1,$$

where c is a constant and we have used the fact that $f_i(0) = 1$. So:

$$\frac{d}{dt}(\ln(f(t)f_{135})) = \frac{d}{dt}(\ln(f(t)f_{146})) = \frac{d}{dt}(\ln(f(t)f_{236})) = \frac{d}{dt}(\ln(f(t)f_{245})) \iff$$
$$\begin{cases} f_{13} = f_{24}, & f_{14} = f_{23}, & f_{15} = f_{26}, \\ f_{16} = f_{25}, & f_{35} = f_{46}, & f_{36} = f_{45}, \end{cases}$$
$$\iff f_1(t)^2 = f_2(t)^2,\ f_3(t)^2 = f_4(t)^2,\ f_5(t)^2 = f_6(t)^2 \iff f_1(t) = f_2(t),\ f_3(t) = f_4(t),\ f_5(t) = f_6(t),$$

where for the last equivalence we have used that $f_i(t)$ are continuous functions satisfying $f_i(0) = 1$. □

We can combine the two previous results to conclude that $f_i(t) = f_j(t)$ for $i, j = 1, \ldots, 6$. If we denote $f_i(t) = F(t)$ for all $i = 1, \ldots, 6$, then $(\omega(t), \psi_\pm(t))$ has the particular form:

$$\omega(t) = F^2(t)\,\omega, \qquad \psi_+(t) = F^3(t)\,\psi_+, \qquad \psi_-(t) = F^3(t)\,\psi_-. \tag{21}$$

Lemma 4.3. *Let* $(\omega(t), \psi_\pm(t))$ *be the one-parameter family of* SU(3)*-structures given in* (21) *where* $(\omega, \psi_\pm)$ *is a nearly Kähler structure. Then* $(\omega(t), \psi_\pm(t))$ *is nearly Kähler for all* t *if and only if* $\sigma_0(t) = \frac{\sigma_0}{F(t)}$*.*

Proof. Equation (21) implies that $d\omega(t) = F^2(t)d\omega$, and $d\psi_-(t) = F^3(t)d\psi_-$. Since $(\omega, \psi_\pm)$ is nearly Kähler, one has

$$d\omega(t) = -\frac{3}{2}\sigma_0 F^2(t)\psi_+, \qquad d\psi_-(t) = \sigma_0 F^3(t)\omega^2,$$

or equivalently

$$d\omega(t) = -\frac{3}{2}\frac{\sigma_0}{F(t)}\psi_+(t) \quad \text{and} \quad d\psi_-(t) = \frac{\sigma_0}{F(t)}\omega^2(t).$$

Therefore, $\big(\omega(t), \psi_\pm(t)\big)$ is nearly Kähler for all t if and only if $\sigma_0(t) = \frac{\sigma_0}{F(t)}$, and the result follows. □

In the next result we show how to solve the Laplacian coflow in this particular case.

Proposition 4.4. *Let M^6 be a manifold endowed with a nearly Kähler structure $(\omega, \psi_\pm)$. Then the one-parameter family of warped* G_2*-structures on $M^6 \times_f S^1$ given by*

$$\varphi(t) = \left(1 - \frac{3\sigma_0^2}{2}\,t\right)^{3/2} (c\,\omega \wedge ds - \psi_-) \quad \textit{and} \quad *_t\,\varphi(t) = \left(1 - \frac{3\sigma_0^2}{2}\,t\right)^2 \left(\frac{1}{2}\omega^2 + c\psi_+ \wedge ds\right)$$

is a solution of the Laplacian coflow for $t \in \left(-\infty, \frac{2}{3\sigma_0^2}\right)$, being $f(t) = c\left(1 - \frac{3\sigma_0^2}{2}\,t\right)^{1/2}$, $c \in \mathbb{R}^*$.

