Gradings and affine group schemes

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Gradings of algebras reflect important symmetries on them, and help us to uncover their structure.

Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading (r being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}/2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.

In 1989, a systematic study of gradings on Lie algebras was started by Patera and Zassenhaus.

Fine gradings on the classical simple complex Lie algebras, other than D_4, by arbitrary abelian groups were considered by Havlícek, Patera, and Pelantova in 1998.

The arguments there are computational and the problem of classification of fine gradings is not completely settled. The complete classification, up to equivalence, of fine gradings on all classical simple Lie algebras (including D_4) over algebraically closed fields of characteristic zero has been obtained quite recently.

For any abelian group G, the classification of all G-gradings, up to isomorphism, on the classical simple Lie algebras other than D_4 over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.

Gradings on associative algebras, on the octonions, on the Albert algebra, and on some other algebraic structures, are instrumental in obtaining a classification of the gradings on the exceptional simple Lie algebras.

The clue to the possibility of 'transfering' gradings from these structures to Lie algebras is to use their affine group schemes of automorphisms.

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Categories

A **category** $\mathbf{C}$ consists of:

- a class of objects,
- for any two objects X, Y, a set $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ of *morphisms* $\alpha : X \to Y$, with a composition law

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{C}}(X,Y) \times \mathrm{Hom}_{\mathbf{C}}(Y,Z) & \longrightarrow & \mathrm{Hom}_{\mathbf{C}}(X,Z) \\ (\alpha, \beta) & \mapsto & \beta\alpha \end{array}$$

  satisfying *associativity* and the existence of *identity morphisms* $\mathrm{id}_X \in \mathrm{Hom}_{\mathbf{C}}(X, X)$.

The notion of *isomorphism* is the natural one.

Examples: Set, Grp, $\mathrm{Alg}_{\mathbb{F}}$.
If we invert the direction of the arrows we obtain the opposite category $\mathbf{C}^{\mathsf{op}}$: $\mathrm{Hom}_{\mathbf{C}^{\mathsf{op}}}(X, Y) = \mathrm{Hom}_{\mathbf{C}}(Y, X)$.

Functors

Given two categories $\mathbf{C}$, $\mathbf{D}$, a (covariant) **functor** $F : \mathbf{C} \to \mathbf{D}$ is a rule that assigns:

- an object $F(X)$ in $\mathbf{D}$ for any object X in $\mathbf{C}$,
- a morphism $F(\alpha) \in \mathrm{Hom}_{\mathbf{D}}\big(F(X), F(Y)\big)$ for any morphism $\alpha \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$.

satisfying

$$F(\alpha\beta) = F(\alpha)F(\beta)$$

when this makes sense, and

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$$

for any object X in $\mathbf{C}$.

A *contravariant functor* swaps the order: $\alpha \in \mathrm{Hom}_{\mathbf{C}}(X, Y)$ is 'mapped' to $F(\alpha) \in \mathrm{Hom}_{\mathbf{D}}(Y, X)$.

Functors

A functor $F : \mathbf{C} \to \mathbf{D}$ is said to be:

- Faithful if its action on morphisms is injective for any objects X, Y in $\mathbf{C}$.
- Full if its action on morphisms is surjective for any objects X, Y in $\mathbf{C}$.
- Fully faithful if it is both faithful and full.
- Dense if any object Y in $\mathbf{D}$ is isomorphic to an object of the form $F(X)$ for some object X in $\mathbf{C}$.
- An equivalence if it is fully faithful and dense.

Natural transformations

Given two functors $F, G : \mathbf{C} \to \mathbf{D}$, a **natural transformation** $T : F \to G$ is a family of morphisms $T_X : F(X) \to G(X)$ such that, for any two objects X, Y in $\mathbf{C}$ and morphism $\alpha : X \to Y$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{T_X} & G(X) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ F(Y) & \xrightarrow{T_Y} & G(Y) \end{array}$$

is commutative.

Natural transformations

- If T has an *inverse*, then T is said to be a **natural isomorphism** and we write $F \simeq G$.
- $\mathbf{C}$ and $\mathbf{D}$ are **equivalent** if there are functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that $GF \simeq 1_{\mathbf{C}}$ and $FG \simeq 1_{\mathbf{D}}$.

Example

The category of finite-dimensional vector spaces over a field $\mathbb{F}$ is equivalent to the category whose objects are the vector spaces $\mathbb{F}^n$, with $n \geq 0$. (The morphisms on both cases are the linear transformations.)

The Axiom of Choice can be used to prove that two categories $\mathbf{C}$ and $\mathbf{D}$ are equivalent if and only if there is an equivalence (fully faithful and dense functor) $F : \mathbf{C} \to \mathbf{D}$.

Representable functors

Any object A in a category $\mathbf{C}$ determines a functor $h^A : \mathbf{C} \to \mathrm{Set}$ that sends any object B in $\mathbf{C}$ to $\mathrm{Hom}_{\mathbf{C}}(A, B)$, and any morphism $\alpha \in \mathrm{Hom}_{\mathbf{C}}(B, C)$ to the map

$$\begin{aligned} \alpha_* : \mathrm{Hom}_{\mathbf{C}}(A, B) &\to \mathrm{Hom}_{\mathbf{C}}(A, C) \\ \beta &\to \alpha\beta. \end{aligned}$$

Moreover, any morphism $\alpha \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ defines a natural transformation $\alpha^* : h^B \to h^A$.

Definition

A functor $F : \mathbf{C} \to \mathrm{Set}$ is said to be **representable** if there is an object A in $\mathbf{C}$ such that F is naturally isomorphic to h^A.
(Then F is said to be represented by A.)

