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Iteration of Partially Specified Target Matrices: Application to the Bi-Factor Case

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ABSTRACT

The current study proposes a new bi-factor rotation method, Schmid-Leiman with iterative target rotation (SLi), based on the iteration of partially specified target matrices and an initial target constructed from a Schmid-Leiman (SL) orthogonalization. SLi was expected to ameliorate some of the limitations of the previously presented SL bi-factor rotations, SL and SL with target rotation (SLt), when the factor structure either includes cross-loadings, near-zero loadings, or both. A Monte Carlo simulation was carried out to test the performance of SLi, SL, SLt, and the two analytic bi-factor rotations, bi-quartimin and bi-geomin. The results revealed that SLi accurately recovered the bi-factor structures across the majority of the conditions, and generally outperformed the other rotation methods. SLi provided the biggest improvements over SL and SLt when the bi-factor structures contained cross-loadings and pure indicators of the general factor. Additionally, SLi was superior to bi-quartimin and bi-geomin, which performed inconsistently across the types of factor structures evaluated. No method produced a good recovery of the bi-factor structures when small samples (N = 200) were combined with low factor loadings (0.30–0.50) in the specific factors. Thus, it is recommended that larger samples of at least 500 observations be obtained.

KEYWORDS
Bi-factor rotation; exploratory factor analysis; Schmid-Leiman; target rotation

The use of bi-factor analysis has dramatically increased in the last decade (e.g. Chen, West, & Sousa, 2006; Reise, 2012). One of the reasons for this rise in popularity is the ability of these models to separate the latent sources of common variance by their degree of broadness, from the more general to the more specific. Bi-factor models may be used to assess the relative strength and potential usefulness of first-order and higher order factors for multitiered constructs (McDonald, 1999; Zinbarg, Revelle, Yovel, & Li, 2005), as well as to determine the impact of multidimensionality (Reise, Cook, & Moore, 2015). Additionally, they can be used to estimate the relative strength of general and specific factors in the prediction of an external criterion (Bandalos & Kopp, 2013). In its typical form, the bi-factor model has one general factor and a number of specific factors, with the latter explaining common variance that is non-accounted for by the general factor.

When there is insufficient prior knowledge for the domain under investigation, an exploratory approach is needed to uncover possible bi-factor structures (Jennrich & Bentler, 2011). Given the specific restrictions of the bi-factor model, traditional rotation methods (e.g. varimax, oblimin) fail to recover this structure, as they are oriented toward finding simple structures (Reise, Moore, & Maydeu-Olivares, 2011). In order to overcome this challenge, three general strategies have been proposed: (1) exploratory bi-factor analysis using a Schmid-Leiman (SL) orthogonalization (Schmid & Leiman, 1957), which involves a reparameterization of a second-order oblique exploratory factor analysis solution (Yung, Thissen, & McLeod, 1999); (2) SL followed by a target rotation applied to the bi-factor structure (Browne, 2001; Reise et al., 2011); and (3) analytic bi-factor rotation methods such as bi-factor quartimin or bi-factor geomin (Jennrich & Bentler, 2011, 2012).

At the moment there is limited information regarding the performance of the bi-factor rotation methods currently available. On the one hand, many of the previous studies have considered a very specific set of models and conditions (e.g. Asparouhov & Muthén, 2012). On the other hand, the three types of rotation methods have neither been tested under similar conditions, nor directly compared (e.g. Bandalos & Kopp, 2013; Reise et al., 2015), making it difficult to ascertain their relative accuracy and to offer practical guidelines. Furthermore, there is reason to believe that each of these rotation methods has inherent shortcomings in their formulation that may not make them optimal to uncover exploratory bi-factor structures. In light of this, in the current paper we will propose and test a novel strategy that has not been applied to the bi-factor case: the iteration of partially specified target matrices (Moore, Reise, Depaoli, & Haviland, 2015). We call this method SL with iterative target rotation (SLi).

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A brief review of the properties and known performance of the aforementioned rotation methods will be presented next, followed by the presentation of the newly proposed bi-factor rotation. In order to better summarize this literature, we will first describe four types of factor structures that may be considered as theoretically and practically relevant for this investigation. Following McDonald (1999, 2000), a factor structure is said to be a perfect *independent cluster* (IC) structure if (1) the specific factors are properly identified (i.e. defined by at least three items on orthogonal structures or by two items on oblique structures) and (2) no cross-loadings are observed. If the former condition is met, but cross-loadings are present on the structure, McDonald (2000, p. 102) named those structures as *independent cluster basis* (ICB) structures. In addition, there can be bi-factor models with variables that represent "pure" indicators of the general factor (i.e. items that have zero loadings on the specific factors) (Mansolf & Reise, 2016), and these will be called *independent cluster pure* (ICP) structures. Finally, bi-factor structures that contain both cross-loadings and pure indicators of the general factor will be referred to as *independent cluster basis pure* (ICBP) structures. The ICBP structures represent realistic factor structures that are often found by practitioners, and are being introduced for the first time in the bi-factor literature in this investigation in order to highlight the relative strengths of the different rotation methods.

Bi-factor rotation methods

Schmid-Leiman rotation (SL)

A brief introduction to the SL transformation (Schmid-Leiman, 1957) is presented in the following section. However, readers interested in a complete description of this procedure and its relationships with the higher order factor models are referred to Yung, Thissen, and McLeod (1999).

The SL method is a multistage procedure. In the first step, the manifest variable correlation matrix ($\mathbf{R}$) is factored with an oblique rotation method (e.g. promax, oblimin, geomin):

$$\mathbf{R} = \boldsymbol{\Lambda}_0\,\boldsymbol{\Phi}\boldsymbol{\Lambda}_0' + \boldsymbol{\Psi}_0^2 \tag{1}$$

where $\boldsymbol{\Lambda}_0$ is the loading matrix of the manifest variables on the first-order factors, $\boldsymbol{\Phi}$ is the first-order factor correlation matrix, and $\boldsymbol{\Psi}_0^2$ is the diagonal matrix of unique variances for the manifest variables. In a second step, the higher order factor solution is obtained by factoring the lower order factor correlation matrix ($\boldsymbol{\Phi}$):

$$\boldsymbol{\Phi} = \boldsymbol{\lambda}_1\boldsymbol{\lambda}_1' + \boldsymbol{\Psi}_1^2 \tag{2}$$

where $\boldsymbol{\lambda}_1$ is a vector with the loadings of the first-order factors on the second-order factor, and $\boldsymbol{\Psi}_1$ is a diagonal matrix with the square root of the unique variances for the first-order factors, which is directly related to $\boldsymbol{\lambda}_1$:

$$\boldsymbol{\Psi}_1 = [\mathbf{I} - diag(\boldsymbol{\lambda}_1\boldsymbol{\lambda}_1')]^{1/2} \tag{3}$$

where *diag* indicates that only the diagonal elements from the second-order factor solution are used. Then, the previous model is parameterized as

$$\mathbf{R} = \boldsymbol{\lambda}_g\boldsymbol{\lambda}_g' + \boldsymbol{\Lambda}_s\boldsymbol{\Lambda}_s' + \boldsymbol{\Psi}_0^2 \tag{4}$$

where $\boldsymbol{\lambda}_g$ ($= \boldsymbol{\Lambda}_0\boldsymbol{\lambda}_1$) and $\boldsymbol{\Lambda}_s$ ($= \boldsymbol{\Lambda}_0\boldsymbol{\Psi}_1$) are called SL transformed loadings of the manifest variables on the general and the residualized first-order factors (i.e. after discounting the effects of the general factor), respectively. In this SL parameterization, latent factors are orthogonal and loadings are linearly dependent.

