Gradings on simple Lie algebras

Instituto Universitario de Matemáticas y Aplicaciones
Universidad de Zaragoza

Alberto Elduque

Tianyuan Mathematical Center
in Southeast China

February 28, 2022

Gradings on algebras reflect important symmetries on them, and help us to uncover their structure.

Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading (r being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}/2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.

Outline

1. Gradings
2. Gradings and diagonalizable group schemes
3. Gradings on simple Lie algebras

Gradings on Lie algebras

The paper

J. Patera and H. Zassenhaus
Gradings on Lie Algebras, I.
Linear Algebra Appl. **112** (1989), 87–159,

represented the beginning of a systematic study of gradings on Lie algebras.

Many of the definitions we use nowadays are given there.

Gradings

Definition

Given an abelian group G, a G-grading on a (finite-dimensional nonassociative) algebra $\mathcal{A}$ is a decomposition $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$, such that $\mathcal{A}_{g_1}\mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_1 g_2}$ for all $g_1, g_2 \in G$.

The support of Γ is the subset $\operatorname{Supp}(\Gamma) := \{g \in G \mid \mathcal{A}_g \neq 0\}$.

Example: Cartan grading

$$\mathcal{L} = \mathcal{H} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha\right)$$

(root space decomposition of a semisimple complex Lie algebra).

This is a grading by $\mathbb{Z}^n$, $n = \operatorname{rank} \mathcal{L}$.

Example: Pauli matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \bigoplus_{(\bar{\imath},\bar{\jmath}) \in (\mathbb{Z}/n)^2} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F}X^iY^j.$$

$\mathcal{A}$ becomes a graded division algebra.

This grading induces a grading on $\mathfrak{sl}_n(\mathbb{F})$.

Universal group

Given a grading $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, with support S, its **universal group** is the abelian group defined by generators and relations as follows:

$$U(\Gamma) := \left\langle S \mid s_1 s_2 = s_3 \ \forall s_1, s_2, s_3 \in S \text{ s.t. } 0 \neq \mathcal{A}_{s_1}\mathcal{A}_{s_2} \subseteq \mathcal{A}_{s_3} \right\rangle.$$

There is a natural one-to-one map $\iota : S \to U(\Gamma)$ taking s to its coset modulo the relations, and this induces a homomorphism $U(\Gamma) \to G$.

The universal group is the natural grading group for Γ.

Example

$\mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}x + \mathbb{F}h + \mathbb{F}y$, $[h, x] = 2x$, $[h, y] = -2y$, $[x, y] = h$.

This is a $\mathbb{Z}/3$-grading with $\deg h = \bar{0}$, $\deg x = \bar{1}$, $\deg y = \bar{2}$.

But $\mathbb{Z}/3$ is not the *natural* grading group. The universal one is $\mathbb{Z}$.

Isomorphism and equivalence

Two G-gradings

$$\Gamma^1 : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g \quad \text{and} \quad \Gamma^2 : \mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$$

are said to be **isomorphic** if there is an isomorphism of algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_g$ for all $g \in G$.

There is a weaker version:

Two gradings

$$\Gamma^1 : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g \quad \text{and} \quad \Gamma^2 : \mathcal{B} = \bigoplus_{h\in H} \mathcal{B}_h$$

are said to be **equivalent** if there is an isomorphism of algebras $\varphi : \mathcal{A} \to \mathcal{B}$ such that for any $g \in \operatorname{Supp}(\Gamma^1)$ there is an $h \in \operatorname{Supp}(\Gamma^2)$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_h$.

Groups attached to a grading

There appear several groups attached to Γ:

- The **automorphism group** (group of self-equivalences)

$$\mathrm{Aut}(\Gamma) = \{\varphi \in \mathrm{Aut}\,\mathcal{A} : \\ \exists \alpha \in \mathsf{Sym}(\mathrm{Supp}\ \Gamma) \text{ s.t. } \varphi(\mathcal{A}_s) = \mathcal{A}_{\alpha(s)}\ \forall s\}.$$

- The **stabilizer group** (group of self-isomorphisms)

$$\mathrm{Stab}(\Gamma) = \{\varphi \in \mathrm{Aut}(\Gamma) : \varphi(\mathcal{A}_g) = \mathcal{A}_g\ \forall g\}.$$

- The **diagonal group**

$$\mathrm{Diag}(\Gamma) = \{\varphi \in \mathrm{Aut}(\Gamma) : \\ \forall s \in \mathrm{Supp}\ \Gamma\ \exists \lambda_s \in \mathbb{F}^\times \text{ s.t. } \varphi|_{\mathcal{A}_s} = \lambda_s \mathrm{id}\}.$$

- The **Weyl group** $W(\Gamma) := \mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma)$.

