Testing for Seasonal Unit Roots in Periodic Integrated Autoregressive Processes

by

Tomas del Barrio Castro
University of Barcelona

and

Denise R Osborn
University of Manchester

October 2003

Keywords: Seasonality, unit root tests, HEGY tests, periodic processes

JEL: C22, C15.

This paper is preliminary. Please do not quote without the permission of the authors. The first author gratefully acknowledges financial assistance from The Agència de Gestió d'Ajusts Universitaris I de Recerca (Generalitat de Catalunya) under grant number 2002BEAI4000, and Comisión Interministerial de Ciencia y Tecnología SEC2002-00165. This research was undertaken whilst the first author was a visitor at the School of Economic Studies, University of Manchester.

ABSTRACT

This paper examines, both theoretically and through Monte Carlo analysis, the implications of applying the HEGY seasonal root tests to a process that is periodically integrated. As an important special case, the random walk process is also considered. It is established that, when the regression is not augmented, the asymptotic distribution of the zero frequency test statistic is shifted to the right in the periodic integration case, but not for the random walk. In practice, however, the HEGY zero frequency statistic performs quite well in terms of capturing the single unit root of the periodic process, but there may be a high probability of incorrectly concluding that the process is periodically integrated.

1.- Introduction.

Periodic autoregressive (*PAR*) models can arise naturally from the application of economic theory when the underlying economic driving forces, such as preferences or technologies, vary seasonally, as shown in Gersovitz and McKinnon (1978), Osborn (1988) or Hansen and Sargent (1993). These PAR models account for seasonality by allowing the parameters of the autoregressive process to change with the seasons of the year. As such, they are generalizations of the dummy variable approach that is widely applied in empirical analyses for seasonal economic data. In recent years a number of papers have contributed to the development of a statistical-kit for inference in *PAR* models and also to the exploration of their usefulness for the analysis of observed macroeconomic time series (mainly at the quarterly frequency); see Ghysels and Osborn (2001) and Franses (1996) for surveys.

Despite the attraction of *PAR* models from the perspective of economic decision-making in a seasonal context, the more prominent approach of empirical workers is to assume that the autoregressive coefficients, except for the intercept, are constant over the seasons of the year. These can be referred to as nonperiodic models, in contrast with the periodic case. Tiao and Grupe (1980) and Osborn (1991) study the properties of stationary *PAR* processes when analyzed as conventional nonperiodic *ARMA* processes, showing that a *PAR* process is (in general) converted into a process with seasonal autoregressive dynamics and a high order moving average component.

However, observed macroeconomic time series are typically nonstationary. Following the dominant use of the nonperiodic approach for empirical analyses, a stream of important research has examined the nature of nonstationarity in a seasonal context through the so-called seasonal unit root tests; see for example, Dickey, Hasza and Fuller (1984), Hylleberg, Engle, Granger and Yoo [HEGY] (1990), Smith and Taylor (1998). These tests take as their null hypothesis the proposition that nonstationary unit root behavior exists not only at the longrun (or zero) frequency, but also at all the seasonal frequencies. Although not always acknowledged in the seasonal unit root literature, the implication of these seasonal unit roots is that the seasons of the year are not cointegrated with each other, and hence "summer may become winter"; see Osborn (1991) or Ghysels and Osborn (2001). From an economic perspective, this implication may be unattractive.

An alternative type of seasonal nonstationary process is the so-called periodically integrated, or *PI,* process. This may be more plausible than the seasonally integrated process studied by HEGY and others, because it allows nonstationarity in conjunction with cointegration between the seasons of the year (Osborn, 1991, Franses, 1994). There is, however, an important gap in the literature, as little attention have been paid to the implications of testing for seasonal unit root when the underlying data generating process is of the *PI* type. To our knowledge, only the works of Boswijk and Franses (1996), Franses (1994, 1996) and Sanso *et al*. (1997) are partially related to this issue. Boswijk and Franses (1996) derive the distributions of Dickey, Hasza and Fuller (1984) test statistics for a *PI*(1) process, while Franses (1994, 1996) and Sanso *et al*. (1996) present Monte Carlo results for the performance of the HEGY test when applied to *PAR* processes.

The purpose of this paper is to analytically study the implications of applying the HEGY seasonal unit root testing procedure to a *PI*(1) data generating process. We obtain the asymptotic distributions of the normalized bias test and t-ratio statistics for the nonseasonal root in the HEGY tests. As a particular, but crucially important, special case of the *PI*(1) process when all the autoregressive coefficients are equal to one, the distributions of these statistics are obtained for a random walk process. In addition, we provide finite sample Monte Carlo results that support our analytical findings and allow us to examine the impact of different *PI* parameter values on the performance of the HEGY test.

The paper is organized as follows, in Section 2 we introduce preliminaries concerning *PI(*1) processes and the HEGY test needed for our analysis. The distribution of the statistics are obtained in Section 3, while Section 4 provides Monte Carlo results. Finally, Section 5 concludes.

2. Preliminaries.

2.1 Periodic Processes

A comprehensive survey on seasonal unit roots and *PAR* models can be found in Ghysels and Osborn (2001). Further, Osborn and Rodrigues (2002) provide a unifying approach for the asymptotics of seasonal unit root tests. Therefore, in the present section we do not repeat this background, but instead focus on the main points needed for the subsequent. Our notation follows that of Ghysels and Osborn (2001) and Osborn and Rodrigues (2002).

For simplicity of exposition, we assume a quarterly series with mean zero. We also restrict our attention to the *PAR*(1) case, since this keeps the analysis as simple as possible without losing any essential features. The *PAR*(1) process for this case is given by

$$y_{s\tau} = \alpha_s y_{s-1,\tau} + \varepsilon_{s\tau}, \qquad s = 1, 2, 3, 4, \tau = 1, 2, \ldots \qquad (1)$$

where, for observation $y_{s\tau}$, the first subscript refers to the season (s) and the second subscript to the year (τ). When $s = 1$, it is understood that $y_{s-1,\tau} = y_{4,\tau-1}$. Also for ease of exposition, we assume that observations are available for precisely N years, so that the total sample size is $T = 4N$. The *PAR* disturbance process $\varepsilon_{s\tau}$ is a zero mean *iid* process that may be heteroscedastic over seasons, but here we assume homoscedasticity with $E(\varepsilon_{s\tau}^2) = \sigma^2$.

The stationarity properties of the *PAR*(1) process are determined by the product $\alpha_1\alpha_2\alpha_3\alpha_4$. Specifically, the process is stationary when $|\alpha_1\alpha_2\alpha_3\alpha_4| < 1$ and is periodically integrated when $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. A periodic process that is integrated of order 1 is referred to as a *PI*(1) process (Osborn, Chiu, Smith and Birchenhall, 1988). Again for simplicity, the *PI*(1) process is assumed to have a zero starting value, so that $y_{04} = 0$.

Franses (1994) exploits the VAR representation of a periodic process, referring to it as the Vector of Quarters [VQ] representation. The VQ representation of the *PAR*(1) in (1) is:

$$\Phi_0 Y_\tau = \Phi_1 Y_{\tau-1} + U_\tau \qquad (2)$$

where $Y_\tau = (y_{1\tau}, y_{2\tau}, y_{2\tau}, y_{4\tau})'$, $U_\tau = (\varepsilon_{1\tau}, \varepsilon_{2\tau}, \varepsilon_{3\tau}, \varepsilon_{4\tau})'$, $E(U_\tau U_\tau') = \sigma^2 I$ and

$$\Phi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\alpha_2 & 1 & 0 & 0 \\ 0 & -\alpha_3 & 1 & 0 \\ 0 & 0 & -\alpha_4 & 1 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As discussed by Ghysels and Osborn (2001), the key to the type of integration exhibited by a *PAR* process is the final equation representation

$$|\Phi_0 - \Phi_1 B| Y_\tau = adj(\Phi_0 - \Phi_1 B) U_\tau, \qquad (3)$$

where *adj*(.) indicates the adjoint matrix for the expression in parentheses and *B* is the annual backshift (or lag) operator such that $BY_\tau = Y_{\tau-1}$.

When the *PAR*(1) is integrated of order 1, *PI*(1), (3) becomes

$$(1 - B)Y_\tau = (\Theta_0 + \Theta_1 B)\, U_\tau \qquad (4)$$

with

$$\Theta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha_2 & 1 & 0 & 0 \\ \alpha_2\alpha_3 & \alpha_3 & 1 & 0 \\ \alpha_2\alpha_3\alpha_4 & \alpha_3\alpha_4 & \alpha_4 & 1 \end{bmatrix}, \quad \Theta_1 = \begin{bmatrix} 1 & \alpha_1\alpha_3\alpha_4 & \alpha_1\alpha_4 & \alpha_1 \\ 0 & 1 & \alpha_1\alpha_2\alpha_4 & \alpha_1\alpha_2 \\ 0 & 0 & 1 & \alpha_1\alpha_2\alpha_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that (4) is a Vector MA [VMA] in the annual difference $(1 - B)Y_\tau = Y_\tau - Y_{\tau-1}$. This VMA process is noninvertible, because the matrix $C(B) = (\Theta_0 + \Theta_1 B)$ has three unit roots. Therefore, the rank of $C(1)$ is one and it is possible to write:

$$C(1) = \Theta_0 + \Theta_1 = ab' \qquad (5)$$

where

$$a = \begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_2\alpha_3 \\ \alpha_2\alpha_3\alpha_4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ \alpha_1\alpha_3\alpha_4 \\ \alpha_1\alpha_4 \\ \alpha_1 \end{bmatrix}.$$

It follows that there are three cointegration relationships between the four quarterly series of the *PI*(1) process, or equivalently there is a single common trend between values corresponding to the four quarters of the year (Boswijk and Franses, 1996, Franses, 1994, Ghysels and Osborn, 2001, Osborn, 1991). An important special case is the random walk process, with $\alpha_s = 1$ (s = 1, 2, 3, 4), which has $a = b = (1, 1, 1, 1)'$.

To summarize the main characteristics of a *PI*(1) process, we use a particular case of Lemma 1 in Boswijk and Franses (1996):

LEMMA 1. *Consider* Y_τ *in* (3) *with* $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. *Assuming* U_τ *is vector white noise with* $E(U_\tau U_\tau') = \sigma^2 I_4$, *then as* $N = T/4 \rightarrow \infty$:

$$\frac{1}{\sqrt{N}} Y_{[rN]} \Rightarrow B(r) = \begin{bmatrix} B_1(r) \\ B_2(r) \\ B_3(r) \\ B_4(r) \end{bmatrix} \qquad r \in [0, 1] \tag{6}$$

where $[rN]$ *is the integer part of* rN *and the* 4×1 *vector Brownian motion process* $B(r)$ *satisfies*

$$B(r) = ab'W(r) = \omega a\, w(r) \tag{7}$$

with $W(r)$ *being* 4×1 *vector Brownian motion with variance matrix* $\sigma^2 I_4$, $\omega = \sigma b'b$ *and* $w(r)$ *is scalar standard Brownian motion. Finally* $\Rightarrow$ *means convergence in distribution.*

For a proof of this Lemma, see Boswijk and Franses (1996).