Proof. From Lemmas 4.1, 4.2 and 4.3, the system (20) with $(\omega(t), \psi_\pm(t))$ nearly Kähler for all t is equivalent to

$$\begin{cases} 4F'(t)F(t) = -3\sigma_0^2, \\ \frac{f'(t)}{f(t)} F^2(t) + 3F'(t)F(t) = -3\sigma_0^2, \end{cases}$$

whose solution is

$$F(t) = \left(1 - \frac{3\sigma_0^2}{2}\,t\right)^{1/2}, \qquad f(t) = c\left(1 - \frac{3\sigma_0^2}{2}\,t\right)^{1/2}$$

and the result follows. □

Corollary 4.5. *In the conditions above, the volume form induced by the one-parameter family of warped* G_2*-structures on $M^6 \times_f S^1$ is such that*

$$\lim_{t \to T^-} vol_7(t) = 0,$$

where $T = \frac{2}{3\sigma_0^2}$ is the maximal existence time of the solution.

Proof. Just observe that, using (15), $vol_7(t) = c\left(1 - \frac{3\sigma_0^2}{2}\,t\right)^{7/2} vol_6 \wedge ds$. □

Remark 4.6. Not many examples of nearly Kähler manifolds are known. Recently, new complete examples on S^6 and $S^3 \times S^3$ have been described in [12] and [21]. Next we solve the Laplacian coflow using an explicit example of nearly Kähler structure appeared in [21].

Example 4.7. Consider the sphere S^3, viewed as the Lie group $\mathrm{SU}(2)$ with the basis of left-invariant one-forms $\{\lambda^1, \lambda^2, \lambda^3\}$ satisfying

$$d\lambda^1 = \lambda^{23}, \quad d\lambda^2 = -\lambda^{13}, \quad d\lambda^3 = \lambda^{12}.$$

Thus, $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is the Lie algebra of $S^3 \times S^3$ and its structure equations are:

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) = (\lambda^{23}, -\lambda^{13}, \lambda^{12}, \nu^{23}, -\nu^{13}, \nu^{12})$$

with $\{\nu^i\}$ the basis of left-invariant 1-forms on the second sphere. The pair (ω, ψ_+) with

$$\omega = \frac{\sqrt{3}}{18}(\lambda^1 \wedge \nu^1 + \lambda^2 \wedge \nu^2 + \lambda^3 \wedge \nu^3),$$

$$\psi_+ = \frac{\sqrt{3}}{54}(\lambda^{23}\wedge\nu^1 - \lambda^1\wedge\nu^{23} - \lambda^{13}\wedge\nu^2 + \lambda^2\wedge\nu^{13} + \lambda^{12}\wedge\nu^3 - \lambda^3\wedge\nu^{12}),$$

where ω is the Kähler form and ψ_+ is the real part of the complex (3,0)-form, defines a nearly Kähler SU(3)-structure on $S^3\times S^3$. Observe that the basis $\{\lambda^i,\,\nu^i\}$ is not adapted to the SU(3)-structure.

Consider $\{h^1,\dots,h^6\}$ the basis of left-invariant 1-forms on $S^3\times S^3$ given by

$$h^1 = \frac{1}{3}\lambda^1 - \frac{1}{6}\nu^1,\quad h^2 = \frac{\sqrt{3}}{6}\nu^1,\quad h^3 = \frac{1}{3}\lambda^2 - \frac{1}{6}\nu^2,\quad h^4 = \frac{\sqrt{3}}{6}\nu^2,\quad h^5 = \frac{\sqrt{3}}{6}\nu^3,\quad h^6 = -\frac{1}{3}\lambda^3 + \frac{1}{6}\nu^3.$$