Yoneda's Lemma

Consider an object A in a category $\mathbf{C}$, a functor $F : \mathbf{C} \to \mathrm{Set}$, and a natural transformation $T : h^A \to F$.
In particular, T induces a map

$$T_A : \mathrm{Hom}_{\mathbf{C}}(A, A) \longrightarrow F(A),$$

and there is a distinguished element $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$.

Therefore T provides a distinguished element $a_T := T_A(\mathrm{id}_A)$ in the set $F(A)$.

It turns out that this single generic element a_T determines completely the natural transformation T.

Yoneda's Lemma

This is due to the commutativity of the diagram:

$$\begin{array}{ccc} h^A(A) = \mathrm{Hom}_{\mathbf{C}}(A, A) & \xrightarrow{T_A} & F(A) \\ \Big\downarrow{\scriptstyle h^A(\alpha)} & & \Big\downarrow{\scriptstyle F(\alpha)} \\ h^A(X) = \mathrm{Hom}_{\mathbf{C}}(A, X) & \xrightarrow{T_X} & F(X) \end{array}$$

$$\begin{array}{ccc} \mathrm{id}_A & \longmapsto & a_T \\ \big\downarrow & & \big\downarrow \\ \alpha & \longmapsto & T_X(\alpha) = F(\alpha)(a_T) \end{array}$$

Yoneda's Lemma

Theorem (Yoneda's Lemma)

The map $T \to a_T$ gives a bijection between the class of natural transformations $h^A \to F$ and the set $F(A)$.

Its inverse takes any element $a \in F(A)$ to the natural transformation $T^a : h^A \to F$ such that, for any object B in $\mathbf{C}$, T^a_B is the map

$$\begin{aligned} T^a_B : \mathrm{Hom}_{\mathbf{C}}(A, B) &\longrightarrow F(B) \\ \alpha \quad &\mapsto \; F(\alpha)(a). \end{aligned}$$

These bijections are natural in A and F.

Yoneda's Lemma

Corollary

Let A and B be two objects in a category $\mathbf{C}$. Then the map $\alpha \to \alpha^*$ gives a natural bijection from $\mathrm{Hom}_{\mathbf{C}}(B, A)$ to the class of natural transformations $h^A \to h^B$.

Composition of natural transformations $h^A \to h^B \to h^C$ corresponds to composition of morphisms $C \to B \to A$.

In particular, if a functor $F : \mathbf{C} \to \mathrm{Set}$ is representable, the object that represents it is unique, up to isomorphism.

Affine group schemes

Definition

An **affine group scheme** over a field $\mathbb{F}$ is a functor $\mathbf{G} : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$ such that the composition with the 'forgetful' functor $\mathrm{Grp} \to \mathrm{Set}$ is representable.

The representing object, unique up to isomorphism, will be denoted by $\mathbb{F}[\mathbf{G}]$.
Therefore, there is a natural isomorphism

$$\theta : \mathbf{G} \simeq \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{G}], .)$$

An affine group scheme $\mathbf{G}$ is said to be

- **abelian** if $\mathbf{G}(R)$ is abelian for any R in $\mathrm{Alg}_{\mathbb{F}}$,
- **finite** if $\dim_{\mathbb{F}} \mathbb{F}[\mathbf{G}]$ is finite,
- **algebraic** if $\mathbb{F}[\mathbf{G}]$ is a finitely generated algebra. In this case, the dimension of $\mathbf{G}$ is defined as the Krull dimension of $\mathbb{F}[\mathbf{G}]$.

Examples

- **Additive group scheme:** $\mathbf{G}_{\mathsf{a}} : R \mapsto (R, +)$; $\mathbb{F}[\mathbf{G}_{\mathsf{a}}] = \mathbb{F}[X]$.
- **Multiplicative group scheme:** $\mathbf{G}_{\mathsf{m}} : R \mapsto R^{\times}$;
  $\mathbb{F}[\mathbf{G}_{\mathsf{m}}] = \mathbb{F}[X, X^{-1}]$.
- **Direct product:** If $\mathbf{G}$ and $\mathbf{H}$ are two affine group schemes, so is the direct product $\mathbf{G} \times \mathbf{H} : R \mapsto \mathbf{G}(R) \times \mathbf{H}(R)$;
  $\mathbb{F}[\mathbf{G} \times \mathbf{H}] = \mathbb{F}[\mathbf{G}] \otimes_{\mathbb{F}} \mathbb{F}[\mathbf{H}]$.
- **Special linear group scheme:**
  $\mathbf{SL}_n : R \mapsto \mathrm{SL}_n(R) = \{x \in \mathrm{Mat}_n(R) \mid \det(x) = 1\}$;
  $\mathbb{F}[\mathbf{SL}_n] = \mathbb{F}[X_{ij} \mid 1 \leq i, j \leq n]/(\det(X_{ij}) - 1)$.
  In a coordinate-free version, if V is a finite-dimensional vector space over $\mathbb{F}$,
  $\mathbf{SL}(V) : R \mapsto \{f \in \mathrm{End}_R(V \otimes_{\mathbb{F}} R) \mid \det(f) = 1\}$.
- **General linear group scheme:** $\mathbf{GL}_n : R \mapsto \mathrm{GL}_n(R)$;
  $\mathbb{F}[\mathbf{GL}_n] = \mathbb{F}[X_{ij}, T \mid 1 \leq i, j \leq n]/(T \det(X_{ij}) - 1)$.