Diagonal group and universal group

Any $\varphi \in \mathrm{Diag}(\Gamma)$ gives a map $\chi : S \to \mathbb{F}^\times$ by the equation $\varphi|_{\mathcal{L}_s} = \chi(s)\mathrm{id}$.

This map induces a character with values in $\mathbb{F}$:

$$\chi : U(\Gamma) \to \mathbb{F}^\times$$

and conversely, any character χ determines a unique element in $\mathrm{Diag}(\Gamma)$.

$$\mathrm{Diag}(\Gamma) \simeq \mathrm{Hom}\big(U(\Gamma), \mathbb{F}^\times\big).$$

Over an algebraically closed field of characteristic 0, the homogeneous components of an abelian group grading Γ are the eigenspaces relative to $\mathrm{Diag}(\Gamma)$.

Fine gradings

$\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g, \quad \Gamma' : \mathcal{A} = \bigoplus_{g'\in G'} \mathcal{A}'_{g'},$ gradings on $\mathcal{A}$.

- Γ is a **refinement** of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
  Then Γ' is a **coarsening** of Γ.
- Γ is **fine** if it admits no proper refinement.

Remark

Any grading is a coarsening of a fine grading.

Over an algebraically closed field of characteristic 0, the diagonal group of a fine abelian group grading is a maximal abelian diagonalizable subgroup of the automorphism group of the algebra.

Affine group schemes

Affine group scheme: Representable functor $\mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$.

$$\mathbf{G} \simeq \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{G}], .), \quad \mathbb{F}[\mathbf{G}] \text{ Hopf algebra.}$$

The generic element of $\mathbf{G}$ is

$$\mathrm{id}_{\mathbb{F}[\mathbf{G}]} \in \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{G}], \mathbb{F}[\mathbf{G}]) \simeq \mathbf{G}(\mathbb{F}[\mathbf{G}]).$$

Diagonalizable group schemes

Definition

An affine group scheme $\mathbf{G}$ over $\mathbb{F}$ is said to be **diagonalizable** if there is an abelian group G such that $\mathbf{G}$ is isomorphic to $\mathbf{D}(G)$ (whose associated Hopf algebra is the group algebra $\mathbb{F}G$).

If, moreover, $\mathbf{G}$ is algebraic (i.e.; G is finitely generated), then it is called a **quasitorus**.

Examples

- $\mathbf{G}_{\mathsf{m}} \simeq \mathbf{D}(\mathbb{Z})$, as $\mathbb{F}[X, X^{-1}]$ is the group algebra of $\mathbb{Z}$.
- $\boldsymbol{\mu}_n \simeq \mathbf{D}(\mathbb{Z}/n)$, as $\mathbb{F}[X]/(X^n - 1)$ is the group algebra of $\mathbb{Z}/n$.

Diagonalizable group schemes

Theorem

1. *The assignment*

$$\begin{array}{ccc} \{\textit{Abelian groups}\} & \to & \{\textit{diagonalizable group schemes}\} \\ G & \mapsto & \mathbf{D}(G), \end{array}$$

   induces an antiequivalence of categories, that restricts to an antiequivalence

$$\{\textit{finitely generated abelian groups}\} \longrightarrow \{\textit{quasitori}\}$$

2. *Any subscheme and any quotient of a diagonalizable group scheme is diagonalizable.*
3. *An affine group scheme is a quasitorus if and only if it isomorphic to a finite direct product of copies of* $\mathbf{G}_{\mathsf{m}}$*'s and* $\boldsymbol{\mu}_n$*'s.*

Diagonalizable group schemes. Representations

By Yoneda's Lemma, any homomorphism $\theta : \mathbf{G} \to \mathbf{H}$ of affine groups schemes is determined by the generic element $\theta(\mathrm{id}_{\mathbb{F}[\mathbf{G}]}) \in \mathbf{H}(\mathbb{F}[\mathbf{G}])$.

In particular, any homomorphism $\theta : \mathbf{D}(G) \to \mathbf{GL}(V)$ is determined by

$$\theta(\mathrm{id}_{\mathbb{F}G}) : V \otimes_{\mathbb{F}} \mathbb{F}G \longrightarrow V \otimes_{\mathbb{F}} \mathbb{F}G,$$

or by its restriction (the comodule map):

$$\rho : V \longrightarrow V \otimes_{\mathbb{F}} \mathbb{F}G.$$

This induces a G-grading on $V = \bigoplus_{g \in G} V_g$, where the homogeneous component of degree g is just the eigenspace for the eigenvalue g:

$$V_g = \{v \in V \mid \rho(v) = v \otimes g\}.$$

Gradings and diagonalizable group schemes

Theorem

Let G be an abelian group, and let $\mathcal{A}$ and $\mathcal{B}$ be two finite-dimensional algebras over $\mathbb{F}$.