This basis is adapted to the SU(3)-structure and (ω,ψ_+) turns out to be nearly Kähler with $\sigma_0 = -2$. Therefore, in view of Proposition 4.4, the one-parameter family of warped G_2-structures on $(S^3\times S^3)\times_f S^1$ given by

$$\varphi(t) = \Big(1-6\,t\Big)^{3/2}\Big[c(h^{12}+h^{34}+h^{56})\wedge ds + h^{246} - h^{235} - h^{136} - h^{145}\Big]$$

and

$$*_t\varphi(t) = \Big(1-6\,t\Big)^{2}\Big[h^{1234}+h^{1256}+h^{3456} + c(h^{135}-h^{146}-h^{236}-h^{245})\wedge ds\Big],$$

where $f(t) = c\,(1-6t)^{\frac{1}{2}}$, is a solution of the Laplacian coflow for all $t\in\big(-\infty,\frac{1}{6}\big)$.

4.2. The symplectic half-flat case ($\mathcal{W}_2^-$)

Recall that a symplectic half-flat SU(3)-structure satisfies

$$d\omega = 0,\quad d\psi_+ = 0,\quad d\psi_- = -\sigma_2\wedge\omega. \tag{22}$$

In particular, $\sigma_0 = \nu_3 = 0$. Particularizing (12) for $\sigma_0(t) = \nu_3(t) = 0$, we get

$$\begin{cases} \dfrac{\partial\omega^2(t)}{\partial t} = 0, \\ \dfrac{f'(t)}{f(t)}\psi_+(t) + \dfrac{\partial\psi_+(t)}{\partial t} = d_6\sigma_2(t). \end{cases} \tag{23}$$

Now, we get necessary conditions in order to solve the Laplacian coflow. Arguing similarly as Lemma 4.1 and providing that $\sigma_0(t) = 0$, it is straightforward to see that the first equation of (23) holds if and only if

$$f_2(t) = \frac{1}{f_1(t)},\quad f_4(t) = \frac{1}{f_3(t)},\quad f_6(t) = \frac{1}{f_5(t)}. \tag{24}$$

In this setting, the behaviour of the induced volumen is $vol_7(t) = f(t)vol_6\wedge ds$ (see (15)).

The following technical result, that makes use of equation (18), states how to solve the coflow in the symplectic half-flat case:

Lemma 4.8. *Consider a warped coclosed* G_2*-structure* φ *on* $M^6\times_f S^1$ *where* $(\omega,\psi_\pm)$ *is a symplectic half-flat* SU(3)*-structure. Then* $\varphi(t)$*, given by* (8)*, is a solution of the coflow* (23) *using the ansatz* (14) *if and only if* $f(t)$*,* $f_1(t)$*,* $f_3(t)$ *and* $f_5(t)$ *satisfy:*

$$\begin{cases} A_{135}(t) = \dfrac{f'(t)}{f(t)} + \dfrac{f_1'(t)}{f_1(t)} + \dfrac{f_3'(t)}{f_3(t)} + \dfrac{f_5'(t)}{f_5(t)}, & A_{146}(t) = \dfrac{f'(t)}{f(t)} + \dfrac{f_1'(t)}{f_1(t)} - \dfrac{f_3'(t)}{f_3(t)} - \dfrac{f_5'(t)}{f_5(t)}, \\ A_{236}(t) = \dfrac{f'(t)}{f(t)} - \dfrac{f_1'(t)}{f_1(t)} + \dfrac{f_3'(t)}{f_3(t)} - \dfrac{f_5'(t)}{f_5(t)}, & A_{245}(t) = \dfrac{f'(t)}{f(t)} - \dfrac{f_1'(t)}{f_1(t)} - \dfrac{f_3'(t)}{f_3(t)} + \dfrac{f_5'(t)}{f_5(t)}, \end{cases} \tag{25}$$

where functions $A_{135}(t), A_{146}(t), A_{236}(t), A_{245}(t)$ are such that

$$d_6\sigma_2(t) = A_{135}(t)x^{135} - A_{146}(t)x^{146} - A_{236}(t)x^{236} - A_{245}(t)x^{245},$$

and $(\omega(t), \psi_\pm(t))$ is symplectic half-flat for all t.