To conclude the preliminaries of *PAR* processes, we present the constant parameter representation of a *PAR*(1) obtained by Osborn (1991) from results initially due to Tiao and Grupe (1980). This constant parameter representation is the one that applies if the *PAR*(1) process is analyzed as a conventional *ARMA* one and is obtained from the final equation representation of (3). For quarter s, equation (3) is:

$$y_{s\tau} = \alpha_s \alpha_{s-1} \alpha_{s-2} \alpha_{s-3} y_{s,\tau-1} + \varepsilon_{s\tau} + \alpha_s \varepsilon_{s-1,\tau} + \alpha_s \alpha_{s-1} \varepsilon_{s-2,\tau} + \alpha_s \alpha_{s-1} \alpha_{s-2} \varepsilon_{s-3,\tau} \qquad (8)$$

which can also be obtained directly from (1) by repeated substitution. As seen from (8), the (annual lag) autoregressive coefficient is $\alpha_1\alpha_2\alpha_3\alpha_4$ for all four quarters, but the MA(3) term is seasonally varying. In effect, the constant parameter representation averages these processes over s = 1, 2, 3, 4, resulting in a time invariant *SAR*(1)×*MA*(3) representation.

When the *PAR*(1) is integrated, then $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ and (8) becomes

$$\Delta_4 y_{s\tau} = \varepsilon_{s\tau} + \alpha_s \varepsilon_{s-1,\tau} + \alpha_s \alpha_{s-1} \varepsilon_{s-2,\tau} + \alpha_s \alpha_{s-1} \alpha_{s-2} \varepsilon_{s-3,\tau}, \qquad (9)$$

which is equivalent to (4). The constant parameter representation is then an *MA*(3) in $\Delta_4 y_{s\tau}$. Although the annual difference autoregressive operator suggests the presence of four unit roots (that is, the zero frequency and all three seasonal unit roots), this is not the case because the VMA of (4) has three noninvertible unit roots, as noted above. The purpose of the analysis of the present paper is to clarify the consequences of applying the HEGY seasonal unit root tests to such a *PI*(1) process.

2.2 The HEGY Test

The basic regression for the HEGY test, with no deterministic terms and no augmentation, is:

$$\Delta_4 y_{s\tau} = \pi_1 y^{(1)}_{s-1\tau} + \pi_2 y^{(2)}_{s-1\tau} + \pi_3 y^{(3)}_{s-2\tau} + \pi_4 y^{(3)}_{s-1\tau} + \varepsilon_{s\tau} \tag{10}$$

where, as in (9) above, $\Delta_4 = 1 - L^4$ with L the usual lag operator ($Ly_{s\tau} = y_{s-1,\tau}$), and $y^{(1)}_{s\tau}$, $y^{(2)}_{s\tau}$, $y^{(3)}_{s\tau}$ are the auxiliary variables associated with the roots of (1 - L), (1 + L) and (1 + L^2) of the seasonal difference operator $\Delta_4 = 1 - L^4 = (1 - L)(1 + L)(1 + L^2)$. Specifically,

$$\begin{aligned} y^{(1)}_{s\tau} &= (1+L)(1+L^2)y_{s\tau} \\ y^{(2)}_{s\tau} &= -(1-L)(1+L^2)y_{s\tau} \\ y^{(3)}_{s\tau} &= -(1-L)(1+L)y_{s\tau} \end{aligned} \tag{11}$$

(Note the distinction made between L operating on the quarter here and B in (3)/(4) above that operates on the year for the vector Y_τ or U_τ.)

The overall HEGY null hypothesis of seasonal integration, $y_{s\tau} \sim SI(1)$, implies the presence of unit roots at the zero frequency (captured through π_1) and at the seasonal frequencies (captured through π_2, π_3 and π_4), so that $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$. Hence, in this simple case, the null hypothesis implies the seasonal random walk $\Delta_4 y_{s\tau} = \varepsilon_{s\tau}$, with the asymptotic distribution of the vector Y_τ therefore given by $N^{-0.5}Y_{[rN]} \Rightarrow W(r)$, where $W(r)$ is the 4×1 vector Brownian motion process of Lemma 1; see, for example, Ghysels and Osborn (2001). Thus, the principal difference between a $PI(1)$ and an $SI(1)$ process is that there are three cointegrating relationships between the four quarters for the $PI(1)$ and no contegration between the quarters for the $SI(1)$ process.

As discussed by HEGY, the regressors in (10) are, by construction, asymptotically orthogonal under the seasonal integration null hypothesis. Thus, the associated asymptotic distributions of the HEGY test statistics can be obtained considering the three factors of Δ_4 one by one. Under the overall HEGY null hypothesis, the normalized bias and *t*-ratio statistics for testing the null of a unit root at the zero frequency, namely $\pi_1 = 0$, are given by:

$$T\hat{\pi}_1 = \frac{T^{-1}\sum_{s=1}^{4}\sum_{t=1}^{\tau}\Delta_4 y_{s\tau} y_{s-1,\tau}^{(1)}}{T^{-2}\sum_{s=1}^{4}\sum_{t=1}^{\tau}\left(y_{s-1,\tau}^{(1)}\right)^2} \quad (12.1)$$

$$t_{\hat{\pi}_1} = \frac{T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{\tau}\Delta_4 y_{s\tau} y_{s-1,\tau}^{(1)}}{\sqrt{\hat{\sigma}^2 T^{-2}\sum_{s=1}^{4}\sum_{\tau=1}^{\tau}\left(y_{s-1,\tau}^{(1)}\right)^2}} \quad (12.2)$$

where $\hat{\sigma}^2$ is the degrees of freedom corrected OLS estimator of σ^2. Osborn and Rodrigues (2002) show that the asymptotic distributions for these statistics under the HEGY null hypothesis can be written as

$$T\,\hat{\pi}_1 \Rightarrow \frac{\int W'(r)\,C_1\,dW(r)}{\int W'(r)C_1\,W(r)dr} = \frac{\int w(r)dw(r)}{\int w(r)^2\,dr} \quad (13.1)$$

$$t_{\hat{\pi}_1} \Rightarrow \frac{\int W'(r)\,C_1\,dW(r)}{\sqrt{\int W'(r)C_1\,W(r)dr}} = \frac{\int w(r)dw(r)}{\sqrt{\int w(r)^2\,dr}} \quad (13.2)$$

where $W(r)$ are $w(r)$ are as defined in Lemma 1 above and C_1 is a 4×4 matrix with all elements equal to one.

3. Asymptotic Distributions

In the section we derive the asymptotic distributions of the HEGY normalized bias and *t*-ratio statistics. Before turning to the HEGY regression (10) when the true DGP is *PI*(1), we analyse (in subsection 3.1) the distributions of the normalized bias and *t*-ratio statistics in the regression

$$\Delta_4 y_{s\tau} = \pi_1 y^{(1)}_{s-1\tau} + v_{s\tau}\,. \tag{14}$$

The regression in (10) provides a test of the zero frequency unit root, while maintaining the presence of all seasonal unit roots. The orthogonality of the regressors in (10) under the seasonal integration null hypothesis implies that, under that null, the same asymptotic distributions apply whether (10) or (14) is used; see, for example, the discussion in Ghysels and Osborn (2001). Therefore, an applied researcher analysing seasonal data may use the regression (14) in an attempt to side-step seasonality issues and concentrate on the zero frequency unit root properties of the data.

Subsequently (subsection 3.2), we consider the important special case of a random walk DGP, where $y_{s\tau} \sim I(1)$, in the context of the complete HEGY regression (10). The general case of $y_{s\tau} \sim PI(1)$ is discussed in subsection 3.3. We find that it is important to distinguish the random walk and *PI*(1) cases, because the distributions of the test statistics $T\hat{\pi}_1$ and $t_{\hat{\pi}_1}$ are the same for a *PI*(1) process whether (10) or (14) is used, but this situation does not hold in the random walk special case. Therefore, although a random walk can be considered as a special case of the *PI*(1) process of (3) with all coefficients equal to unity, this special case has distinctive implications when testing for seasonal unit roots.

3.1 The Zero Frequency Unit Root Test Regression

For $y_{s\tau} \sim PI(1)$, the distribution of $T\hat{\pi}_1$ in (14) is summarized in the following theorem, with details of the proof given in the appendix.

THEOREM 1. *Assuming* $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ *in* (1)*, the asymptotic distribution of the normalized bias and t-ratio test statistics for a unit root test in* (14) *are given by*:

$$T\hat{\pi}_1 \Rightarrow \frac{\int w(r)dw(r) + 4\Gamma/\left(\omega^2\ a'C_1 a\right)}{\int w(r)^2\, dr} \tag{15}$$

and

$$t_{\hat{\pi}_1} \Rightarrow \frac{\sqrt{(\omega^2 a' C_1 a)}\int w(r)\,dw(r) + 4\Gamma/\sqrt{(\omega^2 a' C_1 a)}}{\sqrt{\omega^2 a' a \int [w(r)]^2\, dr}} \tag{16}$$

where $\Gamma = \frac{1}{4}\sum_{s=1}^{4}\sum_{k=1}^{s-1}\gamma_s(k)$ *and* $\gamma_s(j) = E(\Delta_4 y_{s\tau}\ \Delta_4 y_{s-j,\tau})$ *is the periodic autocovariance of* $\Delta_4 y_{s\tau}$ *for season s and lag j.*

We discuss first the result (15) for the normalized bias. An immediate consequence of (15) is that the distribution of $T\hat{\pi}_1$ in (14) when $y_{s\tau} \sim PI(1)$ differs from the usual Dickey-Fuller distribution of the normalized bias by the additional term $4\Gamma/\left(\omega^2\ a'C_1 a\right)$ that appears in the numerator. This term arises from the correlation between $\Delta_4 y_{s\tau}$ and the disturbance $v_{s\tau}$ of (14), and is a function of the autocovariances of $\Delta_4 y_{s\tau}$ and the parameters of the *PI(1)* process .

