Graded-division algebras and Galois extensions

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Graded-division algebras are building blocks in the theory of finite-dimensional associative algebras graded by a group G. If G is abelian, they can be described, using a loop construction, in terms of central simple graded-division algebras.

On the other hand, given a finite abelian group G, any central simple G-graded-division algebra over a field $\mathbb{F}$ is determined, thanks to a result of Picco and Platzeck, by its class in the (ordinary) Brauer group of $\mathbb{F}$ and the isomorphism class of a G-Galois extension of $\mathbb{F}$.

Goal

To explore this connection between graded-division algebras and Galois extensions.

Outline

1. Graded Brauer group
2. Galois extensions
3. From graded-division algebras to Galois extensions
4. Simple abelian Galois extensions

Brauer group

Any finite-dimensional central simple associative algebra over a field $\mathbb{F}$ is isomorphic to the algebra $\mathrm{End}_{\mathcal{D}}(V)$ of endomorphisms of a finite rank right module V for a division algebra $\mathcal{D}$.

The Brauer group $\mathrm{Br}(\mathbb{F})$ is the group of equivalence classes of finite-dimensional central simple algebras, with two such algebras being equivalent if they are isomorphic to matrix algebras over the same division algebra.

The multiplication in $\mathrm{Br}(\mathbb{F})$ is induced by the tensor product of $\mathbb{F}$-algebras: $[\mathcal{A}][\mathcal{B}] = [\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}]$.

Graded Brauer group

The definition of a graded Brauer group is not evident.

One possibility is to fix a bicharacter $\phi : G \times G \to \mathbb{F}^\times$ and define central simple algebras and (twisted) tensor products relative to ϕ.

Here we will focus on the special case of trivial ϕ!!

Definition

The **graded Brauer group** $\mathrm{Br}_G(\mathbb{F})$ consists of the equivalence classes of finite-dimensional associative algebras that are central simple and G-graded, with two such algebras $\mathcal{A}$ and $\mathcal{B}$ being equivalent if there is a central simple G-graded-division algebra $\mathcal{D}$ and G-graded right $\mathcal{D}$-modules V and W such that $\mathcal{A}$ is graded-isomorphic to $\mathrm{End}_{\mathcal{D}}(V)$ and $\mathcal{B}$ to $\mathrm{End}_{\mathcal{D}}(W)$.

The class of $\mathcal{A}$ in $\mathrm{Br}_G(\mathbb{F})$ will be denoted by $[\mathcal{A}]_G$.

Loop algebras

Given an epimorphism of abelian groups $\pi : G \to \overline{G}$ and a nonassociative algebra $\mathcal{A}$ graded by $\overline{G}$, the associated **loop algebra** is the G-graded algebra

$$L_\pi(\mathcal{A}) := \bigoplus_{g \in G} \mathcal{A}_{\pi(g)} \otimes g \subset \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G,$$

$$L_\pi^\tau(\mathcal{A}) := \bigoplus_{g \in G} \mathcal{A}_{\pi(g)} \otimes g \subset \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}^\tau G,$$

Theorem (2019)

Any G-graded-central-simple algebra (not necessarily associative or finite-dimensional) is graded-isomorphic to a cocycle-twisted loop algebra of a central simple $\overline{G}$-graded algebra, for a suitable quotient $\overline{G}$ of G.

Application

Let $\mathcal{L}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{F}$ of characteristic 0, graded by an abelian group G, and let $V = V(\lambda)$ be a finite-dimensional irreducible representation.

A natural question is whether or not $V = V(\lambda)$ admits a G-grading that makes it a graded $\mathcal{L}$-module.

A necessary condition is that λ be invariant under the action of the dual group $\widehat{G} = \mathrm{Hom}(G, \mathbb{F}^\times)$.

If this is the case, one can define a G-grading on $\mathrm{End}_{\mathbb{F}}(V)$ such that the representation $U(\mathcal{L}) \to \mathrm{End}_{\mathbb{F}}(V)$ is a homomorphism of graded algebras.