In order to obtain examples and inspired in the solutions given in Proposition 4.4, we will consider the functions $f_i(t)$ of potential type, i.e.

$$f_i(t) = (1 + kt)^{\alpha_i} \tag{26}$$

with α_i and k real numbers. Thus the solutions of the coflow are of the form:

$$\begin{aligned} \varphi(t) = f(t)\left[(1+kt)^{\alpha_1+\alpha_2}h^{12} + (1+kt)^{\alpha_3+\alpha_4}h^{34} + (1+kt)^{\alpha_5+\alpha_6}h^{56}\right] \wedge ds \\ - (1+kt)^{\alpha_2+\alpha_4+\alpha_6}h^{246} + (1+kt)^{\alpha_2+\alpha_3+\alpha_5}h^{235} + (1+kt)^{\alpha_1+\alpha_4+\alpha_5}h^{145} \\ + (1+kt)^{\alpha_1+\alpha_3+\alpha_6}h^{136}, \end{aligned} \tag{27}$$

where the basis $\{h^1, \ldots, h^6\}$ is defined in (14).

Next we solve the Laplacian coflow on unimodular solvable Lie algebras.

Example 4.9. Consider the Lie algebra $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ whose structure equations are

$$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1) := (0, 0, -h^{14}, -h^{13}, h^{25}, -h^{26}).$$

The corresponding connected and simply connected Lie group G admits a left-invariant symplectic half-flat structure which is given canonically by (1) in basis $\{h^i\}$. Let us consider a one-parameter family of SU(3)-structures given by (13) with $x^i(t) = f_i(t)h^i$ being $f_i(t)$ of potential type as in (26). The structure equations of $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ with respect to the time-dependent basis $\{x^i(t)\}$ are

$$(0, 0, -(1+kt)^{\alpha_3-\alpha_1-\alpha_4}x^{14}, -(1+kt)^{\alpha_4-\alpha_1-\alpha_3}x^{13}, (1+kt)^{-\alpha_2}x^{25}, -(1+kt)^{-\alpha_2}x^{26}).$$

In order to obtain solutions for the Laplacian coflow, and in view of (24), we can set

$$\alpha_2 = -\alpha_1, \qquad \alpha_4 = -\alpha_3, \qquad \text{and} \qquad \alpha_6 = -\alpha_5.$$

With these values, we impose the preservation of the symplectic half-flat condition. It is easy to verify that $d\omega(t) = 0$ for all t; $\psi_+(t)$ remains closed if and only if $\alpha_1 = \alpha_3 = 0$, since

$$d\psi_+(t) = \big(-(1+kt)^{\alpha_1} + (1+kt)^{-\alpha_1-2\alpha_3}\big)x^{1235} + \big(-(1+kt)^{\alpha_1} + (1+kt)^{-\alpha_1+2\alpha_3}\big)x^{1246}.$$

So, $(\omega(t), \psi_\pm(t))$ is symplectic half-flat for all t if and only if $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. Observe that the structure equations are simply:

$$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1) := (0, 0, -x^{14}, -x^{13}, x^{25}, -x^{26}).$$

Finally, to solve the second equation of (23) we make use of (25). Since $(\omega(t), \psi_\pm(t))$ is symplectic half-flat for all t, $\sigma_2(t) = -*_t\, d\psi_-(t)$, see (2), and therefore

$$d\sigma_2(t) = -2x^{135} + 2x^{146} + 2x^{236} + 2x^{245},$$

which means that $A_{ijk}(t) = -2$. We obtain the system

$$\begin{cases} \frac{f'(t)}{f(t)} + k\alpha_5(1+kt)^{-1} = -2, \\ \frac{f'(t)}{f(t)} - k\alpha_5(1+kt)^{-1} = -2, \end{cases}$$

which can be solved taking

$$\alpha_5 = 0 \qquad \text{and} \qquad f(t) = c\,e^{-2t}, \quad c \in \mathbb{R}^*.$$