More examples

- $\boldsymbol{\mu}_n : R \mapsto \{r \in R \mid r^n = 1\}$; $\mathbb{F}[\boldsymbol{\mu}_n] = \mathbb{F}[X]/(X^n - 1)$.
  $\boldsymbol{\mu}_n$ is finite.
- $\mathbf{1} : R \mapsto 1$ (trivial group); $\mathbb{F}[\mathbf{1}] = \mathbb{F}$.
- Given an abelian group G, $\mathbf{D}(G) : R \mapsto \mathrm{Hom}_{\mathrm{Grp}}(G, R^\times)$ (the group of characters of G with values in R); $\mathbb{F}[\mathbf{D}(G)] = \mathbb{F}G$ (the group algebra of G).
- Let $\mathcal{A}$ be a finite-dimensional nonassociative (i.e., not necessarily associative) algebra. Its **automorphism group scheme** is

$$\mathbf{Aut}(\mathcal{A}) : R \mapsto \mathrm{Aut}_{R\text{-alg}}(\mathcal{A} \otimes_{\mathbb{F}} R)$$

Exercise

Compute $\mathbb{F}[\mathbf{Aut}(\mathbb{F} \times \mathbb{F})]$.

From affine group schemes to Hopf algebras

The fact that an affine group scheme $\mathbf{G}$ is representable as a functor $\mathrm{Alg}_{\mathbb{F}} \to \mathrm{Set}$, but it is actually a functor $\mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$, provides the representing algebra $\mathbb{F}[\mathbf{G}]$ with extra structure.

Group multiplication gives a natural transformation $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ which, by Yoneda's Lemma, gives a homomorphism of algebras

$$\Delta : \mathbb{F}[\mathbf{G}] \to \mathbb{F}[\mathbf{G}] \otimes_{\mathbb{F}} \mathbb{F}[\mathbf{G}]$$

called **comultiplication**.

The associativity of group multiplication forces the coassociativity of the comultiplication:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes_{\mathbb{F}} \mathrm{id}) \circ \Delta$$

From affine group schemes to Hopf algebras

The existence of a neutral element gives a natural transformation $\mathbf{1} \to \mathbf{G}$ which, in turn, gives a homomorphism

$$\epsilon : \mathbb{F}[\mathbf{G}] \to \mathbb{F}$$

called the **counit**, which satisfies

$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta$$

Finally, the existence of inverses gives a natural transformation $\mathbf{G} \to \mathbf{G}$ $(g \mapsto g^{-1})$ which, in turn, gives a homomorphism

$$S : \mathbb{F}[\mathbf{G}] \to \mathbb{F}[\mathbf{G}]$$

called the **antipode**, which satisfies

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$$

where m is the multiplication in $\mathbb{F}[\mathbf{G}]$ and η is the unique homomorphism (of unital algebras) $\mathbb{F} \to \mathbb{F}[\mathbf{G}]$.

Hopf algebras

Definition

A **Hopf algebra** over a field $\mathbb{F}$ is a 6-tuple $(\mathcal{A}, m, \eta, \Delta, \epsilon, S)$ such that:

- $(\mathcal{A}, m, \eta)$ is a unital associative algebra ($m : \mathcal{A} \otimes_{\mathbb{F}} \mathcal{A} \to \mathcal{A}$ is the multiplication and $\eta : \mathbb{F} \to \mathcal{A}$ the *unit*).
- $(\mathcal{A}, \Delta, \epsilon)$ is a *counital coassociative coalgebra*:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes_{\mathbb{F}} \mathrm{id}) \circ \Delta,$$
$$(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta.$$

- Δ and ϵ are homomorphisms of algebras.
- The antipode $S : \mathcal{A} \to \mathcal{A}$ is a linear map satisfying the property

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta.$$

Hopf algebras

The ideal (as an algebra) $\ker \epsilon$ is called the **augmentation ideal**.

The antipode of a Hopf algebra is an algebra antihomomorphism and also a coalgebra antihomomorphism.

A Hopf algebra is said to be commutative if the algebra $(\mathcal{A}, m)$ is commutative: $m \circ$ 'twist' $= m$, and it is said to be cocommutative if 'twist' $\circ \Delta = \Delta$.

If a Hopf algebra is commutative or cocommutative, then $S^2 = \mathrm{id}$.

Sweedler notation

$$\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$$

With this notation the coassociativity $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ becomes

$$\sum a_{(11)} \otimes a_{(12)} \otimes a_{(2)} = \sum a_{(1)} \otimes a_{(21)} \otimes a_{(22)},$$

and this is simplified by writing

$$(\Delta \otimes \mathrm{id}) \circ \Delta(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

The properties satisfied by the counit and the antipode become:

$$\sum \epsilon(a_{(1)})a_{(2)} = a = \sum \epsilon(a_{(2)})a_{(1)},$$
$$\sum S(a_{(1)})a_{(2)} = \epsilon(a) = \sum a_{(1)}S(a_{(2)})$$

Hopf algebras. Some remarks

Given a Hopf algebra $\mathcal{A}$:

- A Hopf subalgebra $\mathcal{S}$ is a subspace closed under m, η, Δ (i.e., $\Delta(\mathcal{S}) \subseteq \mathcal{S} \otimes \mathcal{S}$) and S.
- A Hopf ideal $\mathcal{I}$ is an ideal of $(\mathcal{A}, m, \eta)$ with $\Delta(\mathcal{I}) \subseteq \mathcal{I} \otimes \mathcal{A} + \mathcal{A} \otimes \mathcal{I}$, $S(\mathcal{I}) = \mathcal{I}$, and $\epsilon(\mathcal{I}) = 0$.
- The notion of homomorphism of Hopf algebras is the natural one.

Affine group schemes and commutative Hopf algebras

If $\mathbf{G}$ is an affine group scheme, then $\mathcal{A} = \mathbb{F}[\mathbf{G}]$ is a commutative Hopf algebra.