Table 1 collects the parameters and the corresponding values of Γ and $\omega^2 a' C_1 a$ for the *PI*(1) processes used in our Monte Carlo experiments below. With positive *PI*(1) coefficients, the term $4\Gamma/(\omega^2 a'C_1 a)$ is always positive, hence shifting the distribution of $T\hat{\pi}_1$ to the right in relation to the distribution of $T\hat{\pi}_1$ in (13.1), where the latter is also the asymptotic Dickey-Fuller distribution for the normalized bias in a unit root test. To illustrate this effect, Figure 1 shows the empirical distributions of $T\hat{\pi}_1$ in (14) and (10), denoted Tpi1_PI(1)* and Tpi1_PI(1) respectively[1], for the DGPs of Table 1. In addition, the distribution of the normalized bias for a conventional Dickey-Fuller test is shown, with this denoted in the figure as T_pi1_DF. It can be seen that the shift to the right in the empirical distributions of Tpi1_PI(1)* is substantial, compared with the distribution of the for the zero frequency unit root statistic in the HEGY regression (10). However, the distribution for the normalized bias statistics from the full HEGY regression and a DF regression are effectively identical for all cases, except DGP 6.

Thus, if $T\hat{\pi}_1$ from (14) is used to test the null hypothesis of a unit root at the zero frequency when $y_{s\tau} \sim PI(1)$, with the test statistic compared to the corresponding asymptotic Dickey-Fuller distribution, then the null will be rejected less frequently than indicated by the nominal size of the test. This continues to be true if the DGP is a random walk, which is shown as DGP 6 of the figure. In this latter situation, the unit root null hypothesis is clearly true, so that the test will be substantially undersized in relation to a conventional nominal size of, say, 1 or 5 percent.

However, it is also notable from Figure 1 that the random walk case differs from the *PI*(1) processes in that the distributions obtained from (14) and (10), Tpi1_PI(1)* and Tpi1_PI(1) respectively, do not coincide. Therefore, applying the full HEGY test

---

[1] These figure show histograms of the empirical distributions, based in 20 points and obtained with a sample size of T=1000 and s = 4, using 15,000 replications.

regression has an asymptotically non-trivial effect on the distribution of the unit root test for a random walk process, compared with applying a simple Dickey-Fuller test after taking moving seasonal sums in order to account for presumed seasonality. We consider this issue in the next subsection.

The *t*-ratio for $\hat{\pi}_1$ from the OLS estimation of (14) is given by

$$t_{\hat{\pi}_1} = \frac{T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N}\Delta_4 y_{s\tau} y^{(1)}_{s-1,\tau}}{\sqrt{\hat{\sigma}_v^2 T^{-2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}\left(y^{(1)}_{s-1,\tau}\right)^2}}$$

where $\hat{\sigma}_v^2$ is the OLS estimator of the disturbance variance. As shown in the Appendix, $\hat{\sigma}_v^2 \to \sigma_v^2 = Var(\Delta_4 y_{s\tau}) = \frac{\omega^2 a'a}{4}$, leading to the result for the t-ratio given in (16).

In the particular case of a random walk, $a'C_1 a = 4a'a$ and (16) becomes

$$t_{\hat{\pi}_1} \Rightarrow 2\left(\frac{\int w(r)dw(r) + \Gamma/\left(\omega^2\ a'a\right)}{\sqrt{\int w(r)^2\, dr}}\right) \qquad (17)$$

Although this result in (17) formally holds only for the random walk case, in practice it provides an approximation to (16) because the relationship $a'C_1 a \approx 4a'a$ is quite good for many $PAR(1)$ processes that are periodically integrated, as the examples of Table 1 show.

Both Boswijk and Franses (1996) and Taylor (2003) separately obtain the asymptotic distributions of the Dickey, Hasza and Fuller (1984) [DHF] seasonal integration test statistics when the underlying process is a random walk. Note that their distribution is the same as the one in (17) except for the term $\Gamma/\left(\omega^2\ a'a\right)$, this term do not appear in the case of the DHF distribution, because even that $\Delta_4 y_{s\tau}$ is serially correlated is uncorrelated with $\Delta_4 y_{s\tau-1}$, as it is pointed out by Boswijk and Franses (1996).

Figure 2 investigates the nature of the distribution of the *t*-ratio in (16) in comparison with the DF distribution, once again taking the example DGPs of Table 1. This figure shows the empirical distribution of $t_{\hat{\pi}_1}$ from (14) (denoted t_pi1_PI(1)*), from (10) (t_pi1_PI(1)) and from a Dickey-Fuller regression (t_pi1_DF), where the latter is effectively identical to the distribution for the t-ratio in a conventional Dickey-Fuller distribution[2]. From the results collected in Figure 2, it is evident that (like the distribution of the normalized bias) the distribution of (16) is shifted to the right compared to the Dickey-Fuller one, so that the null hypothesis of a unit root at the zero frequency will be rejected substantially fewer times that the nominal size, when this nominal size corresponds to the use of the full HEGY regression (14).

As in the corresponding distributions in Figure 1, the empirical distributions for t_pi1_DF and t_pi1_PI(1) in Figure 2 are coincident for the periodic DGPs of Table 1. However, this AGAIN does not hold for the random walk (DGP 6). Therefore, the next two subsections investigate the asymptotic distributions of the normalized bias and *t*-ratio statistics in the context of the HEGY regression for the random walk and periodically integrated cases, respectively.

---

[2] The number of points used in the histograms for the empirical distributions, the sample size and the number of replications are the same as for Figure 1.

3.2 The Random Walk DGP in the complete HEGY regression

Lemma 2 examines the nature of the variables used in the HEGY regression for a random walk DGP.

LEMMA 2. *Assume* $\alpha_s = 1$ (s = 1, 2, 3, 4) *in* (1)*, so that* $y_{s\tau} = y_{s-1,\tau} + \varepsilon_{s\tau}$. *Then the variables in HEGY test regression* (10) *are given by:*

$$\Delta_4 y_{s\tau} = (1-L)(1+L)(1+L^2)\frac{\varepsilon_{s\tau}}{(1-L)} = (1+L)(1+L^2)\varepsilon_{s\tau} = S(L)\varepsilon_{s\tau}$$

$$y_{s\tau}^{(1)} = (1+L)(1+L^2)\frac{\varepsilon_{s\tau}}{(1-L)} = \frac{S(L)}{(1-L)}\varepsilon_{s\tau}$$

$$y_{s\tau}^{(2)} = -(1-L)(1+L^2)\frac{\varepsilon_{s\tau}}{(1-L)} = -(1+L^2)\varepsilon_{s\tau}$$

$$y_{s\tau}^{(3)} = -(1-L)(1+L)\frac{\varepsilon_{s\tau}}{(1-L)} = -(1+L)\varepsilon_{s\tau}$$

where $S(L) = 1 + L + L^2 + L^3$. *Consequently,* $y_{s\tau}^{(1)} \sim I(1)$, *while* $\Delta_4 y_{s\tau}$, $y_{s\tau}^{(2)}$, $y_{s\tau}^{(3)}$ *are stationary.*

To establish this, note that these expressions for the variables in the HEGY regression follow immediately from the assumption that $y_{s\tau}$ is a random walk, together with the factorization $\Delta_4 = (1 - L)(1 + L)(1 + L^2)$. It immediately follows that $y_{s\tau}^{(1)}$ contains a zero frequency unit root, due to the presence of the autoregressive factor $(1 - L)$. On the other hand, each of $\Delta_4 y_{s\tau}$, $y_{s\tau}^{(2)}$, $y_{s\tau}^{(3)}$ is a simple linear transformation of the white noise disturbances $\varepsilon_{s-j\tau}$ (j = 0, 1, 2, 3) and hence are stationary.

Due to the $I(1)$ property of $y_{s\tau}^{(1)}$ and the stationarity of the vector $Y_{s\tau}^{(2)} = (y_{s\tau}^{(2)}, y_{s\tau}^{(3)}, y_{s-1,\tau}^{(3)})'$, the coefficients corresponding to these variables converge at different rates when (10) is estimated. To reflect this, it is useful to define the 4 × 4 scaling matrix M, where

$$M = \begin{bmatrix} T & 0 & 0 & 0 \\ 0 & T^{1/2} & 0 & 0 \\ 0 & 0 & T^{1/2} & 0 \\ 0 & 0 & 0 & T^{1/2} \end{bmatrix}.$$

It is then straightforward to see that the scaled OLS estimators for the HEGY regression (10) are given by

$$M\hat{\Pi} = \begin{bmatrix} T\hat{\pi}_1 \\ T^{1/2}\hat{\Pi}_2 \end{bmatrix}$$
$$= \begin{bmatrix} T^{-2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}(y_{s-1,\tau}^{(1)})^2 & T^{-3/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N} y_{s-1,\tau}^{(1)}(Y_{s-1,\tau}^{(2)})' \\ T^{-3/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N} y_{s-1,\tau}^{(1)}Y_{s-1,\tau}^{(2)} & T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N} Y_{s-1,\tau}^{(2)}(Y_{s-1,\tau}^{(2)})' \end{bmatrix}^{-1} \begin{bmatrix} T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N} y_{s-1,\tau}^{(1)}\Delta_4 y_{s\tau} \\ T^{-1/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N} Y_{s-1,\tau}^{(2)}\Delta_4 y_{s\tau} \end{bmatrix}$$

(18)

where $\hat{\Pi}_2 = (\hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4)'$.

Our results relating to the distribution of the OLS estimators for the HEGY regression applied to a random walk are contained in Theorem 2.

THEOREM 2. *Assume* $\alpha_s = 1$ (s = 1, 2, 3, 4) *in* (1)*, so that* $y_{s\tau} = y_{s-1,\tau} + \varepsilon_{s\tau}$. *Then:*

(i) *The covariances between* $y_{s\tau}^{(1)}$ *and each of* $y_{s\tau}^{(2)}, y_{s\tau}^{(3)}, y_{s-1,\tau}^{(3)}$ *satisfy*

$$T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N} y_{s-1,\tau}^{(1)} y_{s-1,\tau}^{(2)} \to O_p(1)$$
$$T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N} y_{s-1,\tau}^{(1)} y_{s-1,\tau}^{(3)} \to O_p(1)$$
$$T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N} y_{s-1,\tau}^{(1)} y_{s-2,\tau}^{(3)} \to O_p(1)$$

(ii) *The appropriately scaled OLS estimator* $\hat{\pi}_1$ *is asymptotically orthogonal to* $\hat{\Pi}_2$;

(iii) *The asymptotic distribution of* $T\hat{\pi}_1$ *is given by*:

$$T\hat{\pi}_1 \Rightarrow \frac{1}{4}\frac{\int w(r)dw(r)}{\int w(r)^2 dr}\ ; \qquad (19)$$

(iv) *The asymptotic distribution of the t-ratio for* $\hat{\pi}_1$ *is given by*

$$t_{\hat{\pi}_1} \Rightarrow \frac{\int w(r)dw(r)}{\sqrt{\int w(r)^2 dr}}; \qquad (20)$$

(v) *The OLS estimator of* Π_2 *converges to* (-0.5, -0.5, -0.5)'.