One limitation of the SL method is the assumption that $\boldsymbol{\lambda}_g$ and $\boldsymbol{\Lambda}_s$ follow only one particular structure. Indeed, all the effects from the general factor to the manifest variables are assumed to be indirect. Because of this, Reise et al. (2015) refer to SL as a "semi-restricted" or *hierarchical* bi-factor model. The more general unrestricted bi-factor model follows the same Equation (4), but $\boldsymbol{\lambda}_g$ and $\boldsymbol{\Lambda}_s$ do not follow any specific relationship. These structures that do not contain linearly-dependent general and specific factor loadings are known as *non-hierarchical* bi-factor structures.

Reise et al. (2011, 2015) analyzed the performance of SL under IC and ICB population structures. They found that the "semi-restricted" model produced biased estimates of the factor loadings when proportionality constraints were not met in a simple IC structure. In these cases, loadings on the general factor were either underestimated or overestimated depending on the item. In ICB structures, larger distortions were obtained. For example, for items with large cross-loadings (e.g. .40) that broke the proportionality constraints, SL raised an item's communality, causing an overestimation of the loading on the general factor, whereas loadings on the specific factors were underestimated (Reise et al., 2011, 2015).

Schmid-Leiman with target rotation (SLt)

Despite the expected biases of the SL method, Reise, Moore, and Haviland (2010) predicted that the impact of proportionality on real-world data might be negligible when the goal was only to identify *patterns* of salient and non-salient loadings. Following this line of reasoning, Reise et al. (2011) showed that SL was a good method for identifying the pattern of trivial and non-trivial loadings and, thus, SL could be a useful tool for defining a partially specified pattern matrix for a target rotation (Browne, 1972, 2001). Target rotation, also called Procrustean rotation, requires a partially specified target pattern matrix (**B**) in which zeros indicate that the researcher anticipates that the item will not have a salient loading on the factor, while the remaining target values are not specified. The target rotation minimizes the sum of all the squared differences between each specified target value ($b_{ij} = 0$) and the actual corresponding factor loading (λ_{ij}).

In the first step of the Reise et al. (2011) procedure, a cutoff is applied to obtain the target pattern matrix from the SL loading matrix. For example, if the SL loading is greater than or equal to .20, that target pattern loading is marked as an unspecified element, and if the SL loading is less than .20, that target pattern loading is marked as a specified zero. In the second step, the target rotation is applied. Using a cutoff of .15, Reise et al. (2011) showed that for simple IC structures and sample sizes of 500 or above, target misspecification occurred in a low percentage of cases. Additionally, for conditions where the target transformation matrix was correctly specified, the recovery rates were reasonable. However, further research has shown that under ICB structures, cross-loading presence can impair the performance of SL as a tool for correctly specifying a bi-factor target matrix (Reise et al., 2015).

Analytic bi-factor rotations (bi-quartimin and bi-geomin)

Jennrich and Bentler (2011, 2012) developed an analytic rotation method appropriate for reproducing bi-factor structures. They proposed to minimize the following criterion that measures the departure from the bi-factor structure:

$$B(\boldsymbol{\Lambda}) = \text{qmin}(\boldsymbol{\Lambda}_2) = \sum_{i=1}^{I}\sum_{j=2}^{m}\sum_{j'=j+1}^{m} \lambda_{ij}^2 \lambda_{ij'}^2 \qquad (5)$$

where m is the number of factors, I is the total number of items, and qmin ($\boldsymbol{\Lambda}_2$) is the bi-quartimin rotation criterion, applied to the pattern matrix after excluding the general factor ($\boldsymbol{\Lambda}_2$). When a perfect IC structure is obtained, qmin ($\boldsymbol{\Lambda}_2$) = 0. It follows, therefore, that the bi-quartimin criterion can only be achieved when all cross-loadings in a factor model are zero. It must be noted that $B(\boldsymbol{\Lambda})$ does not depend on the first column of $\boldsymbol{\Lambda}$ (i.e. the general factor), but when $B(\boldsymbol{\Lambda})$ is used for rotation, all the columns in $\boldsymbol{\Lambda}$ are rotated.

Because bi-quartimin rotation attempts to minimize variable complexity by approximating to structures where the items have very low or zero cross-loadings on *all* of the specific factors, it is a method best suited for IC structures. Indeed, Asparouhov and Muthén (2012) simulated IC hierarchical structures under optimal loading size and different number of indicators per specific factor, and concluded that exploratory structural equation models (ESEMs) with bi-quartimin rotation were almost unbiased. However, bi-quartimin is expected to produce biased estimations with ICB structures, and initial studies evaluating its performance in the presence of cross-loadings have supported this expectation (Bandalos & Kopp, 2013).

Another analytic approach developed by Jennrich and Bentler (2012) was the *bi-geomin* rotation method. In this case, the criterion minimized is

$$B(\boldsymbol{\Lambda}) = \text{geomin}(\boldsymbol{\Lambda}_2) = \sum_{i=1}^{I}\prod_{j=2}^{m}\left(\lambda_{ij}^2 + \varepsilon\right)^{1/m} \qquad (6)$$

where ε is a small positive value (i.e. .01) needed to make the function differentiable (Browne, 2001; Jennrich & Bentler, 2012).

The bi-geomin rotation method requires only one specific factor loading of zero per item in order to accomplish the criterion (i.e. $B(\boldsymbol{\Lambda}) = 0$). Thus, this method attempts to minimize variable complexity by approximating to structures that have one zero-element per row in the pattern matrix of the specific factors. Because of this property, Jennrich and Bentler (2012) expected bi-geomin to have better functioning in the presence of cross-loadings. In this line, Mansolf and Reise (2016) showed the theoretical superiority of bi-geomin to bi-quartimin with ICB structures, an advantage that is borrowed from the superior performance of geomin over quartimin with these structures. Additionally, in a preliminary simulation study Bandalos and Kopp (2013) found that bi-geomin rotation provided a good recovery of ICB structures, whereas bi-quartimin failed to recover the true factor structure in these conditions. Nevertheless, for IC structures, higher samples sizes were necessary (e.g. 2,500) in order for bi-geomin to achieve a correct solution.

There are some additional issues regarding the performance of the analytic bi-factor rotations that should be noted. Firstly, as the general factor is not rotated explicitly, both rotation methods are prone to local minima solutions (Mansolf & Reise, 2016). Indeed, these analytic bi-factor rotations can be conceptualized as a mixture of two factor models: a one-factor model defined by the general factor and an m–1 factor model, where m–1 is the number of specific factors. Depending on the starting values, different variance might be shifted to the general factor, and local minima solutions may be obtained. Also, Mansolf and Reise (2016) warn that the analytic bi-factor rotations will tend to shift as much variance onto the general factor as possible, leading in certain cases to the collapse of the specific factors (i.e. for smaller loadings on the specific factors).

Secondly, analytic bi-factor rotations "break down" when the SL constraints are met (Mansolf & Reise, 2016). That is, when there is a perfect linear dependence between the general and specific factor loadings, a first-order model of *m*–1 factors can perfectly represent a bi-factor structure of *m* factors, thus making the latter an overfactored or overparameterized model that can produce Heywood cases and other estimation problems. Therefore, the analytic factor rotations can perform poorly when the SL constraints nearly hold (Mansolf & Reise, 2016).