Therefore, the one-parameter family of G_2-structures on $G \times_f S^1$ given by (27)

$$\varphi(t) = c\,e^{-2t}(h^{12} + h^{34} + h^{56}) \wedge ds - h^{246} + h^{235} + h^{145} + h^{136}$$

is a solution of the Laplacian coflow for all $t \in \mathbb{R}$. Since $\lim_{t\to T} f(t) = 0$, where $T = +\infty$ is the maximal existence time of the solution, we obtain that $\lim_{t\to T} vol_7(t) = 0$.

In [10], the authors classify the 6-dimensional unimodular solvable Lie algebras admitting symplectic half-flat SU(3)-structure and show that all the corresponding solvable Lie groups admit a co-compact discrete subgroup. In addition to the Lie algebra $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$, in terms of an adapted basis $\{h^i\}_{i=1}^6$ to the SU(3)-structure, the structure equations of these algebras are the following:

$$\begin{aligned}
\mathfrak{g}_{5,1} \oplus \mathbb{R} &= (0,0,0,h^{15},0,h^{13}),\\
A_{5,7}^{-1,-1,1} \oplus \mathbb{R} &= (h^{16}, -h^{26}, -h^{36}, h^{46}, 0, 0),\\
A_{5,17}^{-a,-a,1} \oplus \mathbb{R} &= (ah^{15} + h^{35}, -ah^{25} + h^{45}, -h^{15} + ah^{35}, -h^{25} - ah^{45}, 0, 0),\\
\mathfrak{g}_{6,N3} &= (0, -2h^{35}, 0, -h^{15}, 0, h^{13}),\\
\mathfrak{g}_{6,38}^{0} &= (2h^{36}, 0, -h^{26}, h^{25} - h^{26}, -h^{23} - h^{24}, h^{23}),\\
\mathfrak{g}_{6,54}^{0,-1} &= \left(\frac{h^{16}}{\sqrt{2}} + h^{45}, -\frac{h^{26}}{\sqrt{2}}, h^{25} - \frac{h^{36}}{\sqrt{2}}, \frac{h^{46}}{\sqrt{2}}, 0, 0\right),\\
\mathfrak{g}_{6,118}^{0,-1,-1} &= (-h^{15} + h^{36}, h^{25} + h^{46}, -h^{16} - h^{35}, -h^{26} + h^{45}, 0, 0).
\end{aligned}$$

In Table 4 we present long time solutions to the Laplacian coflow for G_2-structures obtained as warped products of solvmanifolds endowed with symplectic half-flat SU(3)-structures. These solutions can be obtained as follows: consider Lemma 4.8 with the potential functions given in (26) and a warping function also of potential type

$$f(t) = c\,(1+kt)^{\beta}, \quad c \in \mathbb{R}^*.$$

Thus, using (25), we obtain a linear system of equations in α_i, β and k that can be easily solved. Known the values of α_i, β and k and considering (27) we can give an explicit description of the solutions of the Laplacian coflow for each example. We also include the value of $d\sigma_2(t)$ in each case, necessary to compute the parameters of the solutions.

In particular, in any case $\lim_{t\to T^-} f(t) = 0$, where $T = \frac{-1}{k}$ is the maximal existence time of the solution, and therefore, $\lim_{t\to T^-} vol_7(t) = 0$.

Table 4
Solutions of the Laplacian coflow in the SHF-case.