Using the properties of the HEGY regressors established in Lemma 2 for a random walk process, (i) follows from standard results on the convergence rates properties of integrated and stationary processes (for example, Hamilton, 1994). This leads immediately to the asymptotic orthogonality result in (ii), when the estimators are scaled by *M*.

The proof of (iii) is given in the appendix. It is important to note from Theorem 2, specifically equation (19), that the distribution of the HEGY normalized bias statistic for the zero frequency unit root leads to a scaling of the distribution of Dickey and Fuller (1984) when the process is a random walk. In this case there is no bias term Γ, which appears in (15), because the additional regressors included in (10), namely

$y^{(2)}_{s-1\tau}$, $y^{(3)}_{s-2\tau}$ and $y^{(3)}_{s-1\tau}$, effectively act in the same way as the augmentation in the ADF test. Part (iv) then follows by using part (iii) of Theorem 2, together with the fact that in regression (10) the perturbation term $\varepsilon_{s\tau}$ is white noise; this latter statement is established in the appendix. We also use the fact that, in the random walk case, $\omega^2 a' C_1 a$ = 64. It is important to appreciate that the distribution in (20) is the usual Dickey-Fuller distribution for the *t*-ratio, implying that this distribution remains valid for testing a zero frequency unit root within the context of the HEGY regression, even though no seasonal unit roots are present in the DGP.

Part (v), which is proved in the appendix, implies that the scaled estimator $T\hat{\pi}_j$ for $j = 2, 3, 4$ diverges to - ∞ as $T \to \infty$ when the HEGY regression (10) is applied to a random walk process. This divergence has also been obtained by Rodrigues (2001) and Taylor (2002), by considering local alternatives to the HEGY seasonal integration null hypothesis.

Therefore, our results can now explain what it has been observed in Figures 1 and 2 for the random walk case. The previous subsection derived the asymptotic distribution of the normalized bias and *t*-ratio unit root test statistics for regression (14), whereas this subsection has considered them in the context of the HEGY regression (14). These are now seen to be different distributions, with the use of the full HEGY distribution delivering the familiar Dickey-Fuller distributions, whereas the application of unit tests in (14) after taking moving seasonal sums does not.

3.3 The *PI*(1) DGP in the complete HEGY regression

We now turn attention to the more general DGP of a *PI*(1) process. From (9), we can write

$$\Delta_4 y_{s\tau} = \alpha_s(L)\varepsilon_{s\tau}$$

and hence

$$\begin{aligned} y_{s\tau}^{(1)} &= -(1+L)(1+L^2)\frac{\alpha_s(L)}{\Delta_4}\varepsilon_{s\tau} = -\frac{\alpha_s(L)}{(1-L)}\varepsilon_{s\tau} \\ y_{s\tau}^{(2)} &= -(1-L)(1+L^2)\frac{\alpha_s(L)}{\Delta_4}\varepsilon_{s\tau} = -\frac{\alpha_s(L)}{(1+L)}\varepsilon_{s\tau} \\ y_{s\tau}^{(3)} &= -(1-L)(1+L)\frac{\alpha_s(L)}{\Delta_4}\varepsilon_{s\tau} = -\frac{\alpha_s(L)}{(1+L^2)}\varepsilon_{s\tau} \end{aligned} \qquad (21)$$

where $\alpha_s(L) = 1 + \alpha_s L + \alpha_s\alpha_{s-1}L^2 + \alpha_s\alpha_{s-1}\alpha_{s-2}L^3$. The polynomial $\alpha_s(L)$ gives the moving average coefficients of the VMA representation in (4), which we have already noted contains three noninvertible unit roots. Unlike the special case of the random walk, no simple cancellation applies in the equations for the HEGY regressors in (10), so that the HEGY regressors of (21) are nonstationary.

As shown in Lemma 1, asymptotically all elements $y_{s\tau}$ (s = 1, 2, 3, 4) can be written in terms of a single common trend, which can be (arbitrarily) identified with $y_{1\tau}$. Thus, from Lemma 1, we can write

$$\begin{bmatrix} y_{1\tau} \\ y_{2\tau} \\ y_{3\tau} \\ y_{4\tau} \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_2\alpha_3 \\ \alpha_2\alpha_3\alpha_4 \end{bmatrix} y_{1\tau} + \begin{bmatrix} v_{1\tau} \\ v_{2\tau} \\ v_{3\tau} \\ v_{4\tau} \end{bmatrix} \qquad (22)$$

and the only stationary linear combinations of $y_{s\tau}$ (s = 1, 2, 3, 4) are those that reduce the coefficient on $y_{1\tau}$ to zero. This does not apply to the HEGY regressors, which are consequently nonstationary. Indeed, it is clear from (21) that the HEGY variables for a *PI*(1) process contain the same unit roots as for a seasonally integrated case, so that the HEGY regressors retain the asymptotic orthogonality as under the seasonal integration null hypothesis. This is the reason why, in Figures 1 and 2, the empirical distributions of the normalized bias and *t*-ratio test statistics are the same for regressions (10) and (14). Indeed, in contrast to the random walk case discussed in subsection 3.2, the results obtained in subsection 3.1 for π_1 in regression (14) also hold for the complete HEGY regression.

To examine the tests associated with remaining variables of the HEGY regression, we use the constant parameter representation of the *PAR* process, studied by Tiao and Grupe (1980) and Osborn (1991). More particularly, when a periodic *MA* process is analyzed as a constant parameter one, the parameters of the constant parameter representation correspond to a process with autocovariances equal to the periodic autocovariances averaged over the four quarters. That is, the constant parameter representation of the MA(3) in (9) has autocovariances

$$\gamma(j) = \frac{1}{4}\sum_{j=1}^{4}\gamma_s(j) \qquad j = 0, 1, 2, 3 \tag{23}$$

where $\gamma_s(j) = E(\Delta_4 y_{s\tau}\ \Delta_4 y_{s-j,\tau})$, and $\gamma(j) = 0, j > 3$. The periodic autocovariances can be obtained from (8), with the autocovariance generating function then applied to the resulting average autocovariances from (23) in order to obtain the constant parameter representation.

In Table 2 we collect the polynomials associated with the roots of the constant parameter *MA*(3) for each DGP of Table 1. Since the HEGY regression treats the process as being nonperiodic, the proximity of the roots in Table 2 to the roots –1, $\pm i$, of the seasonal random walk process, the closer to cancellation between the factors in (21) when the periodic *MA* polynomial $\alpha_s(L)$ is replaced by the corresponding constant parameter one.

Therefore, despite the fact that the variables associated with the seasonal unit roots in the HEGY regression are nonstationary, these variables typically exhibit near cancellation of these nonstationary roots with nearly-noninvertible moving average roots. From a practical point of view, the distribution of the tests associated with the seasonal frequencies when the DGP is a *PI*(1) may be very close to those reported in the random walk case in the previous subsection, due to the near cancellation at the seasonal frequencies. The next section employs Monte Carlo methods to examine this further.

4. Monte Carlo Results

As already noted, Table 1 shows the DGPs used in the Monte Carlo experiment. This table also shows the ratio between the maximum and minimum value of α_j (over j = 1, 2, 3, 4) as a measure of the extent of periodic variation for each DGP, as well as the values of $\omega^2 a' B_1 a$ and Γ (see Lemma 1 and Theorem 1) for each DGP. It can be seen that DGP 4 has the largest periodic variation in its coefficients, followed by DGP 1 and 2. Finally DGP 5 has the least periodic variation, apart from the random walk case of DGP 6.

As already discussed, Table 2 presents the polynomials associated with the periodic moving average polynomials $\alpha_s(L)$ for each DGP of Table 1. It can be seen that DGP 3 has a constant parameter representation with factor $(1 + \theta_1 L) = (1 + .994L)$, which therefore is very close to cancellation with the biannual seasonal AR root of –1. On the other hand, whereas DGP 4 has the lowest value of θ_1, its factor $(1 + a \pm biL)$ is close to cancellation with annual AR unit root of $(1 \pm iL)$. Finally, DGP 1 has the largest value of a and the smallest b, and hence is furthest from cancellation with the annual seasonal unit root of the processes considered. Taking into account both moving average factors $(1 + \theta_1 L)$ and $(1 + a \pm biL)$, DGP 5 is the closest $PI(1)$ process to the random walk case of DGP 6.

For each of the DGP, we generate 5000 replications of 100 observations (corresponding to $N = 25$ years of data), and then apply the HEGY procedure to test for unit roots at the zero and seasonal frequencies. Specifically, we separately consider the null three hypotheses $\pi_1 = 0$, $\pi_2 = 0$ and $\pi_3 = \pi_4 = 0$, corresponding to the presence of unit roots at the zero, biannual and annual frequencies, respectively. The first two hypothesis tests are conducted using the relevant t-ratios ($t_{\pi 1}$ and $t_{\pi 2}$), while the last uses the joint F_{34} statistic proposed by HEGY. It is relevant to note that Burridge and Taylor (2001) emphasize the use of this F_{34} statistic rather than separate t-ratios on these coefficients, since the asymptotic distributions of these t-statistics under the unit root null hypothesis are affected by the presence of stationary serial correlation, whereas the F-statistic is not. In addition, we undertake joint tests of the hypothesis of unit roots at all seasonal frequencies ($\pi_2 = \pi_3 = \pi_4 = 0$) and at the zero frequency and all the seasonal frequencies ($\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$) using the joint F_{234} and F_{1234} tests respectively, as proposed by Ghysels, Lee and Noh (1994). All results are presented as the proportion of times that the relevant null hypothesis is rejected, using a nominal significance level of 5 percent[3].

Tables 3.1 and 3.2 show the results when initial values are set to zero and no seasonal dummies are included in the test regression. For the results in Tables 4.1 and 4.2, initial values are obtained from a standard normal distribution and seasonal dummies are included in the HEGY regression. Tables 3.1 and 4.1 show results for the five $PI(1)$ processes of Table 1, whereas Tables 3.2 and 4.2 show results for the random walk case (DGP 6 of Table 1) and the seasonal random walk, referred to as DGP 7. Results are presented with both fixed augmentation and augmentation selected on the basis on the specific sample. The fixed augmentations considered are no lags, together with augmentation by 4, 8 and 12 lags (that is, one, two and three years, respectively) of $\Delta_4 y_{s\tau}$. The other procedures select the maximum lag based on the AIC criteria and the sequential method recently proposed by Ng and Perron (1995).