Schmid-Leiman with iterative target rotation (SLi)

Moore et al. (2015) recently proposed, based on Browne (2001), a method for exploratory factor rotation grounded on the iteration of partially specified factor structures (ITR). In the ITR method, one begins by performing a standard factor rotation and subsequently defines a partially specified, empirically informed target matrix based on the results of this rotation. For this task, a pre-specified loading cutoff criterion (e.g. .20) is established. Then, an iterative search procedure is used to update the target matrix until convergence is reached.

The ITR rotation strategy appears to be particularly useful for data that have a complex structure, such as those with multiple cross-loadings (Moore et al., 2015). This is due to the iterative nature of the method, which has the potential to solve or help ameliorate the problems of using an initially misspecified target matrix. Indeed, Moore et al. (2015) analyzed the performance of ITR with IC and ICB first-order factor structures and found that ITR always outperformed the "one-shot" classical rotations (e.g. quartimin), especially when more cross-loadings were present.

Even though ITR is a promising strategy for factor rotation, it has yet to be applied to the bi-factor case, where a direct generalization of the method can be made. Therefore, we propose in the current paper the use of ITR in conjunction with SL, and call it the SLi method. SLi bi-factor rotation, which can also be considered as an extension of Reise et al. (2011), starts with an initial SL rotation and the obtained rotated matrix is used to specify an initial target matrix (similar to the SLt method). Then, after the target rotation is performed, this new rotated matrix is used to build a new updated target matrix, and the target rotation is again performed. The procedure is repeated in this fashion until the pattern of specified zeros in the target matrix corresponds to the non-salient loadings (e.g. those <.20) in the latest estimated pattern matrix. Due to its iterative approach, SLi should be less affected than SL and SLt by the presence of cross-loadings and pure indicators of the general factor. Moreover, it is expected to be more robust to the issues that particularly affect analytic bi-factor rotations (e.g. local minima, factor collapse, linear dependence of the general and specific loadings), as a result of it being an SL-based method.

Goals of the current study

The present research had two main goals: (1) to evaluate the performance of the newly proposed bi-factor rotation method, SLi, and (2) to compare it with the four bi-factor rotations currently in use, SL, SLt, bi-quartimin, and bi-geomin. As described earlier, the latter four rotation methods had not been studied together, and the current information on their performance was scarce. Further, each of these methods was known or expected to have important shortcomings for particular types of factor structures (see "Bi-factor rotation methods" section), and there was reason to believe that due to its iterative use of targets SLi would provide a better and more consistent performance, in particular for the more complex structures.

In order to achieve the stated goals, a Monte Carlo simulation study was carried out with the manipulation of a large set of variables that were known to affect the performance of the rotation methods. Also, an empirical application of the five bi-factor rotations with a Quality of Life data set (Chen et al., 2006) was undertaken. It should be noted that only bi-factor structures with *orthogonal* factors were considered. As argued by Morin, Arens, and Marsh (2016), these models: (a) ensure interpretable results, and (b) are the most common form of bi-factor methods, with well-known practical applications (e.g. omega reliability coefficient).

Method

The current study considered a comprehensive set of factors and factor levels for the bi-factor models. The following seven variables were manipulated using Monte Carlo methods: (1) sample size (N: 200, 500, 2,000); (2) number of variables per specific factor (VAR.SF: 4, 5, 6); (3) number of specific factors (NUM.SF: 4, 5, 6); (4) presence of cross-loadings on the specific factors (CROSS.SF: no, yes); (5) factor loadings on the specific factors (LOAD.SF: low, medium, high); (6) factor loadings on the general factor (LOAD.GF: low, medium, high); and (7) presence of pure indicators of the general factor (PURE.GF: no, yes). Therefore, the simulation was based on a $3 \times 3 \times 3 \times 2 \times 3 \times 3 \times 2$ factorial design, for a total of 972 conditions.

The factor loadings had ranges from .30 to .50 for the low condition, from .40 to .60 for the medium condition, and from .50 to .70 for the high loading condition. In each case, the loadings for the IC structures were generated with equal increments between loadings under the

Table 1. Examples of the factor loadings and communalities simulated according to the type of structure.

	Independent cluster (IC)						Independent cluster basis (ICB)						Independent cluster pure (ICP)						Independent cluster basis pure (ICBP)					
Item	gf	sf1	sf2	sf3	sf4	h^2	gf	sf1	sf2	sf3	sf4	h^2	gf	sf1	sf2	sf3	sf4	h^2	gf	sf1	sf2	sf3	sf4	h^2
1	.35	.30				.21	.39	.30				.24	.35	.30				.21	.42	.30				.27
2	.34	.37				.25	.47	.37				.36	.54	*.01*				.29	.59	*.01*				.35
3	.39	.43				.34	.41	.43				.35	.31	.43				.29	.30	.43				.28
4	.30	.50				.34	.10	.41	.40			.34	.47	.50				.47	.13	.41	.40			.35
5	.37		.30			.22	.34		.30			.21	.34		.30			.21	.41		.30			.26
6	.45		.37			.33	.37		.37			.27	.61		*.01*			.37	.49		*.01*			.24
7	.41		.43			.35	.35		.43			.31	.38		.43			.33	.47		.43			.41
8	.49		.50			.49	.16		.41	.40		.36	.45		.50			.45	.40		.41	.40		.49
9	.50			.30		.34	.42			.30		.27	.37			.30		.22	.45			.30		.29
10	.31			.37		.23	.50			.37		.38	.59			*.01*		.35	.51			*.01*		.26
11	.42			.43		.36	.31			.43		.29	.33			.43		.29	.38			.43		.33
12	.47			.50		.47	.35			.41	.40	.45	.42			.50		.43	.27			.41	.40	.40
13	.38				.30	.23	.49				.30	.33	.41				.30	.26	.43				.30	.28
14	.43				.37	.32	.38				.37	.28	.62				*.01*	.38	.52				*.01*	.27
15	.33				.43	.29	.46				.43	.40	.30				.43	.28	.34				.43	.30
16	.46				.50	.46	.33	.40			.41	.44	.43				.50	.44	.41	.40			.41	.50
Avg.						.33						.33						.33						.33

Note. gf = general factor; sf = specific factor; h^2 = communality; Avg. = average. IC: no cross-loadings on the specific factors and no pure indicators of the general factor. ICB: cross-loadings but no pure indicators. ICP: pure indicators but no cross-loadings. ICBP: both cross-loadings and pure indicators; Cross-loadings appear underlined; Near-zero loadings in the specific factors appear in italics.

specified range (e.g. the loadings for a factor containing three items in the high range condition were .5, .6, and .7), When cross-loadings were present, the last item for each specific factor had a cross-loading of .40 in the next specific factor. In order to hold constant the communality of the item after adding the cross-loading, a small value was subtracted from each of the remaining non-zero item loadings. For the condition with pure indicators of the general factor, the item in the middle position of each specific factor (e.g. item 2 for a 4-item factor, item 3 for a 5-item factor) had a near-zero loading of .01 in its corresponding specific factor. Here, the loading of the pure item in the general factor was increased so as to maintain the communality equal to what it was before its loading on the specific factor was approximated to zero. An example of population values for the four types of structures simulated (IC, ICB, ICP, ICBP) is presented in Table 1.

Data generation

For each of the simulated conditions, 50 sample data matrices were simulated according to the common factor model procedure. First, the reproduced population correlation matrix (with communalities in the diagonal) was computed

$$\mathbf{R_R} = \mathbf{\Lambda \Phi \Lambda^T} \tag{7}$$

where $\mathbf{R}_\mathrm{R}$ is the reproduced population correlation matrix, $\mathbf{\Lambda}$ is the population factor loading matrix, and $\mathbf{\Phi}$ is the population factor correlation matrix.