Conversely, if $\mathcal{A}$ is a commutative Hopf algebra, then for any R in $\mathrm{Alg}_{\mathbb{F}}$, $\mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathcal{A}, R)$ is a group with

- multiplication $fg(a) = \sum f(a_{(1)})g(a_{(2)})$,
- neutral element $\epsilon : \mathcal{A} \to \mathbb{F} \hookrightarrow R$,
- inverse element $f^{-1} = f \circ S$.

Therefore the assignments $\mathbf{G} \to \mathbb{F}[\mathbf{G}]$ and $\mathcal{A} \to \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathcal{A}, .)$ give an equivalence from the category of affine group schemes to the (opposite of the) category of commutative Hopf algebras.

The categories of affine group schemes and of commutative Hopf algebras are antiequivalent.

Examples

- $\mathbf{G}_{\mathsf{a}}$, $\mathcal{A} = \mathbb{F}[X]$, $\Delta(X) = X \otimes 1 + 1 \otimes X$, $\epsilon(X) = 0$, $S(X) = -X$.
  The nonzero elements of a Hopf algebra satisfying $\Delta(a) = a \otimes 1 + 1 \otimes a$ are said to be primitive.

- $\mathbf{G}_{\mathsf{m}}$, $\mathcal{A} = \mathbb{F}[X, X^{-1}]$, $\Delta(X) = X \otimes X$, $\epsilon(X) = 1$, $S(X) = X^{-1}$.
  The elements of a Hopf algebra satisfying $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$ are said to be group-like.

- $\boldsymbol{\mu}_n$, $\mathcal{A} = \mathbb{F}[X]/(X^n - 1) = \mathbb{F}[x]$, $\Delta(x) = x \otimes x$, $\epsilon(x) = 1$, $S(x) = x^{-1} = x^{n-1}$.

Examples

- $\mathbf{GL}_n$, $\mathcal{A} = \mathbb{F}[X_{ij}, T \mid 1 \le i, j \le n]/\big(T\det(X_{ij}) - 1\big) = \mathbb{F}[x_{ij}, t \mid 1 \le i, j \le n]$,
  $\Delta(x_{ij}) = \sum_l x_{il} \otimes x_{lj}$, $\epsilon(x_{ij}) = \delta_{ij}$.
  $\det(x_{ij}) = t^{-1}$ is a group-like element.

- G abelian group, $\mathbf{D}(G)$ the affine group scheme of 'characters' of G. Then $\mathcal{A} = \mathbb{F}G$ is the group algebra of G, with $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$, and $S(g) = g^{-1}$ for all $g \in G$.

Homomorphisms

Let $\mathbf{G}$ and $\mathbf{H}$ be two affine group schemes over $\mathbb{F}$.
A **homomorphism** $\theta : \mathbf{G} \to \mathbf{H}$ is simply a natural transformation.

Yoneda's Lemma tells us that θ is determined by a Hopf algebra homomorphism

$$\theta^* : \mathbb{F}[\mathbf{H}] \longrightarrow \mathbb{F}[\mathbf{G}]$$

given by $\theta^* = \theta(\mathrm{id}_{\mathbb{F}[\mathbf{G}]})$.

Note that $\mathrm{id}_{\mathbb{F}[\mathbf{G}]} \in \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{G}], \mathbb{F}[\mathbf{G}]) \simeq \mathbf{G}(\mathbb{F}[\mathbf{G}])$, and
$\theta^* = \theta(\mathrm{id}_{\mathbb{F}[\mathbf{G}]}) \in \mathbf{H}(\mathbb{F}[\mathbf{G}]) \simeq \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{H}], \mathbb{F}[\mathbf{G}])$.

Closed embeddings

$\mathbf{H}$ is said to be a **subgroup scheme** of $\mathbf{G}$ if for all R in $\mathrm{Alg}_{\mathbb{F}}$, $\mathbf{H}(R)$ is a subgroup of $\mathbf{G}(R)$, and this embedding $\iota : \mathbf{H} \to \mathbf{G}$ is a homomorphism.

A homomorphism $\theta : \mathbf{G} \to \mathbf{H}$ is a **closed embedding** if θ^* is surjective.
In this case, $\mathcal{I} := \ker \theta^*$ is a Hopf ideal of $\mathbb{F}[\mathbf{H}]$ and $\mathbb{F}[\mathbf{G}] \cong \mathbb{F}[\mathbf{H}]/\mathcal{I}$.
Conversely, given a Hopf ideal $\mathcal{I}$ of $\mathbb{F}[\mathbf{H}]$, $\mathbb{F}[\mathbf{H}]/\mathcal{I}$ is a commutative Hopf algebra, and hence it represents the affine group scheme:

$$\mathbf{G} : R \mapsto \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{H}]/\mathcal{I}, R) \simeq \{g \in \mathbf{H}(R) \mid g(\mathcal{I}) = 0\} \leq \mathbf{H}(R)$$

The projection $\mathbb{F}[\mathbf{H}] \to \mathbb{F}[\mathbf{H}]/\mathcal{I}$ induces, via Yoneda's Lemma, the closed embedding $\mathbf{G} \to \mathbf{H}$. We can think of $\mathbf{G}$ as a closed subgroup scheme of $\mathbf{H}$.

Quotient maps

A homomorphism $\theta : \mathbf{G} \to \mathbf{H}$ is said to be **surjective** (or a **quotient map**) if θ^* is injective.

Example

$\theta : \mathbf{G}_\mathsf{m} \to \mathbf{G}_\mathsf{m}$, $x \mapsto x^2$, corresponds, by Yoneda's Lemma, to $\theta^* : \mathbb{F}[X, X^{-1}] \to \mathbb{F}[X, X^{-1}]$, $X \mapsto X^2$, and hence θ is surjective.