Our results in Section 3 show that the zero frequency unit root statistic does not follow the usual Dickey-Fuller distribution for $PI(1)$ processes in a HEGY regression without augmentation, due to the term Γ in (16) that shifts the distribution to the right. Therefore, we anticipate empirical rejection frequencies $t_{\pi 1}$ in Tables 3.1 and 4.1 to be lower than the nominal 5 percent. This is indeed the case when no augmentation is applied, with the single exception of DGP 1 in Table 4.1. Especially in the zero starting value case of Table 3.1, as the order of the augmentation increases the proportion of rejections tends to 0.05. Thus, the augmentation here takes account of

[3] Critical values for tests of $\pi_1 = 0$ and $\pi_2 = 0$ are taken from the Dickey-Fuller critical values presented for a sample size of 100 observations in Hamilton (1994). The joint test of $\pi_3 = \pi_4 = 0$ uses the critical values for this sample size in HEGY. Finally, the remaining joint tests use critical values presented by Ghysels *et al*. (1994).

the correlation between $\Delta_4 y_{s\tau}$ and $\varepsilon_{s\tau}$, so the term Γ became less important. It is also worth noting that this test generally has rejection frequency greater than the nominal level of significance when lags are selected according to AIC, with the Ng-Perron procedure resulting in a better performance in this respect.

We also show in Section 3 that the zero frequency unit root statistic asymptotically follows the Dickey-Fuller distribution for a random walk DGP, and from HEGY this is also known to be the case for a seasonally integrated process. The results in Tables are compatible with these theoretical results, although augmentation leads to some undersizing.

Turning to the tests for seasonal unit roots, the proportion of times that the relevant null is rejected when there is no augmentation is close to unity in all tables. The most marked exception is $t_{\pi 2}$ for DGP 4, which has relatively low rejection frequencies in both Tables 3.1 and 4.1. Increasing the order of augmentation leads to dramatic declines in the proportion of rejections of seasonal unit roots by all the test statistics ($t_{\pi 2}$, F_{34}, F_{234}, F_{1234}). For DGPs 1, 2 and 3, these tests are more likely to find evidence of unit roots at the annual frequency $\pi/2$ that at the biannual frequency π, whereas the reverse is true for DGPs 4 and 5.

It is not surprising that the probability of finding evidence of unit roots in these periodic processes depends on the specific parameter values α_j and the corresponding constant parameter moving average components shown in Table 2. Nevertheless, it is also clear that the HEGY test procedure will frequently lead to the wrong conclusion that the $PI(1)$ process is seasonally integrated. For example, basing lag selection on

AIC, Table 4.1 shows that the F_{1234} statistic rejects the seasonal integration null hypothesis only 40 percent of the time for DGP 1. Although this rejection frequency rises to 90 percent for DGP 5, the use of the Ng-Perron lag selection procedure would reduce the rejection frequency for this DGP to 40 percent.

The AIC criterion is frequently used for lag selection by applied researchers. In the present case, this tends to result in rejection frequencies similar to those obtained with an augmentation by 4 lags. On the other hand, the Ng-Perron procedure tends to select a higher order of augmentation and to result in a lower rejection frequency for the null hypothesis. AIC performs particularly well in the random walk case (DGP 6), with the Ng-Perron procedure having less power for rejecting the presence of seasonal unit roots, especially in Table 4.2.

The overall conclusion is that the single unit root of a $PI(1)$ process will be associated with the test for a zero frequency root in the HEGY test, with the corresponding test statistic having size close to the 5 percent nominal level of significance if the test regression is sufficiently augmented. However, depending on the specific parameter values of the DGP and the criterion used for selecting the order of augmentation, the probability can be high that the incorrect conclusion is drawn that the process has unit roots at one or more of the seasonal frequencies.

5. Concluding Remarks

This paper has tackled the consequences of testing for seasonal unit roots in a process that is, in fact, either a random walk or a periodic integrated process. The true process under consideration therefore has a single unit root, with no seasonal unit roots. We show theoretically that the zero frequency unit root test statistics do not follow the usual asymptotic Dickey-Fuller distributions in this case, as the distribution is shifted to the right. However, the Monte Carlo analysis indicates that the size of this test in a sample of 100 observations is relatively close to the nominal size of 5 percent when the HEGY regression is augmented. In other words, therefore, in practice the zero frequency unit root test detects not only a unit root at this frequency in a nonperiodic process, but also the single unit root in a periodic integrated process.

Periodic integrated processes do not contain seasonal unit roots. However, the transformed variables used in the HEGY seasonal unit root test do not remove the nonstationarity in a periodic process. Consequently, the use of these variables in a seasonal unit root test regression may lead to the conclusion that seasonal unit roots are present in the process.

In contrast to the periodic case, we show that the asymptotic Dickey-Fuller distribution continues to apply for the zero frequency unit root test statistic in the important special case of a random walk. Further, the seasonal unit root test statistics diverge to infinity. Therefore, although the HEGY seasonal unit root test regression has been developed under the null hypothesis of a seasonally integrated process, it continues to be applicable when only the zero frequency unit root is present.

References

Boswijk H.P. and P.H. Franses (1996), Unit Roots in Periodic Autoregressions, *Journal of Time Series Analysis*, 17, 221-245.

Burridge P and A.M.R. Taylor (2001), On the Properties of Regression-Based Tests for Seasonal Unit Roots in the Presence of Higher-Order Serial Correlation, *Journal of Business and Economic Statistics*, 19, 374-379.

Dickey, D.A., D.P. Hasza and W.A. Fuller (1984), Testing for Unit Roots in Seasonal Time Series, *Journal of the American Statistical Association*, 79, 355-367.

Franses P.H. (1994), A Multivariate Approach to Modeling Univariate Seasonal Time Series, *Journal of Econometrics*, 63, 133-151.

Franses P.H. (1996), *Periodicity and Stochastic Trends in Economic Time Series*, Oxford, Oxford University Press.

Gersovitz M. and J. G. McKinnon (1978), Seasonality in Regression: An Application of Smoothness Priors, *Journal of the American Statistical Association*, 73, 264-273.

Ghysels E., H.S. Lee and J. Noh (1994) Testing for Unit Roots in Seasonal Time Series: Some Theoretical Extensions and a Monte Carlo Investigation, *Journal of Econometrics,* 62, 415-442.

Ghysels E. and D.R. Osborn (2001), *The Econometric Analysis of Seasonal Time Series*, Cambridge, Cambridge University Press.

Ghysels E., Osborn D.R. and P.M.M. Rodrigues (2001), Seasonal Nonstationarity and Near-Nonstationarity, in B.H. Baltagi (ed.), *A Companion to Theoretical Econometrics*, Massachusetts: Backwell Publishers Inc. 655-677.

Hamilton, J.D. (1994) Time Series Analysis, Princeton, NJ: Princeton University Press.

Hansen L.P. and T.J. Sargent (1993), Seasonality and Approximation Errors in Rational Expectations Models, *Journal of Econometrics*, 55, 21-56.

Hylleberg S., R.F. Engle, C.W.J. Granger and B.S. Yoo (1990) Seasonal Integration and Cointegration, *Journal of Econometrics,* 44, 215-238.

Ng S. and P. Perron (1995) Unit Root Tests in ARMA Models with Data Dependent Methods for the Truncation Lag, *Journa of the American Statistical Association*, Volume 429, 268-281.

Osborn D.R. (1988), Seasonality and Habit Persistence in a Life Cycle Model of Consumption, *Journal of Applied Econometrics*, 3, 255-266.

Osborn D.R. (1991), The Implications of Periodically Varying Coefficients for Seasonal Time Series, *Journal of Econometrics*, 28, 373-384.

Osborn D.R., A.P.L. Chui, J.P. Smith and C.R. Birchenhall (1988), Seasonality and the Order of Integration for Consumption, *Oxford Bulletin of Economics and Statistics*, 50, 361-377.

Osborn D.R. and P.M.M. Rodrigues (2002), Asymptotic Distributions of Seasonal Unit Root Tests: A Unifying Approach, *Econometric Reviews*, 21, 221-241.

Osborn D.R. and J.P. Smith (1989), The Performance of Periodic Autoregressive Models in Forecasting Seasonal U.K. Consumption, *Journal of Business and Economic Statistics*, 7, 117-127.

Phillips, P.C.B. (1988), Weak Convergence to the Matrix Stochastic Integral $\int BdB'$, *Journal of Multivariante Analysis*, 24, 252-264.

Phillips, P.C.B. and S. Oularis (1990), Asymtotic Properties of Residual Based Test for Cointegration, *Econometrica*, 58, 165-193.

Rodrigues, P.M.M. (2001) Near Seasonal Integration, *Econometric Theory,* 17, 70-86.

Sanso, A., M. Artis, and J. Surinyach (1997) Comportamiento en Muestra Finita de los Contrastes de Integración Estacional para Datos Mensuales. *Estadística Espanola,* 39, 141-184.

Simth R.J. and A.M.R. Taylor (1998) Additional Critical Values and Asymptotic Representation for Seasonal Unit Root Tests. *Journal of Econometrics*, 85, 269-288.

Taylor, A.M.R. (2003), On the Asymtotic Properties of Some Seasonal Unit Root Tests, *Econometric Theory,* 19, 311-321.

Taylor, A.M.R. (2002), Regression-Based Unit Root Tests With Recursive Mean Adjustment for Seasonal and Nonseasonal Time Series, *Journal of Business and Economic Statistics,* 20, 269-281.

Tiao G. and M. Grupe (1980), Autoregressive-Moving Average Models in Time Series Data, *Biometrika*, 67, 365-373.

Appendix

Proof of Theorem 1:

Asymptotically, the distribution of $T\hat{\pi}_1$ in (14) is given by

$$\frac{T^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} y_{s-1,\tau}^{(1)}\Delta_4 y_{s\tau}}{T^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left[y_{s-1,\tau}^{(1)}\right]^2}. \tag{A.1}$$

The numerator of this expression is equal to

$$\begin{aligned} T^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} y_{s-1,\tau}^{(1)}\Delta_4 y_{s\tau} &= T^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}(y_{s-1,\tau}+y_{s-2,\tau}+y_{s-3,\tau}+y_{s,\tau-1})v_{s\tau} \\ &= T^{-1}\sum_{\tau=1}^{N}\left\{\left[\sum_{s=1}^{4} y_{s,\tau-1}\sum_{s=1}^{4} v_{s\tau}\right]+v_{1\tau}v_{2\tau}+(v_{1\tau}+v_{2\tau})v_{3\tau}+(v_{1\tau}+v_{2\tau}+v_{3\tau})v_{4\tau}\right\} \\ &= (4N)^{-1}\sum_{\tau=1}^{N}\left\{Y'_{\tau-1}C_1V_\tau+\sum_{s=2}^{4}\sum_{k=1}^{s-1} v_{s\tau}v_{s-k,\tau}\right\} \end{aligned} \tag{A.2}$$

where $v_{s\tau} = \Delta_4 y_{s\tau} = y_{s\tau} - y_{s,\tau\text{-}1}$ and $V_\tau = (v_{1\tau}, v_{2\tau}, v_{3\tau}, v_{4\tau})'$.