The population correlation matrix $\mathbf{R}_\mathrm{P}$ was then obtained by inserting unities in the diagonal of $\mathbf{R}_\mathrm{R}$, thereby raising the matrix to full rank. The next step was performing a Cholesky decomposition of $\mathbf{R}_\mathrm{P}$, such that

$$\mathbf{R_P} = \mathbf{U^T U} \tag{8}$$

where $\mathbf{U}$ is an upper triangular matrix. The sample matrix of continuous variables $\mathbf{X}$ was subsequently computed

$$\mathbf{X} = \mathbf{ZU} \tag{9}$$

where $\mathbf{Z}$ is a matrix of random standard normal deviates with rows equal to the sample size and columns equal to the number of variables.

Accuracy criteria

The accuracy of the rotation methods in the recovery of the population structure was evaluated according to Tucker's congruence coefficient (c.c.; Tucker, 1951)

$$\text{c.c.}_{jj} \frac{\sum_{i=1}^{I} \hat{\lambda}_{ij}\lambda_{ij}}{\sqrt{\sum_{i=1}^{I} \hat{\lambda}_{ij}^2 \sum_{i=1}^{I} \lambda_{ij}^2}} \tag{10}$$

where $\hat{\lambda}_{ij}$ is the estimated loading, λ_{ij} is the population loading, I is the total number of items, i is the item number, and j is the factor number.

The congruence coefficient is an index of similarity between factors that has boundaries of −1 and 1. A congruence coefficient in the range of .85–.94 corresponds to a *fair* similarity between factors, while a coefficient of .95 or higher indicates a *good* level of similarity such that the factors can be considered equal (Lorenzo-Seva, & ten Berge, 2006). The procedure used to align the estimated factors with the population factors before computing the congruence coefficient was as follows.

Firstly, in each sample, the direction of an estimated factor was reverted if its average factor loading was negative, as no true population structure presented negative factor loadings. Secondly, all the possible factor order permutations were computed, retaining the solution that minimized the average absolute deviation between the estimated and the true solutions. Thirdly, an estimated factor was reversed in the final solution if its factor congruence coefficient was negative. All factor analyses were performed in the R environment using the unweighted least squares estimator. In order to obtain the correlated factors solution needed for the initial step of the SL methods a geomin rotation was carried out. The analytic bi-factor rotations bi-quartimin and bi-geomin were performed applying the gradient projection algorithm implemented in the *GPArotation package* (Bernaards & Jennrich, 2005). For each sample, the solution selected for these rotation methods was the one that produced the lowest discrepancy function from a total of 10 random starts. In the case of the SLi method, Moore et al. (2015) reported that ITR rotation converged within 7 iterations; for the current study, a maximum of 20 iterations were computed. In addition, loadings lower than .20 were specified as zeros in the target matrices of the SLt and SLi methods. Analyses of variance (ANOVAs) were carried out with the IBM SPSS Statistics v. 20 program. According to Cohen (1988), partial eta squared (η_p^2) effect sizes of .01 represent small effects, .06 medium effects, and .14 or more, large effects.

Table 2. Average congruence coefficients for the rotation methods across the manipulated variables.

Variable / Level	SL	SLt	SLi	Bi-quartimin	Bi-geomin
N					
200	.926	.934	.938	.866	.899
500	.948	**.967**	**.975**	.907	**.955**
2000	**.957**	**.982**	**.992**	.926	**.985**
VAR.SF					
4	.924	.944	**.959**	.882	.940
5	.944	**.963**	**.970**	.903	.948
6	**.963**	**.976**	**.976**	.914	**.951**
NUM.SF					
4	.944	**.962**	**.969**	.881	.933
5	.945	**.962**	**.969**	.902	**.951**
6	.942	**.959**	**.967**	.917	**.955**
CROSS.SF					
No	**.966**	**.971**	**.969**	**.958**	.937
Yes	.921	**.951**	**.968**	.842	**.955**
LOAD.SF					
Low	.924	.937	.942	.854	.910
Medium	.948	**.967**	**.976**	.911	**.955**
High	**.959**	**.979**	**.987**	.935	**.973**
LOAD.GF					
Low	.934	**.950**	**.958**	.893	.933
Medium	.944	**.961**	**.968**	.900	.945
High	**.953**	**.972**	**.978**	.908	**.960**
PURE.GF					
No	**.977**	**.974**	**.971**	.922	.927
Yes	.910	.949	**.966**	.878	**.966**
STRUCTURE					
IC	**.984**	**.971**	**.968**	**.953**	.910
ICB	**.969**	**.976**	**.974**	.891	.943
ICP	.948	**.972**	**.971**	**.963**	**.964**
ICBP	.872	.926	**.961**	.794	**.968**
TOTAL	.943	**.961**	**.968**	.900	.946

Note. N = sample size; VAR.SF = variables per specific factor; NUM.SF = number of specific factors; CROSS.SF = cross-loadings in the specific factors; LOAD.SF = loadings in the specific factors; LOAD.GF = loadings in the general factor; PURE.GF = pure indicators of the general factor; SL = Schmid-Leiman; SLt = Schmid-Leiman with target rotation; SLi = Schmid-Leiman with iterative target rotation; IC (independent cluster): no cross-loadings and no pure indicators; ICB (independent cluster basis): cross-loadings but no pure indicators; ICP (independent cluster pure): pure indicators but no cross-loadings; ICBP (independent cluster basis pure): both cross-loadings and pure indicators. Congruence coefficients ≥ .95 appear bolded and underlined.

Results

Monte Carlo simulation

An overall assessment of the accuracy of the bi-factor rotation methods is presented in Table 2, which includes the average congruence coefficients across the levels of the independent variables and in total. Additionally, and in order to better understand the performance of the methods, separate ANOVAs were computed for each method where the congruence coefficient was the dependent variable and the manipulated factors were the between-subjects independent variables. The effect sizes resulting from the ANOVAs are shown in Table 3. To limit the number of results shown, only those interactions that attained a large effect size for at least one of the methods were included in Table 3.

The results in Table 2 show that SLi was the most accurate and consistent method in recovering the bi-factor structures. The SLi method produced an overall congruence coefficient of .968, which was followed by SLt (c.c. = .961), bi-geomin (c.c. = .946), SL (c.c. = .943), and lastly, bi-quartimin (c.c. = .900). In addition, SLi obtained a congruence coefficient of "good" (≥.95) for 17 of the 19 factor levels that were evaluated (89.5%), thus exhibiting a more consistently accurate performance than the SLt (78.9%), bi-geomin (52.6%), SL (31.6%), and bi-quartimin (5.3%) rotation methods. As expected, the biggest improvements of SLi in comparison to SL and SLt came with structures that contained cross-loadings (c.c.[SLi] = .968 > c.c.[SLt] = .951 > c.c.[SL] = .921) and pure indicators of the general factor (c.c.[SLi] = .966 > c.c.[SLt] = .949 > c.c.[SL] = .910). Indeed, as Table 3 indicates, whereas these variables had a substantial impact in the performance of SLt (η_p^2 [CROSS.SF] = .124; η_p^2 [PURE.GF] = .177), and particularly SL (η_p^2 [CROSS.SF] = .548; η_p^2 [PURE.GF] = .725), their effect was very small for SLi (η_p^2 [CROSS.SF] = .001; η_p^2 [PURE.GF] = .012). In general, all three SL methods were highly affected by the sample size ($.283 \le \eta_p^2 \le .431$) and the loadings in the specific factors ($.305 \le \eta_p^2 \le .349$), but the effect on the

Table 3. Univariate analysis of variance (ANOVA) effect sizes for the rotation methods.