V admits a G-grading if and only if the class of $\mathrm{End}_{\mathbb{F}}(V)$ in the graded Brauer group $\mathrm{Br}_G(\mathbb{F})$ is trivial.

From now on, all the algebras will be unital, associative, and finite-dimensional over a field $\mathbb{F}$.

The group G will always be abelian and finite.

G-algebras

A ***G*-algebra** is an algebra $\mathcal{C}$ endowed with an action of the group G by automorphisms:

$$\begin{aligned} G &\longrightarrow \mathrm{Aut}_{\mathbb{F}}(\mathcal{C}) \\ g &\mapsto (c \mapsto g \cdot c), \end{aligned}$$

The fixed subalgebra $\{c \in \mathcal{C} \mid g \cdot c = c \ \forall g \in G\}$ will be denoted by $\mathcal{C}^G$.

A **homomorphism of *G*-algebras** is an algebra homomorphism $\psi : \mathcal{C}_1 \to \mathcal{C}_2$ such that $\psi(g \cdot c) = g \cdot \psi(c)$ for all $g \in G$ and $c \in \mathcal{C}_1$.

Galois extensions

Definition

Let G be a finite group. A **G-Galois extension of $\mathbb{F}$** is a finite-dimensional unital G-algebra $\mathcal{C}$ over $\mathbb{F}$ such that the action of G on $\mathcal{C}$ is faithful, $\mathcal{C}^G = \mathbb{F}1$, and the following equivalent conditions hold:

(a) The homomorphism

$$\Phi : \mathcal{C}\#\mathbb{F}G \longrightarrow \mathrm{End}_{\mathbb{F}}(\mathcal{C}), \quad cg \mapsto \big(x \mapsto c(g\cdot x)\big)$$

is an isomorphism.

(b) The linear map

$$\mathcal{C} \otimes_{\mathbb{F}} \mathcal{C} \longrightarrow \mathrm{Maps}(G, \mathcal{C}), \quad a \otimes b \mapsto \big(g \mapsto a(g\cdot b)\big)$$

is bijective.

Group of Galois extensions

The isomorphism class of a G-algebra $\mathcal{C}$ will be denoted by $[\mathcal{C}]_{G\text{-alg}}$.

The set of isomorphism classes

$$\mathrm{E}_G(\mathbb{F}) = \{[\mathcal{C}]_{G\text{-alg}} \mid \mathcal{C} \text{ is a } G\text{-Galois extension of } \mathbb{F}\}$$

turns out to be a group with neutral element

$$(\mathbb{F}G)^* \simeq \mathrm{Maps}(G, \mathbb{F}),$$

where G acts by $(g \cdot f)(h) = f(hg)$.

Picco-Platzeck's exact sequence

Theorem (1970)

There is a split exact sequence

$$1 \longrightarrow \mathrm{Br}(\mathbb{F}) \xrightarrow{\iota} \mathrm{Br}_G(\mathbb{F}) \xrightarrow{\zeta} \mathrm{E}_G(\mathbb{F}) \longrightarrow 1$$

where $\zeta([\mathcal{A}]_G) = [\mathrm{Cent}_{\mathcal{A}\#(\mathbb{F}G)^*}(\mathcal{A})]_{G\text{-alg}}$

- $(\mathbb{F}G)^* = \bigoplus_{g\in G} \mathbb{F}\epsilon_g$ is a Hopf algebra:

$$\Delta(\epsilon_g) = \sum_{h\in G} \epsilon_{gh^{-1}} \otimes \epsilon_h.$$

- $\mathcal{A}\#(\mathbb{F}G)^*$ is the smash product:

$$(a\epsilon_g)(b\epsilon_h) = (ab_{gh^{-1}})\epsilon_h,$$

  where b_g is the g-component of b.
- G-action on $\mathcal{A}\#(\mathbb{F}G)^*$: $g \cdot (a\epsilon_h) = a\epsilon_{hg^{-1}}$.