Asymptotically

$$(4N)^{-1}\sum_{\tau=1}^{N}\left\{\sum_{s=2}^{4}\sum_{k=1}^{s-1} v_{s\tau}v_{s-k,\tau}\right\}\to\Gamma_1=\frac{1}{4}\sum_{s=2}^{4}\sum_{k=1}^{s-1}E(v_{s\tau}v_{s-k,\tau})=\frac{1}{4}\sum_{s=2}^{4}\sum_{k=1}^{s-1}\gamma_s(k) \tag{A.3}$$

where $\gamma_s(k) = E(v_{s\tau}v_{s\text{-}k,\tau})$ is the autocovariance of $v_{s\tau}$ at lag k. This autocovariance is, in general, periodic. Returning to (A.2) and noting that V_τ is autocorrelated, Theorem 2.6 of Phillips (1988) implies that

$$(4N)^{-1}\sum_{\tau=1}^{N} Y'_{\tau-1}C_1V_\tau \Rightarrow \frac{1}{4}\int B'(r)C_1 dB(r)+\Gamma_2 \tag{A.4}$$

where

$$\Gamma_2=\frac{1}{4}E(V_{\tau-1}C_1V'_\tau)=\frac{1}{4}\sum_{s=1}^{3}\sum_{j=s+1}^{4}E(v_{s\tau}v_{j,\tau-1})=\frac{1}{4}\sum_{s=1}^{3}\sum_{k=s}^{3}\gamma_s(k) \tag{A.5}$$

and we have used the fact that $v_{s\tau}$ is an MA(3). Notice that Γ_1 is equal to the sum of the intra-year covariances between the elements of V_τ, while Γ_2 captures the corresponding inter-year covariances. The sum of these, namely the term $\Gamma = \Gamma_1 + \Gamma_2$ arises from (A.2) due to the covariance between the regressor $y_{s-1,\tau}^{(1)}$ and the dependent variable $\Delta_4 y_{s\tau}$; Γ here plays a similar role to Ω_1 in Theorem 2.6 of Phillips (1988).

From (A.2) to (A.4), and using Lemma 1,

$$
\begin{aligned}
(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} y_{s-1,\tau}^{(1)}\Delta_4 y_{s\tau} &\Rightarrow \frac{1}{4}\int B'(r)C_1 dB(r) + \Gamma \\
&= \frac{\omega^2\ a'C_1 a}{4}\int w(r)\,dw(r) + \Gamma
\end{aligned} \tag{A.6}
$$

where, as before, $\omega^2 = \sigma^2(b'b)$, while

$$
\Gamma = \frac{1}{4}\sum_{s=1}^{4}\{\gamma_s(1) + \gamma_s(2) + \gamma_3(3)\} = \frac{1}{4}\sum_{s=1}^{4}\sum_{k=1}^{3} E(\Delta_4 y_{s\tau}\Delta_4 y_{s-k,\tau}) \tag{A.7}
$$

The denominator of (A.1) is

$$
\begin{aligned}
(4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left[y_{s-1,\tau}^{(1)}\right]^2 &= (4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left[y_{s-1\tau} + y_{s-2,\tau} + y_{s-3,\tau} + y_{s,\tau-1}\right]^2 \\
&= (4N)^{-2}\sum_{\tau=1}^{N} 4[y_{1\tau-1} + y_{2\tau-1} + y_{3\tau-1} + y_{4\tau-1}]^2 + d \\
&= (4N)^{-2}\sum_{\tau=1}^{N} 4\{Y_\tau' C_1 Y_\tau\} + d
\end{aligned}
$$

where:

$$
\begin{aligned}
d = (4N)^{-2}\sum_{\tau=1}^{N}\ &[2(v_{3\tau} + 2v_{2\tau} + 3v_{1\tau})(y_{1,\tau-1} + y_{2,\tau-1} + y_{3,\tau-1} + y_{4,\tau-1}) \\
&+ (v_{1\tau} + v_{2\tau} + v_{3\tau})^2 + (v_{1\tau} + v_{2\tau})^2 + v_{1\tau}^2] \\
\rightarrow 0.
\end{aligned}
$$

Consequently d is asymptotically negligible and hence

$$
(4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left[y_{s-1,\tau}^{(1)}\right]^2 \Rightarrow \frac{1}{4}\int B'(r)C_1 B(r)dr = \frac{\omega^2 a'C_1 a}{4}\int [w(r)]^2\,dr \tag{A.8}
$$

where we have used Lemma 1 to substitute for $B(r)$.

Therefore, the asymptotic distribution of the normalized bias is

$$
\frac{(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} y_{s-1,\tau}^{(1)}\Delta_4 y_{s\tau}}{(4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left[y_{s-1,\tau}^{(1)}\right]^2} \Rightarrow \frac{\int w(r)dw + 4\Gamma/(\omega^2 a'C_1 a)}{\int [w(r)]^2\,dr} \tag{A.9}
$$

The general results in (A.6) to (A.9) hold for any *PI*(1) process. Specific expressions can then be obtained for any given set of *PI*(1) coefficients. For the special (nonperiodic) case of the random walk, $b'b = 4$ while $a'C_1a = 16$ so that (A.6) becomes

$$N^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} y_{s-1,\tau}^{(1)}\Delta_4 y_{s\tau} \Rightarrow \frac{64}{4}\sigma^2 \int w(r)\,dw(r) + \Gamma$$

with $\Gamma = \{\gamma(1) + \gamma(2) + \gamma(3)\} = 6\sigma^2$. Using the corresponding special case of (A.8), the required distribution for the normalized unit root test statistic for the random walk can be seen to be

$$T\hat{\pi}_1 = \frac{(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} y_{s-1,\tau}^{(1)}\Delta_4 y_{s\tau}}{(4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}\left[y_{s-1,\tau}^{(1)}\right]^2} \Rightarrow \frac{\int w(r)dw + 0.375}{\int [w(r)]^2\,dr} \qquad \text{(A.10)}$$

From (15) in Theorem 1 we have that $T\hat{\pi}_1 = O_p(1)$, so $T^{-1}T\hat{\pi}_1 = o(1)O_p(1) = o_p(1)$ and hence $\hat{\pi}_1 \to 0$. Therefore, $\hat{\sigma}_v^2 = \frac{1}{T-1}\sum\left(\Delta_4 y_{s\tau} - \hat{\pi}_1 y_{s-1\tau}^{(1)}\right)^2 = \frac{1}{T-1}\sum\left(\Delta_4 y_{s\tau}\right)^2 + o_p(1)$ and consequently, $\hat{\sigma}_v^2 \to \sigma_v^2$.

Therefore,

$$t_{\hat{\pi}_1} = \frac{T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N}\Delta_4 y_{s\tau}y_{s-1,\tau}^{(1)}}{\sqrt{\hat{\sigma}_v^2 T^{-2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}\left(y_{s-1,\tau}^{(1)}\right)^2}} \Rightarrow \frac{\frac{\omega^2 a'C_1 a}{4}\int w(r)sw(r) + \Gamma}{\sqrt{\left(\frac{\omega^2 a'a}{4}\right)\frac{\omega^2 a'C_1 a}{4}\int [w(r)]^2\,dr}} \qquad \text{(A.11)}$$

Proof of Theorem 2 (iii):

$T\hat{\pi}_1$ in (10) can be expressed as:

$$T\hat{\pi}_1 = \frac{(4N)^{-1}Y_{-1}^{(1)}{}'Q\Delta_4 Y}{(4N)^{-2}Y_{-1}^{(1)}{}'QY_{-1}^{(1)}} \qquad \text{(A.12)}$$

Where $Y_{-1}^{(1)}$ and $\Delta_4 Y$ are 4N*1 vectors with generic elements $y_{s-1,\tau}^{(1)}$ and $\Delta_4 y_{s\tau}$ respectively and Q is a 4N*4N matrix $Q = I - X(X'X)^{-1}X'$ with the columns of with generic elements $y_{s-1,\tau}^{(2)}$ $y_{s-2,\tau}^{(3)}$ and $y_{s-1,\tau}^{(3)}$.

As in Phillips and Oularis (1990) Theorem 4.2 it follows that:

$$(4N)^{-2}Y_{-1}^{(1)}{}'QY_{-1}^{(1)} = (4N)^{-2}Y_{-1}^{(1)}{}'Y_{-1}^{(1)} + op(1) \qquad \text{(A.13)}$$

and it can be seen that $Q\Delta_4 Y$ are the residuals of the regression:

$$\Delta_4 y_{s\tau} = \pi_2 y^{(2)}_{s-1\tau} + \pi_3 y^{(3)}_{s-2\tau} + \pi_4 y^{(3)}_{s-1\tau} + u_{s\tau} \qquad \text{(A.14)}$$

It follows from theorem 1 (iv) that in (A.14) the estimates of the parameters are asymptotically equal to –0.5. then using lemma 2 it can be seen that the residuals of (A.14) and hence $Q\Delta_4 Y$ follows a white noise process.

$$\begin{aligned} \hat{u}_{s\tau} &= \Delta_4 y_{s\tau} - \pi_2 y^{(2)}_{s-1\tau} - \pi_3 y^{(3)}_{s-2\tau} - \pi_4 y^{(3)}_{s-1\tau} = \\ &= S(L)\varepsilon_{s\tau} - 0.5(1+L^2)\varepsilon_{s-1\tau} - 0.5(1+L)\varepsilon_{s-2\tau} - 0.5(1+L)\varepsilon_{s-1\tau} = \varepsilon_{s\tau} \end{aligned} \qquad \text{(A.15)}$$

The denominator of (A.12) is then

$$(4N)^{-2} Y^{(1)}_{-1}{}' Q Y^{(1)}_{-1} = (4N)^{-2} Y^{(1)}_{-1}{}' Y^{(1)}_{-1} + op(1) \approx (4N)^{-2} \sum_{\tau=1}^{N}\sum_{s=1}^{4} \left[y^{(1)}_{s-1,\tau}\right]^2 \Rightarrow \frac{\omega^2 a' C_1 a}{4} \int w(r)^2 dr$$