But this does not mean that θ_R is surjective for all R!!
For instance, for $\mathbb{F} = \mathbb{R}$, $\theta_\mathbb{C}$ is surjective, but $\theta_\mathbb{R}$ is not.

If θ is surjective, the group homomorphism $\theta_{\overline{\mathbb{F}}}$ is always surjective.

Kernel and image

A homomorphism $\theta : \mathbf{G} \to \mathbf{H}$ is an **isomorphism** if it has an inverse or, alternatively, if θ_R is a group isomorphism for all R in $\mathrm{Alg}_{\mathbb{F}}$.
θ is an isomorphism if and only if θ^* is an isomorphism of Hopf algebras.

The **kernel** of a homomorphism $\theta : \mathbf{G} \to \mathbf{H}$ is defined in the natural way: $\ker \theta : R \to \ker(\theta_R)$, and it is representable, and hence an affine group scheme.

Let $\mathcal{I} = \ker(\theta^*)$, so that θ^* factors as $\mathbb{F}[\mathbf{H}] \longrightarrow \mathbb{F}[\mathbf{H}]/\mathcal{I} \longrightarrow \mathbb{F}[\mathbf{G}]$, where the first arrow is surjective and the second one injective. This corresponds to a chain of homomorphisms of group schemes

$$\mathbf{G} \longrightarrow \mathbf{S} \longrightarrow \mathbf{H}$$

where $\mathbf{S}$ is the affine group scheme represented by $\mathbb{F}[\mathbf{H}]/\mathcal{I}$. The first arrow is surjective and the second one is a closed embedding. $\mathbf{S}$ is called the **image** of θ.

Representations

Definition

Let $\mathbf{G}$ be an affine group scheme and let V be a finite-dimensional vector space over $\mathbb{F}$.
A homomorphism $\theta : \mathbf{G} \to \mathbf{GL}(V)$ is called a **(linear) representation** of $\mathbf{G}$ on V.

By Yoneda's Lemma, θ is determined by the generic automorphism $\theta_{\mathbb{F}[\mathbf{G}]}(\mathrm{id}_{\mathbb{F}[\mathbf{G}]}) \in \mathrm{GL}(V \otimes_{\mathbb{F}} \mathbb{F}[\mathbf{G}])$, and hence by its restriction

$$\rho : V \longrightarrow V \otimes_{\mathbb{F}} \mathbb{F}[\mathbf{G}]$$
$$v \mapsto \theta_{\mathbb{F}[\mathbf{G}]}(\mathrm{id}_{\mathbb{F}[\mathbf{G}]})(v \otimes 1).$$

Representations

If $\{v_1, \ldots, v_n\}$ is a basis of V, and

$$\rho(v_j) = \sum_i v_i \otimes a_{ij}$$

($a_{ij} \in \mathbb{F}[\mathbf{G}]$), then for any R in $\mathrm{Alg}_{\mathbb{F}}$ and any $g \in \mathbf{G}(R) \simeq \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{G}], R)$,

$$\theta_R(g)(v_j \otimes 1) = \sum_i v_i \otimes g(a_{ij}).$$

In other words, $\Big(g(a_{ij})\Big)$ is the coordinate matrix of $\theta_R(g)$.

Comodules

Write, using Sweedler notation, $\rho(v) = \sum v_{(1)} \otimes a_{(2)}$.
For R in $\mathrm{Alg}_{\mathbb{F}}$, $g, h \in \mathbf{G}(R)$, $v \in V$:

$$\begin{aligned}\theta_R(g)\theta_R(h)(v \otimes 1) &= \theta_R(g)\Big(\sum v_{(1)} \otimes h(a_{(2)})\Big) \\ &= \sum v_{(11)} \otimes g(a_{(12)})h(a_{(2)})\end{aligned}$$

$$\begin{aligned}\theta_R(gh)(v \otimes 1) &= \sum v_{(1)} \otimes gh(a_{(2)}) \\ &= \sum v_{(1)} \otimes g(a_{(21)})h(a_{(22)})\end{aligned}$$

and this gives $(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \rho$.

From $\theta_R(e) = \mathrm{id}$, we get $(\mathrm{id} \otimes \epsilon) \circ \rho = \mathrm{id}$.

Comodules

Definition

Let $\mathcal{A}$ be a Hopf algebra and let V be a vector space over $\mathbb{F}$. V is said to be a **comodule** for $\mathcal{A}$ if there is a linear map $\rho : V \to V \otimes_{\mathbb{F}} \mathcal{A}$ such that the conditions

$$(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \rho \qquad \text{and} \qquad (\mathrm{id} \otimes \epsilon) \circ \rho = \mathrm{id}$$

hold.

Conversely, if $\rho : V \to V \otimes_{\mathbb{F}} \mathbb{F}[\mathbf{G}]$ is a *comodule map*, then the natural transformation $\theta : \mathbf{G} \to \mathbf{GL}(V)$, given by $\theta_R(g)(v \otimes r) = \sum v_{(1)} \otimes g(a_{(2)})r$ is a representation of $\mathbf{G}$.

Representations of $\mathbf{G} \longleftrightarrow$ finite-dimensional comodules for $\mathbb{F}[\mathbf{G}]$

Diagonalizable group schemes

Definition

An affine group scheme $\mathbf{G}$ over $\mathbb{F}$ is said to be **diagonalizable** if there is an abelian group G such that $\mathbf{G}$ is isomorphic to $\mathbf{D}(G)$ (whose associated Hopf algebra is the group algebra $\mathbb{F}G$.)
If, moreover, $\mathbf{G}$ is algebraic, then it is called a **quasitorus**.