Effect type/variables	SL	SLt	SLi	Bi-quartimin	Bi-geomin
Main effects					
N	**.283**	**.365**	**.431**	**.294**	**.372**
VAR.SF	**.373**	**.192**	.063	.104	.009
NUM.SF	.003	.002	.001	.129	.042
CROSS.SF	**.548**	.124	.001	**.688**	.038
LOAD.SF	**.332**	**.305**	**.349**	**.438**	**.246**
LOAD.GF	.124	.097	.087	.026	.051
PURE.GF	**.725**	**.177**	.012	**.241**	**.148**
Two-way interactions					
VAR.SF × CROSS.SF	**.220**	.102	.009	.052	.001
CROSS.SF × PURE.GF	**.351**	**.189**	.024	**.323**	.024
N × LOAD.SF	**.145**	**.200**	**.257**	.128	**.145**
VAR.SF × PURE.GF	**.296**	.113	.013	.002	.000
Three-way interactions					
VAR.SF × CROSS.SF × PURE.GF	**.168**	.104	.013	.002	.000

Note. N = sample size; VAR.SF = variables per specific factor; NUM.SF = number of specific factors; CROSS.SF = cross-loadings in the specific factors; LOAD.SF = loadings in the specific factors; LOAD.GF = loadings in the general factor; PURE.GF = pure indicators of the general factor; SL = Schmid-Leiman; SLt = Schmid-Leiman with target rotation; SLi = Schmid-Leiman with iterative target rotation. The dependent variable in the ANOVAs was the congruence coefficient. The effect size statistic used was partial eta squared. Large effect sizes (≥ .14) appear bolded and underlined. Only interactions with large effect sizes for at least one method are shown.

number of variables per specific factors was much lower for SLi ($\eta_p^2 = .063$), in comparison to SL ($\eta_p^2 = .373$) and SLt ($\eta_p^2 = .192$).

Regarding the performance of the analytic bi-factor rotations, bi-quartimin produced the worst results of any other method evaluated. This rotation performed poorly for the majority of the factor levels, but was especially sensitive to the cross-loading factor ($\eta_p^2 = .688$), a finding that is line with the theoretical expectations. Bi-geomin, on the other hand, performed much better than bi-quartimin, particularly when the factor structures contained cross-loadings (c.c.[bi-geomin] = .955 >> c.c.[bi-quartimin] = .842) or pure indicators of the general factor (c.c.[bi-geomin] = .966 >> c.c.[bi-quartimin] = .878). In fact, and contrary to the behavior of the other four methods, bi-geomin actually performed better with cross-loadings or pure indicators than without them (Table 2). Despite these results, bi-geomin still produced subpar estimations with small samples of 200 observations (c.c. = .899) or with low loadings in the specific factors (c.c. = .910).

As can be seen in Table 3, the rotation methods were affected by several interactions of the manipulated variables. The two-way interaction of sample size × loadings on the specific factors was the most consistently salient one, producing a large or near-large effect for all of the methods ($.128 \leq \eta_p^2 \leq .257$). The results of this interaction are plotted in Figure 1 and they show that the performance of all the methods improved with larger samples, but that the improvement was greater for lower loadings in the specific factors. In the case of bi-quartimin and bi-geomin, the accuracy in the recovery of the factor structures was particularly poor when small samples of 200 observations were combined with low factor loadings in the specific factors (c.c. <.85).

An additional interaction that affected particularly the SL ($\eta_p^2 = .168$) and SLt ($\eta_p^2 = .104$) methods was the three-way interaction of variables per specific factor × cross-loadings × pure indicators (Figure 2). The 3 two-way interactions (VAR.SF × CROSS.SF, VAR.SF × PURE.GF, and CROSS.SF × PURE.GF) contained in this three-way interaction all had large or near-large effect sizes for the aforementioned rotations ($.102 \leq \eta_p^2 \leq .351$), so they were analyzed in the context of the higher order interaction. Also, and in order to better understand the differences in performance between the methods, the three-way interaction was plotted for the other three methods (SLi, bi-quartimin, and bi-geomin), where it had a small or negligible effect ($\eta_p^2 \leq .013$). It should be noted that the two-way interaction of cross-loadings x pure indicators did produce a large effect for bi-quartimin ($\eta_p^2 = .323$).

The three-way interaction contained in Figure 2 can be explained as a function of the two-way interaction of cross-loading × pure indicators that in turn interacts with the third factor, number of variables per specific factor.

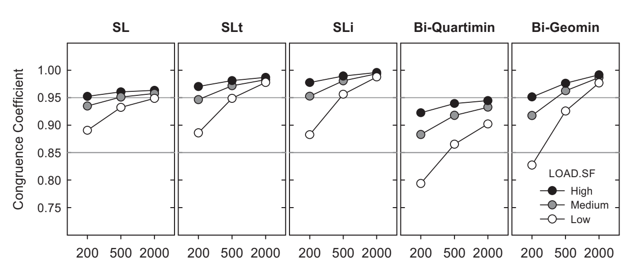

Figure 1. Two-way interaction of N x LOAD.SF with congruence coefficient as dependent variable. *Note.* N, sample size; LOAD.SF, loadings on the specific factors; SL, Schmid-Leiman; SLt, SL target; SLi, SL with iterative target.