(A.16)

while the numerator is

$$(4N)^{-1} Y^{(1)}_{-1}{}' Q\Delta_4 Y = (4N)^{-1} Y^{(1)}_{-1}{}' U = (4N)^{-1} \sum_{\tau=1}^{N}\sum_{s=1}^{4} y^{(1)}_{s-1,\tau}\varepsilon_{s\tau}$$

where U is a 4N*1 vector with generic elements $\varepsilon_{s\tau}$ .Finally as in theorem 1 and knowing that now Γ is zero because $\varepsilon_{s\tau}$ is white noise, we have

$$(4N)^{-1} \sum_{\tau=1}^{N}\sum_{s=1}^{4} y^{(1)}_{s-1,\tau}\varepsilon_{s\tau} = (4N)^{-1} \sum_{\tau=1}^{N} Y'_{\tau-1} C_1 U_\tau \Rightarrow \frac{1}{4}\int B'(r) C_1 dW(r) = \int W'(r) C_1 dW(r)$$

where $U_\tau = (\varepsilon_{1\tau}, \varepsilon_{2\tau}, \varepsilon_{3\tau}, \varepsilon_{4\tau})'$, and in the last step we use $B'(r) = W'(r)ba'$ from the definition of $B(r)$ in lemma 1, and in this particular case we have $b'a=C_1$ and $b'aC_1=4C_1$. Furthermore as $\int W'(r) C_1 dW(r) = 4\int wr)dw(r)$ then:

$$(4N)^{-1} \sum_{\tau=1}^{N}\sum_{s=1}^{4} y^{(1)}_{s-1,\tau}\varepsilon_{s\tau} \Rightarrow 4\int w(r) dw(r) \qquad \text{(A.17)}$$

Finally $\omega^2 a' C_1 a = 64$, so that

$$T\hat{\pi}_1 \Rightarrow \frac{4\int w(r) dw(r)}{\frac{64}{4}\int w(r)^2 dr} \qquad \text{(A.18)}$$

Proof of Theorem 2(v):

Using asymptotic orthogonality between the estimators of π_1 and Π_2, it can be seen that $-\hat{\Pi}_2$ converges to

$$\begin{bmatrix} \sum_{s=1}^{4}\sum_{\tau=1}^{N}[y_{s-1,\tau}^{(2)}]^2/N & \sum_{s=1}^{4}\sum_{\tau=1}^{N}y_{s-1,\tau}^{(2)}y_{s-2,\tau}^{(3)}/N & \sum_{s=1}^{4}\sum_{\tau=1}^{N}y_{s-1,\tau}^{(2)}y_{s-1,\tau}^{(3)}/N \\ & \sum_{s=1}^{4}\sum_{\tau=1}^{N}[y_{s-1,\tau}^{(3)}]^2/N & \sum_{s=1}^{4}\sum_{\tau=1}^{N}y_{s-2,\tau}^{(3)}y_{s-1,\tau}^{(3)}/N \\ & & \sum_{s=1}^{4}\sum_{\tau=1}^{N}[y_{s-1,\tau}^{(4)}]^2/N \end{bmatrix}^{-1} \times \begin{bmatrix} \sum_{s=1}^{4}\sum_{\tau=1}^{N}y_{s-1,\tau}^{(2)}\Delta_4 y_{s\tau}/N \\ \sum_{s=1}^{4}\sum_{\tau=1}^{N}y_{s-2,\tau}^{(3)}\Delta_4 y_{s\tau}/N \\ \sum_{s=1}^{4}\sum_{\tau=1}^{N}y_{s-1,\tau}^{(3)}\Delta_4 y_{s\tau}/N \end{bmatrix}$$

$$\to \begin{bmatrix} Var(y_{s-1,\tau}^{(2)}) & Cov(y_{s-1,\tau}^{(2)}, y_{s-2,\tau}^{(3)}) & Cov(y_{s-1,\tau}^{(2)}, y_{s-1,\tau}^{(3)}) \\ & Var(y_{s-2,\tau}^{(3)}) & Cov(y_{s-2,\tau}^{(3)}, y_{s-1,\tau}^{(3)}) \\ & & Var(y_{s-1,\tau}^{(3)}) \end{bmatrix}^{-1} \begin{bmatrix} Cov(y_{s-1,\tau}^{(2)}, \Delta_4 y_{s\tau}) \\ Cov(y_{s-2,\tau}^{(3)}, \Delta_4 y_{s\tau}) \\ Cov(y_{s-1,\tau}^{(3)}, \Delta_4 y_{s\tau}) \end{bmatrix}$$

$$= \begin{bmatrix} 2\sigma^2 & \sigma^2 & \sigma^2 \\ & 2\sigma^2 & \sigma^2 \\ & & 2\sigma^2 \end{bmatrix}^{-1} \begin{bmatrix} -2\sigma^2 \\ -2\sigma^2 \\ -2\sigma^2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

Figure 1. Simulated asymptotic distributions of the HEGY normalized bias statistic for a unit root at the zero frequency for the DGPs of Table 1

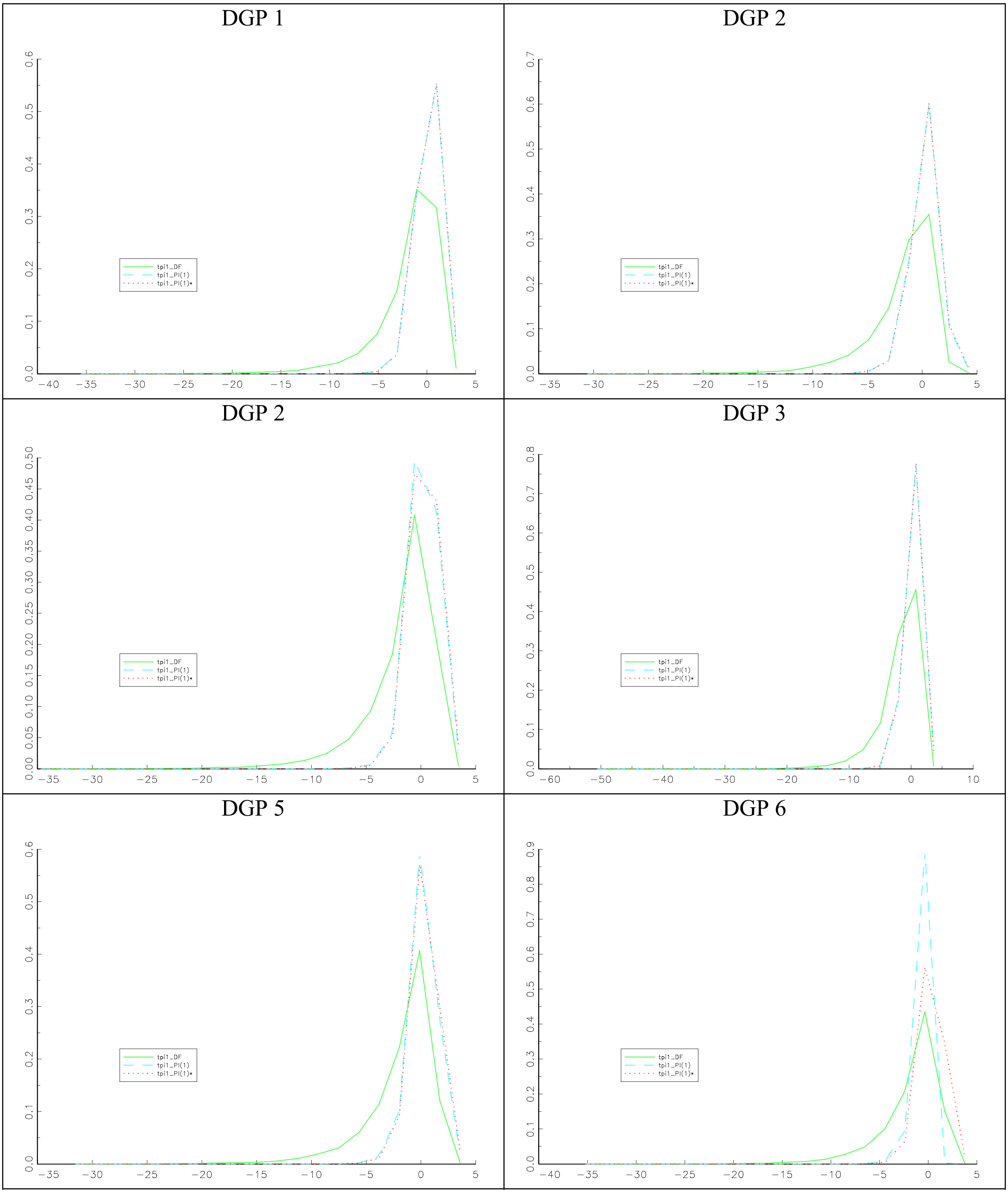

Figure 2. Simulated asymptotic distributions of the HEGY *t*- statistic for a zero frequency unit root for the DGPs of Table 1