Lie algebra	$d\sigma_2(t)$	$(\alpha_1, \ldots, \alpha_6)$	β	k
$\mathfrak{g}_{5,1} \oplus \mathbb{R}$	$A_{135} = -2(1+kt)^{-2\alpha_1-2\alpha_3-2\alpha_5}$	$(\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6})$	$\frac{1}{6}$	-3
$A_{5,7}^{-1,-1,1} \oplus \mathbb{R}$	$A_{146} = A_{236} = -4(1+kt)^{2\alpha_5}$	$(0, 0, 0, 0, -\frac{1}{2}, \frac{1}{2})$	$\frac{1}{2}$	-4
$A_{5,17}^{-a,-a,1} \oplus \mathbb{R}$	$A_{135} = A_{245} = -4a^2(1+kt)^{-2\alpha_5}$	$(0, 0, 0, 0, \frac{1}{2}, -\frac{1}{2})$	$\frac{1}{2}$	$-4a^2$
$\mathfrak{g}_{6,N3}$	$A_{135} = -6(1+kt)^{-2\alpha_1-2\alpha_3-2\alpha_5}$	$(\frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6})$	$\frac{1}{6}$	-9
$\mathfrak{g}_{6,38}^{0}$	$A_{236} = -6(1+kt)^{2\alpha_1-4\alpha_3}$	$(-\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{6})$	$\frac{1}{6}$	-9
$\mathfrak{g}_{6,54}^{0,-1}$	$A_{146} = A_{236} = -2(1+kt)^{2\alpha_5}$ $A_{245} = -2(1+kt)^{2\alpha_1+2\alpha_3-2\alpha_5}$	$(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$	$\frac{3}{2}$	-1
$\mathfrak{g}_{6,118}^{0,-1,-1}$	$A_{135} = A_{245} = -4(1+kt)^{-2\alpha_5}$ $A_{146} = A_{236} =$ $-2(1+kt)^{2\alpha_5}(-1+(1+kt)^{2\alpha_1-2\alpha_3})$	$(0, 0, 0, 0, \frac{1}{2}, -\frac{1}{2})$	$\frac{1}{2}$	-4

4.3. The balanced case ($\mathcal{W}_3$)

Recall that a balanced SU(3)-structure satisfies

$$d\omega = \nu_3, \quad d\psi_+ = 0, \quad d\psi_- = 0. \tag{28}$$

In particular, $\sigma_0 = \sigma_2 = 0$. Particularizing (12) for $\sigma_0(t) = \sigma_2(t) = 0$, we get

$$\begin{cases} \dfrac{\partial \omega^2(t)}{\partial t} = 2d_6(*_6\nu_3(t)), \\ \dfrac{f'(t)}{f(t)}\psi_+(t) + \dfrac{\partial \psi_+(t)}{\partial t} = 0. \end{cases} \tag{29}$$

In this case, we can apply Lemma 4.2 with $\sigma_0(t) = 0$ (compare the second equations in (20) and (29)) obtaining the same conclusion, i.e., $f_{2k}(t) = f_{2k-1}(t)$ for $k = 1, 2, 3$. Now, the behaviour of the induced volumen is $vol_7(t) = f_1(t)^2 f_3(t)^2 f_5(t)^2 f(t) vol_6 \wedge ds$.

Similarly to Lemma 4.8, we can set:

Lemma 4.10. *Consider a warped coclosed* G_2*-structure* φ *on* $M^6 \times_f S^1$ *where* $(\omega, \psi_\pm)$ *is a balanced* SU(3)*-structure. Then* $\varphi(t)$*, given by* (8)*, is a solution of the coflow* (29) *using the ansatz* (14) *if and only if* $f(t)$, $f_1(t)$, $f_3(t)$ *and* $f_5(t)$ *satisfy:*

$$B_{1234}(t) = 2\left(\frac{f_1'(t)}{f_1(t)} + \frac{f_3'(t)}{f_3(t)}\right), \quad B_{1256}(t) = 2\left(\frac{f_1'(t)}{f_1(t)} + \frac{f_5'(t)}{f_5(t)}\right), \quad B_{3456}(t) = 2\left(\frac{f_3'(t)}{f_3(t)} + \frac{f_5'(t)}{f_5(t)}\right),$$

where functions $B_{1234}(t)$, $B_{1256}(t)$, $B_{3456}(t)$ *are such that*

$$d_6(*\nu_3(t)) = B_{1234}(t)x^{1234} + B_{1256}(t)x^{1256} + B_{3456}(t)x^{3456},$$

and $(\omega(t), \psi_\pm(t))$ *is balanced for all* t.