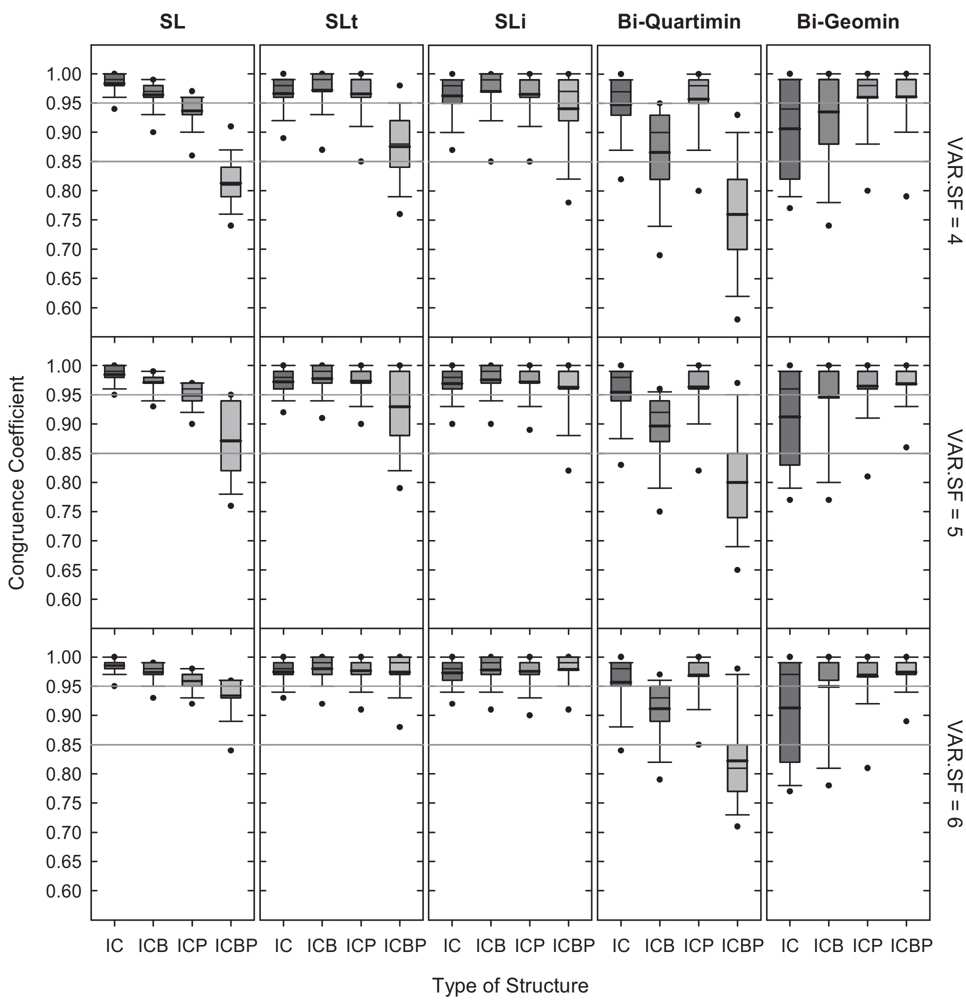

Figure 2. Box plots corresponding to the three-way interaction of VAR.SF x CROSS.SF x PURE.GF with congruence coefficient as dependent variable. *Note*. VAR.SF, variables per specific factor; CROSS.SF, cross-loadings on the specific factors; PURE.GF, pure indicators of the general factor; SL, Schmid-Leiman; SLt, SL target; SLi, SL with iterative target; IC (independent cluster): no cross-loadings and no pure indicators; ICB (independent cluster basis): cross-loadings but no pure indicators; ICP (independent cluster pure): pure indicators but no cross-loadings; ICBP (independent cluster basis pure): both cross-loadings and pure indicators. The thick and thin horizontal lines within each box denote the mean and median. The top and bottom black circles denote the 95th and 5th percentiles.

Firstly, the combined levels of cross-loadings (no, yes) and pure indicators (no, yes) generate the four types of structures considered in this study (IC, ICB, ICP, and ICBP), and their two-way interaction can be clearly seen within each rectangle in Figure 2 for the SL ($\eta_p^2 = .351$), SLt ($\eta_p^2 = .189$), and bi-quartimin ($\eta_p^2 = .323$) rotations. This interaction is evidenced by the substantial differences in accuracy that these rotation methods produce for the four types of structures evaluated. In particular, it can be seen that there was a notably poorer recovery of the ICBP factor structures for these rotations in comparison to their recovery of the IC, ICB, and ICP structures. Additionally, in the case of bi-quartimin, there was also a marked decrease in accuracy for ICB in comparison to the congruence coefficients obtained for IC or ICP. Secondly, the three-way interaction emerged for the SL ($\eta_p^2 = .168$) and SLt ($\eta_p^2 = .104$) methods because the differences in accuracy in the recovery of the four types of structures were greatly diminished as the number of variables per specific factor increased. In contrast, bi-quartimin produced similar results for each level of number of variables per specific factor, which is why the three-way interaction was not salient for this method ($\eta_p^2 = .002$).

The SLi and bi-geomin methods, on the other hand, did not produce important interactions between the factors considered in Figure 2 ($\eta_p^2 \leq .024$ for the two-way interactions and $\eta_p^2 \leq .013$ for the three-way interaction). This was because their recovery accuracy was fairly similar for the four types of structures, regardless of the number of variables per specific factor. The SLi method, in particular, showed the most stable estimations across the IC, ICB, ICP, and ICBP structures, as bi-geomin showed much greater variability in the congruence coefficients that it produced for the IC structures.

Quality of life data set

An empirical study was conducted by factor analyzing a Quality of Life data set (Chen et al., 2006). This Quality of Life data set encompasses 403 observations and 17 items that are hypothesized to reflect a common general factor (Quality of Life) and four specific factors (Cognition, Vitality, Mental Health, and Disease Worry).

There is some controversy regarding the possible bi-factor structure underlying the Quality of Life data set, in particular regarding the third specific factor Mental Health. Using a confirmatory approach, Chen et al. (2006) concluded that this specific factor could be absorbed by the general factor and recommended that it be dropped. In contrast, Jennrich and Bentler (2011) suggested based on a bi-quartimin rotation that it might be retained as two of its items produced salient loadings on this specific factor. For the current study, the five bi-factor rotation methods under investigation were applied to the Quality of Life data set by factorizing the covariance matrix provided in Chen et al. (2006) with the package *psych* (Revelle, 2016). In the case of the SLt and SLi methods, a cutoff of .20 (Jennrich & Bentler, 2011; Moore et al., 2015) was used to distinguish between salient and non-salient loadings for the specification of the target matrices.

The factor loading matrices corresponding to the five bi-factor rotation methods are shown in Table 4. As evidenced by Table 4, the factor loadings for the general factor (Quality of Life) and the first (Cognition) and fourth (Disease Worry) specific factors are consistently high, with no cross-loadings ($\geq$.20) for these items in any of the rotations. Indeed, when congruence coefficients were computed between each pair of rotation methods for these factors, they were extremely high: between .997 and 1.000 for Quality of Life, .983 and 1.000 for Cognition, and between .972 and 1.000 for Disease Worry. In the case of the second specific factor (Vitality), the congruence coefficients between the SL, SLt, SLi, and bi-geomin methods were also especially high (.978 $\leq$ c.c. $\leq$ .997); however, they were somewhat lower for the four pairs that contained the bi-quartimin rotation (.936 $\leq$ c.c. $\leq$ .957). Interestingly, this factor had the only item ("Feel full of pep?") that achieved a cross-loading of at least .20 for any of the rotations, and the previous simulation results had shown that bi-quartimin was highly affected by the presence of cross-loadings.

As with previous factor analyses of this data set, the greatest differences in factor loadings were obtained for the third specific factor (Mental Health). Here, the congruence coefficients between seven pairs of methods were notably low (.290 $\leq$ c.c. $\leq$ .685). The only three congruence coefficients that showed good agreement were between SLt and SLi (c.c. = .958), SLt and bi-geomin (c.c. = .967), and between SLi and bi-geomin (c.c. = .995). It is noteworthy in this case that the factor loadings suggested by SLi and bi-geomin for this specific factor include three items ("Feel downhearted and blue?," "Feel very nervous?," and "Feel so down in the dumps nothing could cheer you up?") that are essentially pure indicators of the general factor, as they produced negligible loadings on the specific factor. If indeed the population structure had these characteristics, the findings would be in line with the simulation results of this study, which showed that SLi and bi-geomin were the two most likely methods to produce accurate recoveries of the factor loadings when pure indicators were present. If researchers are interested in reproducing the presented analysis, the R code necessary for computing the SLi rotation of the Quality of Life data set can be found in the Supplementary Materials of this article.

Discussion

For hierarchically structured constructs that operate at various levels of generality, bi-factor analysis has become an essential modeling technique as a result of its capability to separate the general and specific variances underlying the observed data (Brunner, Nagy, & Wilhelm, 2012). Several rotation methods have been proposed for exploratory bi-factor analysis, including the SL orthogonalization (SL; Schmid & Leiman, 1957), SLt(SLt; Reise et al., 2011), and two analytic bi-factor rotations, bi-quartimin and bi-geomin (Jennrich & Bentler, 2011, 2012). However, at the moment, there is limited information regarding the performance of these rotations under varying data characteristics and in comparison to each other. Furthermore, there are concerns regarding the efficacy of these rotations for certain types of factor structures that are based on their theoretical formulations and the empirical evidence that is available (Bandalos & Kopp, 2013; Mansolf & Reise, 2016; Reise et al., 2011, 2015). Taking into consideration the issues outlined previously, a new bi-factor rotation was proposed in the current study based the use of iterative targets in conjunction with an initial SL orthogonalization: The SLi method. To test the accuracy of this new rotation, an extensive simulation study was undertaken where seven relevant variables were manipulated, thus permitting an in-depth comparison of the accuracy of SLi against the other four bi-factor rotations. The most important findings from this Monte Carlo study will be discussed next, as well as the results obtained with a Quality of Life data set.