Picco-Platzeck's exact sequence

Proposition

Let G be a finite abelian group and let $\mathcal{A}$ be a G-graded algebra such that, for every $g \in G$, the homogeneous component $\mathcal{A}_g$ contains an invertible element. Then $\mathrm{Cent}_{\mathcal{A}\#(\mathbb{F}G)^*}(\mathcal{A})$ is antiisomorphic to $\mathrm{Cent}_{\mathcal{A}}(\mathcal{A}_e)$ as a G-algebra.

The action of G on $\mathrm{Cent}_{\mathcal{A}}(\mathcal{A}_e)$ by automorphisms is given by:

$$\sigma_g(c) := g \cdot c = u_g c u_g^{-1}$$

for any invertible element $u_g \in \mathcal{A}_g$.

Picco-Platzeck's exact sequence splits

Proposition

Let G be a finite abelian group and let $\mathcal{C}$ be a G-Galois extension of $\mathbb{F}$. Then $\mathrm{Cent}_{(\mathcal{C}\#\mathbb{F}G)\#(\mathbb{F}G)^*}(\mathcal{C}\#\mathbb{F}G)$ is isomorphic to $\mathcal{C}$ as a G-algebra.

Therefore, the map

$$\begin{aligned}\vartheta : \mathrm{E}_G(\mathbb{F}) &\longrightarrow \mathrm{Br}_G(\mathbb{F})\\ [\mathcal{C}]_{G\text{-alg}} &\mapsto [\mathcal{C}\#\mathbb{F}G]_G,\end{aligned}$$

is a right inverse of ζ.
Note that, since the algebra $\mathcal{C}\#\mathbb{F}G$ is isomorphic to $\mathrm{End}_{\mathbb{F}}(\mathcal{C})$, its class in $\mathrm{Br}(\mathbb{F})$ is trivial, which means that $\vartheta([\mathcal{C}]_{G\text{-alg}})$ is in the kernel of the *forgetful* map $\varphi : \mathrm{Br}_G(\mathbb{F}) \to \mathrm{Br}(\mathbb{F})$.
It follows that ϑ is an isomorphism $\mathrm{E}_G(\mathbb{F}) \simeq \ker\varphi$.

Picco-Platzeck's exact sequence splits

Corollary

Let G be a finite abelian group. Then the map

$$\begin{aligned} \mathrm{Br}_G(\mathbb{F}) &\longrightarrow \mathrm{Br}(\mathbb{F}) \times \mathrm{E}_G(\mathbb{F}) \\ [\mathcal{A}]_G &\mapsto \Big([\mathcal{A}], [\mathrm{Cent}_{\mathcal{A}\#(\mathbb{F}G)^*}(\mathcal{A})]_{G\text{-alg}}\Big) \end{aligned}$$

is a group isomorphism, and its inverse is the map

$$\begin{aligned} \mathrm{Br}(\mathbb{F}) \times \mathrm{E}_G(\mathbb{F}) &\longrightarrow \mathrm{Br}_G(\mathbb{F}) \\ ([\mathcal{B}], [\mathcal{C}]_{G\text{-alg}}) &\mapsto [\mathcal{B} \otimes_{\mathbb{F}} (\mathcal{C}\#\mathbb{F}G)]_G, \end{aligned}$$

where the G-grading on $\mathcal{B} \otimes_{\mathbb{F}} (\mathcal{C}\#\mathbb{F}G)$ is given by

$$\big(\mathcal{B} \otimes_{\mathbb{F}} (\mathcal{C}\#\mathbb{F}G)\big)_g = \mathcal{B} \otimes_{\mathbb{F}} (\mathcal{C}g).$$

From graded-division algebras to Galois extensions

Recall that the homomorphism on the right of Picco-Platzeck's exact sequence sends $[\mathcal{A}]_G$ to $[\mathrm{Cent}_{\mathcal{A}\#(\mathbb{F}G)^*}(\mathcal{A})]_{G\text{-alg}}$.