Examples

- $\mathbf{G}_{\mathsf{m}} \simeq \mathbf{D}(\mathbb{Z})$, as $\mathbb{F}[X, X^{-1}]$ is the group algebra of $\mathbb{Z}$.
- $\boldsymbol{\mu}_n \simeq \mathbf{D}(\mathbb{Z}/n)$, as $\mathbb{F}[X]/(X^n - 1)$ is the group algebra of $\mathbb{Z}/n$.

Group-like elements

Proposition

Let $(\mathcal{A}, m, \eta, \Delta, \epsilon, S)$ *be a Hopf algebra over* $\mathbb{F}$*, and let* $G(\mathcal{A})$ *be the set of group-like elements of* $\mathcal{A}$*.*

1. $G(\mathcal{A})$ *is a subgroup of* $\mathcal{A}^\times$*.*
2. *The elements of* $G(\mathcal{A})$ *are linearly independent.*
3. *If* $\mathcal{A}$ *is spanned by* $G(\mathcal{A})$*, then* $\mathcal{A}$ *is isomorphic, as a Hopf algebra, to the group algebra* $\mathbb{F}G(\mathcal{A})$*.*

Diagonalizable group schemes

Theorem

1. *The assignment*

$$\begin{array}{ccc} \{\textit{Abelian groups}\} & \to & \{\textit{diagonalizable group schemes}\} \\ G & \mapsto & \mathbf{D}(G), \end{array}$$

induces an antiequivalence of categories, that restricts to an antiequivalence

$$\{\textit{finitely generated abelian groups}\} \longrightarrow \{\textit{quasitori}\}$$

2. *Any subscheme and any quotient of a diagonalizable group scheme is diagonalizable.*
3. *An affine group scheme is a quasitorus if and only if it isomorphic to a finite direct product of copies of* $\mathbf{G}_\mathsf{m}$*'s and* $\boldsymbol{\mu}_n$*'s.*

Diagonalizable group schemes

Definition

Let $\mathbf{G}$ be an affine group scheme and let V be a finite-dimensional vector space over $\mathbb{F}$. Let $\theta : \mathbf{G} \to \mathbf{GL}(V)$ be a representation. θ is said to be **diagonalizable** if there is a basis $\{v_1, \ldots, v_n\}$ of V such that, for all R in $\mathrm{Alg}_{\mathbb{F}}$ and all $g \in \mathbf{G}(R)$, the coordinate matrix of $\theta_R(g)$ in the basis $\{v_1 \otimes 1, \ldots, v_n \otimes 1\}$ is diagonal.

Theorem (the reason behind the name 'diagonalizable group scheme')

Let $\mathbf{G}$ be an affine group scheme over $\mathbb{F}$.

1. *If $\mathbf{G}$ is diagonalizable and $\theta : \mathbf{G} \to \mathbf{GL}(V)$ is any representation, then θ is diagonalizable.*
2. *If $\theta : \mathbf{G} \to \mathbf{GL}(V)$ is a diagonalizable representation of $\mathbf{G}$, then the image of θ is a diagonalizable group scheme.*

Gradings

G abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

G-grading on $\mathcal{A}$:

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,$$

$$\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

Examples

Cartan grading:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\right)$$

(root space decomposition of a semisimple complex Lie algebra).

This is a grading by $\mathbb{Z}^n$, $n = \operatorname{rank}\mathfrak{g}$.

Examples

Pauli matrices: $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \bigoplus_{(\bar{\imath},\bar{\jmath})\in\mathbb{Z}/n\times\mathbb{Z}/n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F}X^iY^j.$$

$\mathcal{A}$ becomes a graded division algebra.

This grading induces a grading on $\mathfrak{sl}_n(\mathbb{F})$.

Basic definitions (Patera-Zassenhaus)

Let $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ be a grading on an algebra $\mathcal{A}$:

- The **support** of Γ is $\mathrm{Supp}\,\Gamma = \{g \in G : \mathcal{A}_g \neq 0\}$.
- The **universal grading group** of Γ is the group $U(\Gamma)$ generated by $\mathrm{Supp}\,\Gamma$ subject to the relations $g_1 g_2 = g_3$ if $0 \neq \mathcal{A}_{g_1}\mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_3}$.

The grading Γ is then a grading too by $U(\Gamma)$.

Basic definitions (Patera-Zassenhaus)

Two G-gradings $\Gamma^1 : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma^2 : \mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ are said to be **isomorphic** if there is an isomorphism of algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_g$ for all $g \in G$.

There is a weaker version:

Two gradings $\Gamma^1 : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ and $\Gamma^2 : \mathcal{B} = \bigoplus_{h \in H} \mathcal{B}_h$ are said to be **equivalent** if there is an isomorphism of algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that for any $g \in G$ there is an $h \in H$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_h$.

Basic definitions (Patera-Zassenhaus)

There appear several groups attached to Γ:

- The **automorphism group** (group of self-equivalences)

$$\mathrm{Aut}(\Gamma) = \{\varphi \in \mathrm{Aut}\,\mathcal{A} : \exists \alpha \in \mathsf{Sym}(\mathrm{Supp}\ \Gamma) \text{ s.t. } \varphi(\mathcal{A}_g) = \mathcal{A}_{\alpha(g)}\ \forall g\}.$$

- The **stabilizer group** (group of self-isomorphisms)

$$\mathrm{Stab}(\Gamma) = \{\varphi \in \mathrm{Aut}(\Gamma) : \varphi(\mathcal{A}_g) = \mathcal{A}_g\ \forall g\}.$$

- The **diagonal group**

$$\mathrm{Diag}(\Gamma) = \{\varphi \in \mathrm{Aut}(\Gamma) : \forall g \in \mathrm{Supp}\ \Gamma\ \exists \lambda_g \in \mathbb{F}^\times \text{ s.t. } \varphi|_{\mathcal{A}_g} = \lambda_g \mathrm{id}\}.$$

- $W(\Gamma) := \mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma)$ is the **Weyl group** of Γ.