Table 4. Bi-factor rotation methods applied to the Chen et al. (2006) quality of life data.

	Schmid-Leiman (SL)					SL with target rotation (SLt)					SL with iterative target rotation (SLi)					Bi-quartimin					Bi-Geomin				
Item	gf	sf1	sf2	sf3	sf4	gf	sf1	sf2	sf3	sf4	gf	sf1	sf2	sf3	sf4	gf	sf1	sf2	sf3	sf4	gf	sf1	sf2	sf3	sf4
diffreas	**.53**	**.67**	.00	.01	− .01	**.55**	**.65**	.00	− .12	− .02	**.59**	**.62**	− .03	− .11	− .04	**.56**	**.64**	.02	.11	.00	**.58**	**.62**	− .02	− .12	− .04
sloract	**.45**	**.47**	.06	− .01	.04	**.45**	**.47**	.07	− .03	.04	**.47**	**.46**	.06	− .03	.03	**.48**	**.45**	.06	.00	.03	**.47**	**.46**	.06	− .03	.03
confsed	**.54**	**.64**	.01	.01	.02	**.54**	**.66**	.03	.08	.04	**.55**	**.64**	.03	.07	.04	**.60**	**.60**	− .02	− .09	.00	**.55**	**.65**	.03	.07	.04
forget	**.44**	**.67**	.00	− .02	− .03	**.44**	**.68**	.02	.03	− .02	**.46**	**.66**	.01	.02	− .02	**.49**	**.64**	− .01	− .06	− .04	**.45**	**.67**	.02	.01	− .02
diffconc	**.56**	**.63**	.00	.04	.00	**.57**	**.61**	.01	− .03	.00	**.60**	**.59**	− .01	− .02	− .01	**.59**	**.58**	.00	.04	.01	**.59**	**.59**	− .01	− .03	− .01
tired	**.65**	.02	**.54**	.04	− .03	**.65**	.05	**.54**	− .02	− .03	**.65**	.03	**.54**	− .01	− .04	**.70**	− .01	**.47**	.00	− .07	**.67**	.02	**.52**	.02	− .06
enrgtic	**.55**	− .01	**.38**	.04	.07	**.55**	.00	**.38**	− .01	.06	**.55**	− .01	**.38**	.00	.06	**.58**	− .04	**.33**	.00	.03	**.57**	− .02	**.36**	.01	.04
wornout	**.65**	.06	**.57**	− .04	.06	**.64**	.09	**.58**	− .09	.06	**.65**	.07	**.56**	− .10	.04	**.68**	.04	**.54**	.00	.02	**.67**	.06	**.55**	− .07	.02
peppy	**.67**	− .03	**.39**	.15	− .02	**.67**	− .02	**.42**	.17	− .01	**.66**	− .02	**.44**	**.20**	.01	**.74**	− .11	**.28**	− .10	− .08	**.67**	− .03	**.40**	**.21**	− .01
atpeace	**.72**	.02	− .04	**.36**	.02	**.77**	− .04	− .06	**.34**	.00	**.73**	− .03	− .02	**.42**	.04	**.80**	− .12	− **.22**	− .04	− .04	**.73**	− .02	− .07	**.41**	.02
feelblue	**.75**	.05	.00	**.31**	.08	**.83**	− .05	− .08	− .04	.01	**.83**	− .08	− .08	.05	.00	**.78**	− .05	− .08	**.29**	.07	**.83**	− .07	− .11	.03	− .02
happy	**.66**	− .03	.06	**.33**	− .02	**.70**	− .07	.05	**.28**	− .03	**.67**	− .07	.09	**.35**	.00	**.74**	− .15	− .10	− .04	− .08	**.68**	− .06	.04	**.35**	− .02
nervous	**.64**	**.24**	.01	.17	.09	**.68**	.19	− .02	− .03	.06	**.69**	.17	− .02	.02	.05	**.66**	.18	− .03	.16	.09	**.69**	.17	− .04	.01	.04
down	**.70**	.13	.05	**.27**	.01	**.81**	.01	− .05	− **.20**	− .10	**.83**	− .03	− .06	− .10	− .12	**.73**	.05	.01	**.42**	.00	**.82**	− .02	− .09	− .12	− .14
afraid	**.66**	.03	− .02	.00	**.60**	**.66**	.03	− .05	− .07	**.60**	**.67**	.00	− .06	− .06	**.59**	**.63**	.02	− .03	.06	**.63**	**.68**	.00	− .07	− .08	**.57**
frust	**.68**	.00	.13	.02	**.44**	**.67**	.02	.12	.02	**.45**	**.67**	.00	.12	.02	**.45**	**.68**	− .02	.09	− .01	**.44**	**.68**	− .01	.10	.02	**.43**
hlthwry	**.61**	− .02	.02	.01	**.52**	**.59**	.01	.02	.08	**.56**	**.59**	.00	.02	.07	**.56**	**.61**	− .04	− .03	− .08	**.54**	**.60**	− .01	.01	.07	**.55**

Note. gf = general factor (Quality of life); sf = specific factor (sf1: Cognition; sf2: Vitality; sf3: Mental health; sf4: Disease worry); diffreas: "Have difficulty reasoning and solving problems?"; sloract: "React slowly to things that were said or done?"; confsed: "Become confused and start several actions at a time?"; forget: "Forget where you put things or appointments?"; diffconc: "Have difficulty concentrating?"; tired: "Feel tired?"; enrgtic: "Have enough energy to do the things you want?"; wornout: "Feel worn out?"; peppy: "Feel full of pep?"; atpeace: "Feel calm and peaceful¿'; feeleblue: "Feel downhearted and blue?"; happy: "Feel very happy?"; nervous: "Feel very nervous?"; down: "Feel so down in the dumps nothing could cheer you up?"; afraid: "Were you afraid because of your health?"; frust: "Were you frustrated about your health?"; healthwry: "Was your health a worry in your life?"; The dotted lines indicate the item groupings according to the theoretical dimensions. Factor loadings ≥ .20 in absolute value appear bolded and underlined.

Main findings

The results pertaining to the SL rotation showed that it produces the highest levels of accuracy of any method for IC structures, but that it is much less effective with complex structures, in particular those that combine cross-loadings with pure indicators of the general factor. These results are in line with the theoretical expectations, as the latter structure presents the greatest departure from the hierarchical model that is the basis for its formulation. In particular, pure indicators constitute severe violations of the hierarchical model assumed by SL rotation where the general and specific factor loadings are linearly dependent or proportional. Previous research had also shown that SL produces biased estimates of the factor loadings in the presence of cross-loadings (Reise et al., 2011, 2015). Additionally, the number of variables per specific factor, the factor loadings in the specific factors, and the sample size, affect the performance of SL, which produces substantially lower levels of accuracy when these variables have smaller values.