Theorem

Let G be a finite abelian group and let $\mathcal{D}$ be a central simple G-graded-division algebra with support G. Then $\mathrm{Cent}_{\mathcal{D}\#(\mathbb{F}G)^*}(\mathcal{D})$ is a simple algebra, and it is antiisomorphic to $\mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$ as a G-algebra.

It is important to understand the G-algebras $\mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$!

From graded-division algebras to Galois extensions

If the support of $\mathcal{D}$ is not the whole G, then we need *induced algebras*.

Theorem

Let G be a finite abelian group and let $\mathcal{D}$ be a central simple G-graded-division algebra with support T. Then $\mathrm{Cent}_{\mathcal{D}\#(\mathbb{F}G)^*}(\mathcal{D})$ is antiisomorphic to $\mathrm{Ind}_T^G(\mathcal{C})$ as a G-algebra, where $\mathcal{C} = \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$ is a simple T-Galois extension of $\mathbb{F}$.

$$\begin{aligned}\mathrm{Ind}_T^G(\mathcal{C}) &:= \mathrm{Hom}_{\mathbb{F}T}(\mathbb{F}G, \mathcal{C}) \\ &\simeq \{f : G \to \mathcal{C} \mid f(tg) = t \cdot f(g)\ \forall t \in T\}.\end{aligned}$$

Twisted group algebras

Given a group K, a field $\mathbb{L}$ and a 2-cocycle $\tau \in \mathsf{Z}^2(K, \mathbb{L}^\times)$ (with trivial action of K on $\mathbb{L}^\times$), the **twisted group algebra** $\mathbb{L}^\tau K$ is the $\mathbb{L}$-algebra with basis $\{X_k \mid k \in K\}$ and multiplication given by

$$X_{k_1} X_{k_2} = \tau(k_1, k_2) X_{k_1 k_2}$$

for any $k_1, k_2 \in K$.

$\mathbb{L}^\tau K$ is naturally K-graded, and the graded-isomorphism class is determined by the class of τ in the second cohomology group: $[\tau] \in \mathsf{H}^2(K, \mathbb{L}^\times)$.

Any graded-division algebra over $\mathbb{L}$ with support K and 1-dimensional homogeneous components is, up to a graded isomorphism, a twisted group algebra $\mathbb{L}^\tau K$.

The G-algebra $\mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$

Proposition

Let G be a finite group, and let $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$ be a graded-division ring with support G. Assume that $\mathcal{D}$ is finite-dimensional over the field $\mathbb{F} := Z(\mathcal{D}) \cap \mathcal{D}_e$. Denote $\mathbb{L} := Z(\mathcal{D}_e)$,

$$K := \{k \in G \mid \mathcal{D}_k \cap \mathrm{Cent}_{\mathcal{D}}(\mathbb{L}) \neq 0\},$$

$\mathcal{D}_K := \bigoplus_{k\in K} \mathcal{D}_k$, and $\mathcal{C} := \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$. Then the following assertions hold:

1. *K is a normal subgroup of G and $\mathrm{Cent}_{\mathcal{D}}(\mathbb{L}) = \mathcal{D}_K$.*
2. *The extension $\mathbb{L}/\mathbb{F}$ is a Galois field extension and*

$$\begin{aligned} \bar{\sigma} : G &\longrightarrow \mathrm{Aut}_{\mathbb{F}}(\mathbb{L}) = \mathrm{Gal}(\mathbb{L}/\mathbb{F}) \\ g &\mapsto (\mathrm{Int}\, u_g)|_{\mathbb{L}} \quad \text{for any } 0 \neq u_g \in \mathcal{D}_g \end{aligned}$$

is a surjective group homomorphism with kernel K.