$W(\Gamma)$ acts by automorphisms on $U(\Gamma)$

Each $\varphi \in \operatorname{Aut}(\Gamma)$ determines a self-bijection α of $\operatorname{Supp} \Gamma$ that induces an automorphism of the universal grading group $U(\Gamma)$. Then, there appears a natural group homomorphism:

$$\operatorname{Aut}(\Gamma) \to \operatorname{Aut}(U(\Gamma))$$

with kernel $\operatorname{Stab}(\Gamma)$.

Thus, the Weyl group embeds naturally in $\operatorname{Aut}(U(\Gamma))$, i.e., there is a natural action of the Weyl group on $U(\Gamma)$ by automorphisms.

Remark

$\operatorname{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$.
The definition can be extended naturally to give an affine group scheme $\mathbf{Diag}(\Gamma) \simeq \mathbf{D}(U(\gamma))$.

Fine gradings

$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g' \in G'} \mathcal{A}'_{g'}, \quad$ gradings on $\mathcal{A}$.

- Γ is a **refinement** of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
  Then Γ' is a **coarsening** of Γ.
- Γ is **fine** if it admits no proper refinement.

Remark

Any grading is a coarsening of a fine grading.

Gradings and diagonalizable group schemes

Theorem

Let G be an abelian group, and let $\mathcal{A}$ and $\mathcal{B}$ be two finite-dimensional algebras over $\mathbb{F}$.

1. *Any G-grading $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ induces a homomorphism $\theta : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{A}) \left(\leq \mathbf{GL}(\mathcal{A})\right)$, whose associated comodule map is $\rho : \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G$, $a \in \mathcal{A}_g \mapsto a \otimes g$. ρ is a homomorphism of algebras.*

   Conversely, given a homomorphism $\theta : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{A})$, then the associated comodule map is an algebra homomorphism, and it induces a G-grading $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, where $\mathcal{A}_g = \{a \in \mathcal{A} \mid \rho(a) = a \otimes g\}$.
   (The homogeneous components are the eigenspaces for the 'generic automorphism'!!)

Gradings and diagonalizable group schemes

Theorem (continued)

2. *Given G-gradings $\Gamma^1 : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma^2 : \mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$, with associated homomorphisms $\theta^1 : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{A})$ and $\theta^2 : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{B})$; Γ^1 and Γ^2 are isomorphic if and only if there is an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that for any R in $\mathrm{Alg}_{\mathbb{F}}$ and $\chi \in \mathbf{D}(G)(R)$, $\theta^2_R(\chi) = \varphi_R \circ \theta^1_R(\chi) \circ \varphi_R^{-1}$.*
3. *A grading Γ on $\mathcal{A}$ is fine if and only if $\mathbf{Diag}(\Gamma)$ is a maximal quasitorus of $\mathbf{Aut}(\mathcal{A})$.*
4. *Given two fine gradings $\Gamma^1 : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma^2 : \mathcal{B} = \bigoplus_{h\in H} \mathcal{B}_h$, Γ^1 and Γ^2 are equivalent if and only if there is an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\mathbf{Diag}(\Gamma^2) = \varphi \circ \mathbf{Diag}(\Gamma^1) \circ \varphi^{-1}$.*

Gradings and diagonalizable group schemes

Given an abelian group and a finite-dimensional vector space over $\mathbb{F}$, a homomorphism $\theta : \mathbf{D}(G) \to \mathbf{GL}(V)$ is equivalent to a 'grading' $V = \bigoplus_{g\in G} V_g$, that is, to the existence of a homogeneous basis $\{v_1, \ldots, v_n\}$:

$$\rho(v_i) = v_i \otimes g_i,$$

$i = 1, \ldots, n$, $g_1, \ldots, g_n \in G$, not necessarily different.

Exercise

Find analogous equivalences if $\mathbf{GL}(V)$ is replaced by $\mathbf{O}(V, q)$ (orthogonal group), $\mathbf{SO}(V, q)$ (special orthogonal group), or $\mathbf{Sp}(V, b)$ (symplectic group).

Transfer

If $\mathcal{A}$ and $\mathcal{B}$ are two finite-dimensional algebras over a field $\mathbb{F}$, and $\theta : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$ is a homomorphism of group schemes, any grading Γ on $\mathcal{A}$ by the abelian group G is determined by a homomorphism

$$\eta_\Gamma : \mathbf{D}(G) \longrightarrow \mathbf{Aut}(\mathcal{A}).$$

Composing with θ we get a homomorphism

$$\theta \circ \eta_\Gamma : \mathbf{D}(G) \longrightarrow \mathbf{Aut}(\mathcal{B}),$$

which induces a grading on $\mathcal{B}$ denoted by $\theta(\Gamma)$.

Transfer

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite-dimensional algebras over $\mathbb{F}$, and let $\theta : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$ be an isomorphism of affine group schemes.

1. *Let G be an abelian group and let Γ and Γ' be two G-gradings on $\mathcal{A}$. Then Γ and Γ' are isomorphic if and only if so are $\theta(\Gamma)$ and $\theta(\Gamma')$.*
2. *Let Γ and Γ' be two fine abelian group gradings on $\mathcal{A}$. Then Γ and Γ' are equivalent if and only so are $\theta(\Gamma)$ and $\theta(\Gamma')$.*

Orthogonal and symplectic Lie algebras

Consider the Lie algebra $\mathcal{L} := \mathrm{Skew}(\mathcal{R}, \varphi)$ of the skew-symmetric elements of a central simple associative algebra $\mathcal{R}$ relative to an involution of the first kind φ, and let $\theta : \mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$ be the homomorphism of affine group schemes obtained by restriction.