The examples that we present in this case are the 6-dimensional nilpotent Lie algebras admitting balanced SU(3)-structures, that are classified in [22]. In terms of an adapted basis to the balanced SU(3)-structure, the structure equations are:

$$\begin{aligned} \mathfrak{h}_2 &= (0, 0, 0, 0, 2h^{12} + \left(2\sqrt{2} - 2\right) h^{13} + \left(-2 - 2\sqrt{2}\right) h^{24} - 2h^{34}, 4\sqrt{2}h^{12} + 4\sqrt{2}h^{23} - 4\sqrt{2}h^{34}), \\ \mathfrak{h}_3 &= (0, 0, 0, 0, 0, -2h^{12} + 2h^{34}), \\ \mathfrak{h}_4 &= (0, 0, 0, 0, 2h^{13}, h^{14} + h^{23}), \end{aligned}$$

$$\mathfrak{h}_5 = (0,0,0,0,h^{13} - h^{24}, h^{14} + h^{23}),$$
$$\mathfrak{h}_6 = (0,0,0,0,h^{13}, h^{14}),$$
$$\mathfrak{h}_{19}^- = (0,0,-h^{15}, -h^{25}, 0, -h^{13} - h^{24}).$$

We present long time solutions for the Laplacian coflow of G_2-structures obtained as warped products of balanced nilmanifolds endowed with SU(3)-structures. These solutions remain balanced for any t. As before, with the notation in Lemma 4.10 and functions of potential type (26) giving an explicit description of these solutions is equivalent to obtain the values of the parameters α_i, β and k. Solving the corresponding linear equations these values are given in Table 5. The solutions $\varphi(t)$ of the coflow are of the form (27). We also include the value of $d * \nu_3(t)$ in each case, necessary to compute the parameters of the solutions.

Table 5
Solutions of the Laplacian coflow in the balanced case.

Lie algebra	$d * \nu_3(t)$	$(\alpha_1, \ldots, \alpha_6)$	β	k
$\mathfrak{h}_2$	$B_{1234} = -128(1+kt)^{-4\alpha_1+2\alpha_5}$	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$	$-\frac{1}{6}$	-192
$\mathfrak{h}_3$	$B_{1234} = -8(1+kt)^{-4\alpha_1+2\alpha_5}$	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$	$-\frac{1}{6}$	-12
$\mathfrak{h}_4$	$B_{1234} = -6(1+kt)^{-2\alpha_1-2\alpha_3+2\alpha_5}$	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$	$-\frac{1}{6}$	-9
$\mathfrak{h}_5$	$B_{1234} = -4(1+kt)^{-2\alpha_1-2\alpha_3+2\alpha_5}$	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$	$-\frac{1}{6}$	-6
$\mathfrak{h}_6$	$B_{1234} = -2(1+kt)^{-2\alpha_1-2\alpha_3+2\alpha_5}$	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$	$-\frac{1}{6}$	-3
$\mathfrak{h}_{19}^-$	$B_{1234} = -2(1+kt)^{-2\alpha_1-2\alpha_3+2\alpha_5}$ $B_{1256} = -2(1+kt)^{-2\alpha_1+2\alpha_3-2\alpha_5}$	$(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0)$	$-\frac{1}{2}$	-2

Observe that in these cases, $\lim_{t\to T^-} vol_7(t) = \lim_{t\to T^-}(1+kt)^{2\alpha_1+2\alpha_3+2\alpha_5+\beta} = \lim_{t\to T^-}(1+kt)^{-\beta} = 0$, where $T = \frac{-1}{k}$ is the maximal existence time of the solution.

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