The G-algebra $\mathcal{C} = \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$

Proposition (continued)

3. *$\mathcal{C}$ is a graded subalgebra of $\mathcal{D}$ with support K and $\mathcal{C}_e = \mathbb{L}$, hence graded-isomorphic to the twisted group algebra $\mathbb{L}^\tau K$ for some $\tau \in \mathsf{Z}^2(K, \mathbb{L}^\times)$.*
4. *$\mathcal{D}_K \simeq \mathcal{D}_e \otimes_{\mathbb{L}} \mathcal{C}$.*

Corollary

Assume further that $\mathcal{D}$ is simple and $Z(\mathcal{D}) = \mathbb{F}$. Then $\mathcal{C}$ is simple with $Z(\mathcal{C}) = \mathbb{L}$ and the order $|K|$ is a square.

The G-algebra $\mathcal{C} = \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$

Proposition

Identify $\mathcal{C}$ with $\mathbb{L}^\tau K = \bigoplus_{k \in K} \mathbb{L} X_k$. For any $k \in K$ and $g \in G$, define the element $f_k(g) \in \mathbb{L}^\times$ by

$$\sigma_g(X_k) = f_k(g) X_k.$$

Then we have the following:

1. *For any $k \in K$, $f_k : G \to \mathbb{L}^\times$ is a 1-cocycle: $f_k \in \mathsf{Z}^1(G, \mathbb{L}^\times)$.*
2. *Replacing the element X_k by $X_k' = l X_k$, $l \in \mathbb{L}^\times$, changes f_k to the cohomologous 1-cocycle $f_k' = (\mathsf{d}l) f_k$, where $\mathsf{d}l : G \to \mathbb{L}^\times$ is the 1-coboundary $g \mapsto \bar{\sigma}_g(l) l^{-1}$. In particular, the class $[f_k]$ of f_k in the cohomology group $\mathsf{H}^1(G, \mathbb{L}^\times) = \mathsf{Z}^1(G, \mathbb{L}^\times)/\mathsf{B}^1(G, \mathbb{L}^\times)$ does not depend on the choice of X_k.*

The G-algebra $\mathcal{C} = \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$

Proposition (continued)

3. *The alternating bicharacter* $\beta : K \times K \to \mathbb{L}^\times$ *given by*

$$\beta(k_1, k_2) = \tau(k_1, k_2)\tau(k_2, k_1)^{-1}$$

takes values in $\mathbb{F}^\times$*, depends only on the class* $[\tau] \in \mathsf{H}^2(K, \mathbb{L}^\times)$*, and satisfies* $f_k(g) = \beta(g, k) \quad \forall k, g \in K$*.*

4. *For any* $k_1, k_2 \in K$*,*

$$f_{k_1} f_{k_2} = \mathsf{d}\big(\tau(k_1, k_2)\big) f_{k_1 k_2}.$$

5. *The map* $f : K \longrightarrow \mathsf{H}^1(G, \mathbb{L}^\times)$*,* $k \mapsto [f_k]$*, is a group homomorphism whose kernel is the support of the graded subalgebra* $Z(\mathcal{D})$*.*
6. *The following are equivalent:* (a) $\mathcal{D}$ *is central simple over* $\mathbb{F}$*,* (b) $\mathcal{C}$ *is central simple over* $\mathbb{L}$*, and* (c) β *is nondegenerate.*

The G-algebra $\mathcal{C} = \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$

Corollary

If $\mathcal{D}$ is central simple over $\mathbb{F}$ (i.e., if $\mathcal{C}$ is central simple over $\mathbb{L}$), then $\mathbb{F}$ contains the primitive roots of unity of degree $\exp(K)$, the exponent of the finite abelian group K, and K is isomorphic to $A \times A$ for some finite abelian group A.