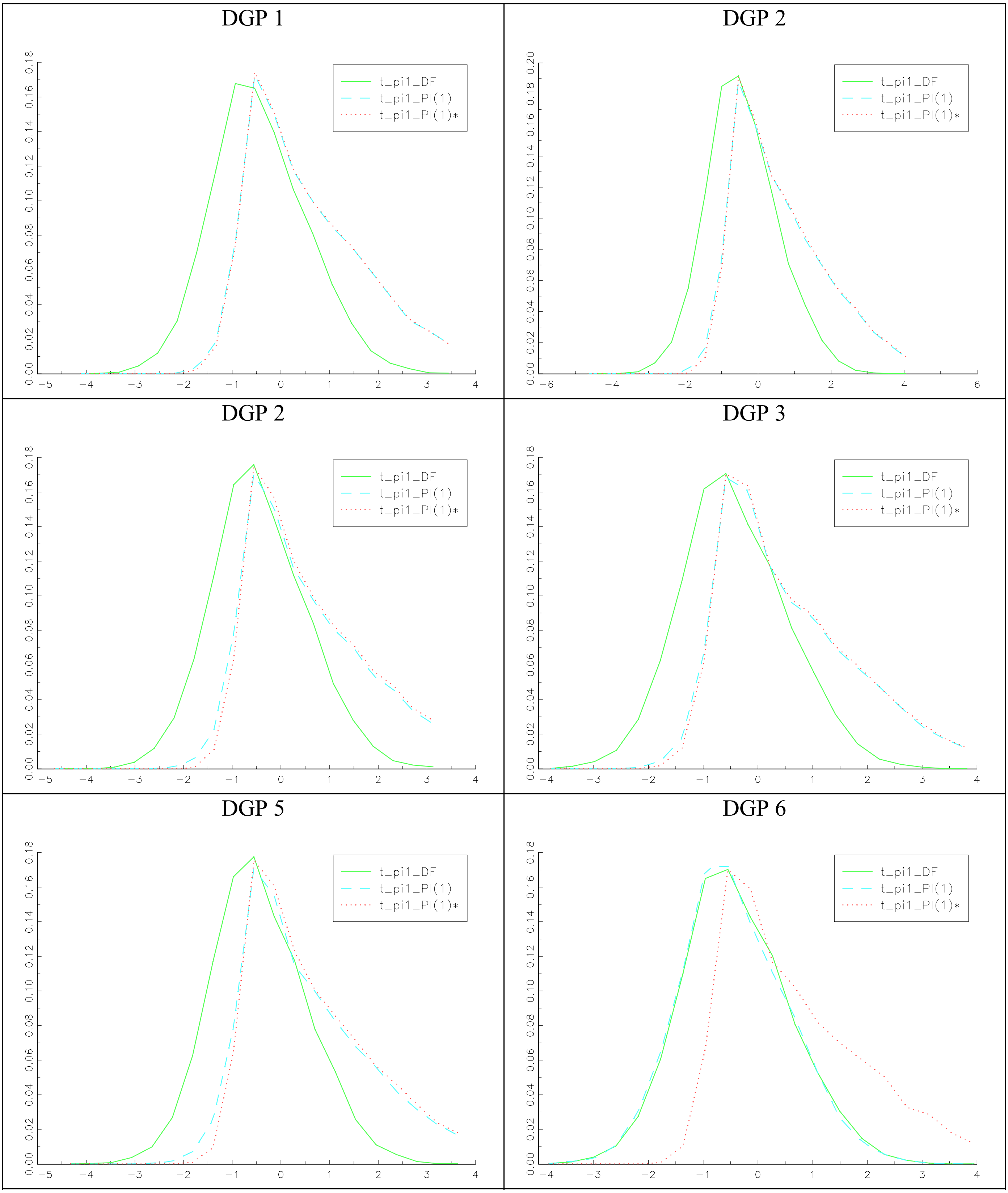

Table 1. Data Generating Processes Considered

DGP	α_1	α_2	α_3	α_4	$\alpha_{max}/\ \alpha_{min}$	$\omega^2 a'C_1 a$	Γ
1	0.500	0.900	1.500	1.481	3.000	83.604	7.557
2	1.200	0.600	1.000	1.389	2.315	73.568	6.783
3	1.250	0.800	0.900	1.111	1.563	66.730	6.224
4	2.000	0.500	1.500	0.667	4.000	81.507	7.410
5	1.000	0.800	1.200	1.042	1.500	65.705	6.114
6	1.000	1.000	1.000	1.000	1.000	64.000	6.000

Note: α_j (j = 1, 2, 3, 4) are the coefficients of the $PAR(1)$ process of (1) used in the Monte Carlo analysis; $\alpha_{max}/\alpha_{min}$ is the ratio of the largest to smallest of these coefficients; see text for definitions of $\omega^2 a'C_1 a$ and Γ.

Table 2. Moving Average Components of Constant Parameter Representation

DGP	Real factor $(1 + \theta_1 L)$	Complex conjugate root factor $1 + (a \pm bi)L$	$1 + 2aL + (a^2 + b^2)L^2$
1	$1 + 0.985L$	$1 + (0.12 \pm 0.805i)L$	$1 + 0.239L + 0.662L^2$
2	$1 + 0.984L$	$1 + (0.068 \pm 0.9i)L$	$1 + 0.135L + 0.815L^2$
3	$1 + 0.994L$	$1 + (0.026 \pm 0.973i)L$	$1 + 0.053L + 0.947L^2$
4	$1 + 0.589L$	$1 + (0.018 \pm 0.989i)L$	$1 + 0.036L + 0.979L^2$
5	$1 + 0.983L$	$1 + (0.012 \pm 0.987i)L$	$1 + 0.024L + 0.973L^2$
6	$1 + L$	$1 \pm iL$	$1 + L^2$

Note: Constant parameter representation derives from equation (9). All DGPs considered have one real factor and one factor with a pair of complex roots in the moving average component of this constant parameter representation.

Table 3.1. Rejection Frequency for HEGY Null Hypothesis and *PI*(1) DGPs with Zero Starting Values

DGP	AUG	t_{π_1}	t_{π_2}	F_{34}	F_{234}	F_{1234}
1	0	0.018	0.886	0.732	0.863	0.927
	4	0.038	0.196	0.153	0.231	0.217
	8	0.050	0.070	0.082	0.125	0.138
	12	0.055	0.038	0.065	0.090	0.112
	NP	0.047	0.090	0.106	0.136	0.151
	AIC	0.061	0.279	0.203	0.279	0.278
2	0	0.023	0.995	0.877	0.986	1.000
	4	0.038	0.510	0.253	0.450	0.411
	8	0.042	0.196	0.112	0.207	0.198
	12	0.039	0.095	0.072	0.122	0.131
	NP	0.051	0.172	0.125	0.179	0.187
	AIC	0.065	0.517	0.322	0.462	0.441
3	0	0.038	1.000	0.997	1.000	1.000
	4	0.035	0.732	0.534	0.717	0.690
	8	0.036	0.365	0.239	0.390	0.349
	12	0.041	0.184	0.124	0.218	0.205
	NP	0.056	0.272	0.199	0.286	0.283
	AIC	0.060	0.676	0.575	0.661	0.643
4	0	0.024	0.187	0.991	0.981	0.999
	4	0.039	0.052	0.502	0.441	0.408
	8	0.041	0.040	0.231	0.214	0.201
	12	0.045	0.035	0.128	0.133	0.136
	NP	0.054	0.053	0.207	0.207	0.212
	AIC	0.056	0.099	0.604	0.573	0.550
5	0	0.039	0.998	1.000	1.000	1.000
	4	0.038	0.596	0.782	0.811	0.793
	8	0.037	0.274	0.425	0.482	0.446
	12	0.040	0.145	0.223	0.290	0.261
	NP	0.047	0.220	0.310	0.355	0.333
	AIC	0.045	0.655	0.723	0.734	0.715

Note: Based on 5000 replications, with T=100 (N = 25 years) for periodic DGPs of Table 1 with zero starting values. t_{π_1} and t_{π_2} are tests for $\pi_1 = 0$ and $\pi_2 = 0$ respectively, F_{34}, F_{234} and F_{1234} are joint tests for $\pi_3 = \pi_4 = 0$, $\pi_2 = \pi_3 = \pi_4 = 0$ and $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$ respectively, all in HEGY regression (10). AUG is the order of augmentation, where NP indicates use of the Ng and Perron (1995) sequential procedure (maximum 12 lags) and AIC the Akaike Information Criterion (maximum 12 lags).

Table 3.2. Rejection Frequency for HEGY Null Hypothesis and Nonperiodic DGPs with Zero Starting Values

DGP	AUG	t_{π_1}	t_{π_2}	F_{34}	F_{234}	F_{1234}
6	0	0.048	1.000	1.000	1.000	1.000
	4	0.040	0.998	1.000	1.000	1.000
	8	0.041	0.901	0.994	1.000	0.999
	12	0.033	0.667	0.890	0.981	0.961
	NP	0.044	0.802	0.922	0.980	0.970
	AIC	0.049	0.999	1.000	1.000	1.000
7	0	0.043	0.043	0.044	0.048	0.050
	4	0.043	0.043	0.044	0.047	0.046
	8	0.039	0.037	0.042	0.046	0.044
	12	0.034	0.038	0.041	0.038	0.042
	NP	0.045	0.049	0.053	0.054	0.058
	AIC	0.047	0.058	0.055	0.061	0.058

Notes: As for Table 3.1, except that the DGPs considered are the random walk (DGP 6) and seasonal random walk (DGP 7).

Table 4.1. Rejection Frequency for HEGY Null Hypothesis and *PI*(1) DGPs with Random Starting Values

DGP	AUG	t_{π_1}	t_{π_2}	F_{34}	F_{234}	F_{1234}
1	0	0.065	0.961	0.865	0.961	0.949
	4	0.065	0.155	0.161	0.279	0.258
	8	0.071	0.046	0.082	0.138	0.150
	12	0.067	0.022	0.061	0.097	0.114
	NP	0.038	0.097	0.125	0.173	0.196
	AIC	0.052	0.344	0.290	0.387	0.395
2	0	0.038	0.999	0.960	0.999	0.998
	4	0.040	0.446	0.271	0.540	0.480
	8	0.044	0.125	0.107	0.224	0.216
	12	0.042	0.050	0.058	0.116	0.130
	NP	0.049	0.182	0.146	0.229	0.240
	AIC	0.066	0.630	0.461	0.623	0.600
3	0	0.035	1.000	1.000	1.000	1.000
	4	0.029	0.636	0.586	0.818	0.753
	8	0.033	0.204	0.202	0.398	0.350
	12	0.034	0.076	0.084	0.170	0.168
	NP	0.049	0.248	0.233	0.336	0.332
	AIC	0.071	0.802	0.767	0.844	0.825
4	0	0.030	0.139	1.000	0.997	0.994
	4	0.033	0.030	0.586	0.513	0.455
	8	0.033	0.023	0.230	0.230	0.222
	12	0.039	0.019	0.105	0.128	0.139
	NP	0.051	0.059	0.280	0.275	0.289
	AIC	0.087	0.138	0.784	0.731	0.693
5	0	0.047	1.000	1.000	1.000	1.000
	4	0.042	0.513	0.846	0.916	0.860
	8	0.042	0.172	0.376	0.519	0.443
	12	0.040	0.060	0.157	0.229	0.214
	NP	0.044	0.210	0.346	0.398	0.394
	AIC	0.060	0.806	0.903	0.917	0.905

Notes: As Table 3.1, except that initial values are selected randomly from the standard normal distribution and seasonal dummy variables are included in the HEGY regression.

Table 4.2.
Proportion of times that the null is reject.

DGP	AUG	t_{π_1}	t_{π_2}	F34	F234	F1234
6.000	0.000	0.052	1.000	1.000	1.000	1.000
	4.000	0.045	0.867	0.998	1.000	1.000
	8.000	0.043	0.428	0.786	0.958	0.895
	12.000	0.034	0.186	0.383	0.614	0.501
	np	0.041	0.466	0.669	0.787	0.746
	I	0.052	0.986	0.999	1.000	0.999
7.000	0.000	0.045	0.047	0.047	0.046	0.044
	4.000	0.037	0.040	0.045	0.043	0.043
	8.000	0.032	0.035	0.037	0.036	0.035
	12.000	0.026	0.028	0.040	0.035	0.028
	np	0.033	0.034	0.054	0.048	0.047
	I	0.047	0.044	0.065	0.066	0.066

Based in 5000 replications T=25, s=4. Original DGP PAR(1): $y_{s\tau} = \alpha_s y_{s-1\tau} + \varepsilon_{s\tau} \quad y_{s0} \sim N(0,1)$.
t_{π_1} and t_{π_2} tests for $H_0 : \pi_1 = 0$ and $H_0 : \pi_2 = 0$ in HEGY.
F 34, F 234 and F 1234 joint tests for $H_0 : \pi_3 = \pi_4 = 0$, $H_0 : \pi_2 = \pi_3 = \pi_4 = 0$ and $H_0 : \pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$ in HEGY.
AUG: Lag truncation in the HEGY augmentation.
NP: Sequential determination of the truncation lag Ng and Perron (1995). maximum lag 12.
AIC: Akaike Information Criterium used to choose the lag truncation, maximum lag 12.