Theorem

If φ is orthogonal, assume the degree of $\mathcal{R}$ is ≥ 5 and $\neq 8$, and if φ is symplectic, assume it is ≥ 4.
Then the homomorphism $\theta : \mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$ is an isomorphism.

Special linear Lie algebras

Call an algebra of the form $\mathcal{S} \times \mathcal{S}^{\mathsf{op}}$, with $\mathcal{S}$ a finite-dimensional central simple associative algebra, a central simple associative algebra over $\mathbb{F} \times \mathbb{F}$. Its degree is the degree of the algebra $\mathcal{S}$ (the square root of its dimension).

Theorem

Let $\mathbb{K}$ be an étale quadratic algebra of dimension 2 over a field $\mathbb{F}$ of characteristic not 2. Let $\mathcal{R}$ be a finite-dimensional central simple associative algebra over $\mathbb{K}$ of degree $n \geq 3$, endowed with an involution φ of the second kind. Let $\mathcal{K} = \mathrm{Skew}(\mathcal{R}, \varphi)$ and $\mathcal{L} = [\mathcal{K}, \mathcal{K}]/([\mathcal{K}, \mathcal{K}] \cap Z(\mathcal{R}))$, which is a simple Lie algebra. Assume that the characteristic of $\mathbb{F}$ is not 3 if $n = 3$. Then the natural homomorphism $\theta : \mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$ is an isomorphism.

Known results

Patera, Zassenhaus et al. (1989, 1998) obtained a description of the fine gradings on the simple classical Lie algebras over $\mathbb{C}$ (other than D_4) by describing the corresponding quasitori.

The complete classification of the fine gradings, up to equivalence, including D_4, was obtained in 2010.

Over algebraically closed fields of characteristic $\neq 2$, Bahturin et al. (2001–2010) dealt with gradings on matrix algebras (with or without involution) and transferred the results to the classical simple Lie algebras (except D_4, which has a larger automorphism group scheme, because of the triality phenomenon).

Gradings on the classical simple Lie algebras, other than D_4 over algebraically closed fields are obtained by combining Pauli gradings and coarsenings of Cartan gradings.

Known results

Remark

D_4 requires a different treatment (E.-Kochetov).

Over $\mathbb{R}$, gradings on the simple classical Lie algebras have been classified, up to isomorphism, recently in works by Bahturin, E., Kochetov, and Rodrigo-Escudero.

The preprint by E., Kochetov, and Rodrigo-Escudero, with the classification of fine gradings on classical simple Lie algebras over $\mathbb{R}$, was posted a few weeks ago!

Octonions and G_2

$$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}(\mathbb{O}).$$

There are, up to equivalence, two fine gradings on the octonions (E. 1998, $\operatorname{char}\mathbb{F} \neq 2$):

- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\mathbf{Aut}(\mathbb{O})$.
- A $(\mathbb{Z}/2)^3$-grading that appears naturally while constructing $\mathbb{O}$ from the ground field using the Cayley-Dickson doubling process.

The induced $(\mathbb{Z}/2)^3$-grading on the simple Lie algebra of type G_2 satisfies that $\mathcal{L}_0 = 0$ and $\mathcal{L}_\alpha$ is a Cartan subalgebra of $\mathcal{L}$ for any $0 \neq \alpha \in (\mathbb{Z}/2)^3$.

The situation in characteristic 0 was first settled by Draper and Martín (2006) and, independently, by Bahturin and Tvalavadze (2009).

The Albert algebra and F_4

$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}(\mathbb{A})$, where $\mathbb{A} = H_3(\mathbb{O})$ is the Albert algebra (exceptional simple Jordan algebra).

There are, up to equivalence, four fine gradings on the Albert algebra –Draper-Martín ($\mathrm{char}\,\mathbb{F} = 0$, 2009); E.-Kochetov (2012, $\mathrm{char}\,\mathbb{F} \neq 2$)–:

- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\mathbf{Aut}(\mathbb{A})$.
- A $\mathbb{Z} \times (\mathbb{Z}/2)^3$-grading related to the fine $(\mathbb{Z}/2)^3$-grading on the octonions.
- A $(\mathbb{Z}/2)^5$-grading obtained by combining a natural $(\mathbb{Z}/2)^2$-grading on 3×3 hermitian matrices with the fine grading over $(\mathbb{Z}/2)^3$ of $\mathbb{O}$.
- A $(\mathbb{Z}/3)^3$-grading with $\dim \mathbb{A}_g = 1\ \forall g$ ($\mathrm{char}\,\mathbb{F} \neq 3$).

The induced $(\mathbb{Z}/3)^3$-grading on the simple Lie algebra of type F_4 satisfies that $\mathcal{L}_0 = 0$ and $\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}$ is a Cartan subalgebra of $\mathcal{L}$ for any $0 \neq \alpha \in (\mathbb{Z}/3)^3$.

The E-series

E_6: Draper-Viruel (2016, over $\mathbb{C}$).

E_7, E_8: Recent work by Jun Yu classifying conjugacy classes of certain subgroups of the compact Lie groups classifies, in particular, the fine gradings on E_7 and E_8 over $\mathbb{C}$.
This is enough to classify these gradings over arbitrary algebraically closed fields of characteristic 0 (E. 2016).

Open problem:

Fine gradings on E_6, E_7, E_8 in the modular case?

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Thank you!