Inflation-restriction exact sequence

To get more precise information, we recall the
inflation-restriction exact sequence
(coming from the Lyndon-Hochschild-Serre spectral sequence):

$$1 \longrightarrow \mathsf{H}^1\big(G/K, (\mathbb{L}^\times)^K\big) \xrightarrow{\text{inf}} \mathsf{H}^1(G, \mathbb{L}^\times) \xrightarrow{\text{res}} \mathsf{H}^1(K, \mathbb{L}^\times)^{G/K}$$
$$\xrightarrow{\rho} \mathsf{H}^2\big(G/K, (\mathbb{L}^\times)^K\big) \xrightarrow{\text{inf}} \mathsf{H}^2(G, \mathbb{L}^\times)$$

Inflation-restriction exact sequence

- $\mathsf{H}^1(G/K, (\mathbb{L}^\times)^K) = \mathsf{H}^1(G/K, \mathbb{L}^\times) = 1$, by Hilbert's Theorem 90.
- $\mathsf{H}^1(K, \mathbb{L}^\times)^{G/K} = \mathsf{H}^1(K, \mathbb{L}^\times)^G = \mathsf{H}^1(K, (\mathbb{L}^\times)^G) = \mathsf{H}^1(K, \mathbb{F}^\times) = \mathrm{Hom}(K, \mathbb{F}^\times)$.

The inflation-restriction exact sequence becomes:

$$\begin{array}{ccccccc} 1 \longrightarrow & \mathsf{H}^1(G, \mathbb{L}^\times) & \xrightarrow{\text{res}} & \mathrm{Hom}(K, \mathbb{F}^\times) & \xrightarrow{\rho} \cdots \\ & [\gamma] & \mapsto & \gamma|_K & \end{array}$$

Inflation-restriction exact sequence

Corollary

The diagram

$$
\begin{array}{ccc}
\mathsf{H}^1(G, \mathbb{L}^\times) & \xrightarrow{\text{res}} & \mathrm{Hom}(K, \mathbb{F}^\times) \\
 & \nwarrow^{f} & \uparrow \acute{\beta} \\
 & & K
\end{array}
$$

where $\acute{\beta}$ is induced by the bicharacter β: $k \mapsto \beta(\cdot, k)$, is commutative.

If $\mathcal{D}$ is central simple over $\mathbb{F}$, then all homomorphisms in this diagram are isomorphisms, and σ is an isomorphism from G onto the group of automorphisms of $\mathcal{C}$ as a K-graded algebra over $\mathbb{F}$.

G-Galois extensions

Our results so far show that any G-Galois extension of $\mathbb{F}$ is isomorphic to an algebra of the form $\mathrm{Ind}_T^G(\mathcal{C})$ where $\mathcal{C} = \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$ and $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ is a central simple G-graded-division algebra with support T.

Any simple G-Galois extension of $\mathbb{F}$ is isomorphic to an algebra of the form $\mathcal{C} = \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$ where $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ is a central simple G-graded-division algebra with support G.

Simple G-Galois extensions

Given $\mathcal{C} = \mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$ simple, the following ingredients appear:

- $\mathbb{L} = Z(\mathcal{C})$ is a Galois field extension of $\mathbb{F}$.
- $\theta : G \to \mathrm{Gal}(\mathbb{L}/\mathbb{F})$, the surjective group homomorphism induced by the action of G on $\mathcal{C}$.
  Its kernel K is isomorphic to $A \times A$ for some abelian group A, and $\mathbb{F}$ contains a primitive root of unity of degree $\exp(K)$.
  Moreover, the restriction map
  $\mathrm{res} : \mathsf{H}^1(G, \mathbb{L}^\times) \to \mathrm{Hom}(K, \mathbb{F}^\times)$ is bijective.
- $[\tau] \in \mathsf{H}^2(K, \mathbb{L}^\times)$ satisfies that the alternating form $\beta(k_1, k_2) := \tau(k_1, k_2)\tau(k_2, k_1)^{-1}$ is nondegenerate, plus a technical condition $\delta(\mathrm{res}^{-1} \circ \acute{\beta}) = \pi_*([\tau])$, for
  $\delta : \mathrm{Hom}(K, \mathsf{H}^1(G, \mathbb{L}^\times)) \to \mathsf{H}^2(K, \mathbb{L}^\times/\mathbb{F}^\times)$ and
  $\pi_* : \mathsf{H}^2(K, \mathbb{L}^\times) \to \mathsf{H}^2(K, \mathbb{L}^\times/\mathbb{F}^\times)$.