Using a one-shot target rotation with the SLt method improves the performance of SL slightly for structures with cross-loadings and more substantially for structures with pure indicators. In particular, SLt is advantageous for very complex structures that combine cross-loadings with pure indicators. It appears, therefore, that SL can be a useful tool to define a partially specified pattern matrix for target rotation, as suggested by Reise et al. (2010, 2011), and that using the SLt method can correct some of the misspecifications that SL produces with these complex structures. However, the performance of SLt with structures that combine cross-loadings and pure indicators is still very variable and mostly below the levels that are considered to represent a good factorial recovery. This is true, especially in those cases where there are also a small number of variables per specific factor. Regarding the other independent variables, the accuracy of SLt is affected by the sample size and the loadings in the specific factors in a similar way as the SL method.

The SLi method was introduced in this study with the aim of improving the performance provided by SLt with the more complex structures, and the findings from the Monte Carlo simulation suggest that indeed it is capable of accomplishing this goal. The performance of SLi is nearly identical to that of SLt for the majority of the data structures, except for the most complex ones that combine cross-loadings and pure indicators. In these cases, SLi is much less variable and produces substantially higher levels of accuracy than SLt, in particular when these types of structures also have a small number of variables per factor. Therefore, it can be concluded that using iterative targets is a useful strategy for bi-factor rotation, particularly when the population structures have diverse departures from the IC model. These findings extend those of Moore et al. (2015), which had proposed and evaluated the use of iterative targets with first-order factor models. In general, the evidence suggests that SLi is the most consistent and accurate of the bi-factor rotations considered here.

The performances of the two analytic bi-factor rotations, bi-quartimin and bi-geomin, are distinctly different. Bi-quartimin produces good accuracy levels for structures that contain ICs or that deviate from them only due to pure indicators of the general factor. However, its performance is notably poor when cross-loadings are introduced and even worse when they are combined with pure indicators. The poor results of bi-quartimin with cross-loadings are in line with Bandalos and Kopp (2013) and reflect the major theoretical shortcoming of the quartimin rotation: it attempts to minimize variable complexity by searching for structures where the items have very low or zero cross-loadings on *all* of the factors. Here, the evidence suggests that when the population structure deviates from the IC model (as it often does in practice) the impact on the accuracy of bi-quartimin is extreme. Bi-geomin, on the other hand, produces a unique pattern of results that is unlike that of the other methods. With IC structures bi-geomin obtains its worst accuracy levels, which are also substantially below the ones of the other methods evaluated. This result was expected, as the bi-geomin criterion is minimized for structures that contain at least *one* non-zero cross-loading. When cross-loadings are introduced, the performance of bi-geomin shows a notable improvement that includes much higher levels of accuracy than bi-quartimin, in line with Mansolf and Reise (2016), but that are still considerably lower than those of the SL-based methods. Surprisingly, bi-geomin produces its best performance with pure indicators, achieving its highest accuracy and that of any method for the most complex structures that combine cross-loadings and pure indicators. This is a unique finding of this study that points to the usefulness of this rotation for these types of structures. Nonetheless, it should be mentioned that bi-geomin is a method that performs considerably different across sample sizes, and that is not really suited for small samples.

From the results of this simulation study it is not possible to determine if the poor performance of the analytic rotations for certain factor structures can be attributed in part to the more correct rotation being contained in a local minimum rather than the global minimum (chosen here). Analytic bi-factor rotations are prone to local minima solutions because the general factor is not rotated explicitly (Mansolf & Reise, 2016). That is, analytic bi-factor rotations utilize a two-stage process where in the first "gradient descent" step only the specific factors are rotated, excluding the general factor. Then, in the second "projection" step the obtained solution is projected in order to obtain a proper factor loading matrix. Thus, the general factor is only rotated implicitly (i.e. by projection to a proper solution). For practical use, it has been suggested that researchers examine the different local minima and global minimum solutions (as there is no mathematical reason to prefer one over the other) and to select the most interpretable one (Asparouhov & Muthén, 2009). However, it is unknown if this process would lead to actually choosing the more correct or replicable solution in practice more often or not.

An empirical study based on a Quality of Life data set (Chen et al., 2006) included in the study appears to support the results of the Monte Carlo simulation. The findings related to the congruence of the factor solutions obtained by the different rotation methods show that for complex factors (those that appear to have items with cross-loadings or pure indicators) bi-quartimin and SL are the methods that have the least agreement with the others, suggesting that their solutions may not be accurate estimations of the population structure. Additionally, for these factors the methods with the highest agreement are SLi and bi-geomin, which in the simulation were the ones that performed the best for structures that contained both cross-loadings and pure indicators. For strong factors that contained items without notable cross-loadings and that had substantial loadings in both the general factor and their respective specific factors, all the methods showed very high agreement with each other.

Limitations

As with any Monte Carlo simulations, the findings of this study are only generalizable to the conditions that were analyzed. Some additional limitations, as well as recommendations for future research, will be addressed next. First, the factor analyses in this study were carried out using the unweighted least squares estimator over Pearson correlations obtained from continuous variables. More research is needed to understand how the bi-factor rotations perform with other estimators (e.g. maximum likelihood, weighted least squares), types of variables (e.g. ordered-categorical), and measures of association (e.g. polychoric correlations). Second, all the simulated structures were balanced, with equal numbers per factor of variables, cross-loadings, and pure indicators. The study of unbalanced structures could provide further insight regarding the accuracy of the bi-factor rotations. Third, iterative targets were evaluated in conjunction with an initial SL orthogonalization based on a geomin rotation for the first-order factor analysis. Additional research is needed to determine how bi-factor rotations with iterative targets would perform with an initial target based on other methods, such as a bi-geomin rotation, or with a SL orthogonalization based on other oblique rotations like oblimin, which is implemented in the SCHMID routine of the *psych* package (Revelle, 2016).

Another issue of importance related to iterative target rotation is the selection of the factor loading cutoff value needed to specify the target matrices. In the present study, a theoretical cutoff value of .20 was used to determine if a loading was to be considered as salient or non-salient. At this moment it is unknown how using other cutoff values would affect the recovery of the bi-factor structures. A possible alternative to this issue could be the use of empirically derived cutoff values, like it is done with promin rotation (Lorenzo-Seva, 1999) or with the standard error method (Moore, 2013). The combination of empirical specifications of the target transformation with empirical cutoff values could ultimately lead to an application of target rotation methods that does not require any additional input from the researcher.

Practical implications

The findings from this study suggest that there are important differences in the levels of accuracy with which the different rotation methods currently available can recover exploratory bi-factor structures. In light of this, it is important for applied researchers to be cognizant of the methods that can best aid them in uncovering these structures, those that should be avoided, and the conditions where none are likely to perform well.

The SLi method, proposed for the first time in this study, was the most accurate and consistent bi-factor rotation across the wide range of conditions that were explored, thus making it one of the methods that can be recommended for applied research. In contrast, the original SL rotation is not recommended due to its notable poorer performance with structures that deviate from the IC model. In the case of the SLt method, its performance was mainly as good as or worse than SL with iterative targets, making the latter the obvious choice for general applied use. Regarding the analytic bi-factor rotations, bi-quartimin is clearly a method that should be avoided due it is markedly poor accuracy across the majority of factor structures. Bi-geomin, on the other hand, can be recommended for cases where the researchers expect complex structures that contain both cross-loadings and pure indicators of the general factor. It should be used with caution, however, because bi-geomin requires larger samples and tends to perform very poorly when the structures contain ICs. Finally, no method is likely to produce a good recovery of the bi-factor structures when small samples (N =

200) are combined with low factor loadings (0.30–0.50) in the specific factors. If this situation is expected, it is recommended that larger samples be obtained in order to offset the detrimental effects of the low item loadings.

Article information

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