Simple G-Galois extensions

The (isomorphism class of the) triple $(\mathbb{L}, \theta, [\tau])$ classifies $\mathcal{C}$ up to a weaker condition than isomorphism.
Isomorphism classes require a slight change.

Theorem

The isomorphism classes of simple G-Galois extensions are classified by the isomorphism classes of triples

$$(\mathbb{L}, \theta, \xi)$$

with $\mathbb{L}$ and θ as before, but with $\xi \in \mathsf{Z}^2(K, \mathbb{L}^\times)/\mathsf{B}^2(K, \mathbb{F}^\times)$, with properties as above.

Examples

(a) Galois field extensions $\mathbb{L}$ of $\mathbb{F}$ with $\mathrm{Gal}(\mathbb{L}/\mathbb{F}) \simeq G$: these correspond to the case $K = 1$.

(b) Central simple graded-division algebras over $\mathbb{F}$ with support G and 1-dimensional homogeneous components: these correspond to the case $K = G$.
If $\mathbb{F}$ is algebraically closed, these are the only simple G-Galois extensions, and they are parametrized by nondegenerate alternating bicharacters.

(c) Suppose $\mathrm{Br}(\mathbb{F})$ is trivial (for example, $\mathbb{F}$ is finite). Then, for any subgroup K admitting a nondegenerate alternating bicharacter and any Galois field extension $\mathbb{L}$ with $\mathrm{Gal}(\mathbb{L}/\mathbb{F}) \simeq G/K$, every central simple graded-division algebra over $\mathbb{L}$ with support K and 1-dimensional homogeneous components admits a G-action that makes it a G-Galois extension of $\mathbb{F}$.
All simple G-Galois extensions have this form.

Characterization of G-Galois extensions

Theorem

Let G be a finite abelian group and $\mathbb{F}$ a field. Let $\mathcal{A}$ be a G-algebra over $\mathbb{F}$. Then $\mathcal{A}$ is a G-Galois extension of $\mathbb{F}$ if and only if the following conditions hold:

1. $\dim_{\mathbb{F}} \mathcal{A} = |G|$;
2. $\mathbb{L} := Z(\mathcal{A})$ *is a* G/K*-Galois extension of* $\mathbb{F}$ *where* K *is the kernel of the homomorphism* $\bar{\sigma} : G \to \mathrm{Aut}_{\mathbb{F}}(\mathbb{L})$, $g \mapsto \sigma_g|_{\mathbb{L}}$;
3. $\mathbb{F}$ *contains a primitive root of unity of degree* $\exp(K)$;
4. *For every* $\chi \in \widehat{K} := \mathrm{Hom}(K, \mathbb{F}^{\times})$, *the eigenspace*

$$\mathcal{A}_{\chi} := \{a \in \mathcal{A} \mid \sigma_k(a) = \chi(k)a \ \forall k \in K\}$$

contains an invertible element.

Back to graded-division algebras

Theorem

Let $\mathcal{D}$ and $\mathcal{D}'$ be finite-dimensional G-graded-division algebras with supports T and T'. Assume that $\mathcal{D}$ and $\mathcal{D}'$ are central simple over $\mathbb{F}$. Then $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic as G-graded algebras if and only if the following conditions are satisfied:

1. $T = T'$;
2. $\mathrm{Cent}_{\mathcal{D}}(\mathcal{D}_e)$ *and* $\mathrm{Cent}_{\mathcal{D}'}(\mathcal{D}'_e)$ *are isomorphic as* T*-algebras;*
3. $[\mathcal{D}] = [\mathcal{D}']$ *in* $\mathrm{Br}(\mathbb{F})$. □

In order to suppress the condition on central simplicity, the loop algebra construction must be invoked.

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Thank you!