1. *Any G-grading $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ induces a homomorphism $\theta : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{A}) \left(\leq \mathbf{GL}(\mathcal{A})\right)$, whose associated comodule map is* $\rho : \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G, \quad a \in \mathcal{A}_g \mapsto a \otimes g.$
   ρ *is a homomorphism of algebras.*

   Conversely, given a homomorphism $\theta : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{A})$, then the associated comodule map is an algebra homomorphism, and it induces a G-grading $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, where

   $$\mathcal{A}_g = \{a \in \mathcal{A} \mid \rho(a) = a \otimes g\}.$$

   (The homogeneous components are the eigenspaces for the 'generic automorphism'!!)

Gradings and diagonalizable group schemes

Theorem (continued)

2. *Given G-gradings*

$$\Gamma^1 : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g \quad \textit{and} \quad \Gamma^2 : \mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g,$$

with associated homomorphisms

$$\theta^1 : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{A}) \quad \textit{and} \quad \theta^2 : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{B}),$$

Γ^1 *and* Γ^2 *are isomorphic if and only if there is an algebra isomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ *such that for any* R *in* $\mathrm{Alg}_{\mathbb{F}}$ *and* $\chi \in \mathbf{D}(G)(R)$*,*

$$\theta^2_R(\chi) = \varphi_R \circ \theta^1_R(\chi) \circ \varphi_R^{-1}.$$

Gradings and diagonalizable group schemes

Theorem (continued)

3. *A grading* Γ *on* $\mathcal{A}$ *is fine if and only if* $\mathbf{Diag}(\Gamma)$ *(the image of* $\mathbf{D}(U(\Gamma))$*) is a maximal quasitorus of* $\mathbf{Aut}(\mathcal{A})$*.*

4. *Given two fine gradings*

$$\Gamma^1 : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g \quad \textit{and} \quad \Gamma^2 : \mathcal{B} = \bigoplus_{h\in H} \mathcal{B}_h,$$

   Γ^1 *and* Γ^2 *are equivalent if and only if there is an algebra isomorphism* $\varphi : \mathcal{A} \to \mathcal{B}$ *such that*

$$\mathbf{Diag}(\Gamma^2) = \varphi \circ \mathbf{Diag}(\Gamma^1) \circ \varphi^{-1}.$$

Transfer of gradings

If $\mathcal{A}$ and $\mathcal{B}$ are two finite-dimensional algebras over a field $\mathbb{F}$, and $\theta : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$ is a homomorphism of group schemes, any grading Γ on $\mathcal{A}$ by the abelian group G is determined by a homomorphism

$$\eta_\Gamma : \mathbf{D}(G) \longrightarrow \mathbf{Aut}(\mathcal{A}).$$

Composing with θ we get a homomorphism

$$\theta \circ \eta_\Gamma : \mathbf{D}(G) \longrightarrow \mathbf{Aut}(\mathcal{B}),$$

which induces a grading on $\mathcal{B}$ denoted by $\theta(\Gamma)$.

Transfer of gradings

Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be two finite-dimensional algebras over $\mathbb{F}$, and let $\theta : \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$ be an isomorphism of affine group schemes.

1. *Let G be an abelian group and let Γ and Γ' be two G-gradings on $\mathcal{A}$. Then Γ and Γ' are isomorphic if and only if so are $\theta(\Gamma)$ and $\theta(\Gamma')$.*
2. *Let Γ and Γ' be two fine abelian group gradings on $\mathcal{A}$. Then Γ and Γ' are equivalent if and only so are $\theta(\Gamma)$ and $\theta(\Gamma')$.*

Classical simple Lie algebras

The clue to classify gradings on the classical simple Lie algebras is to reduce this classification to the associative setting.

Orthogonal and symplectic Lie algebras

Consider the Lie algebra $\mathcal{L} := \mathrm{Skew}(\mathcal{R}, \varphi)$ of the skew-symmetric elements of a central simple associative algebra $\mathcal{R}$ relative to an involution of the first kind φ, and let $\theta : \mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$ be the homomorphism of affine group schemes obtained by restriction.

Theorem

If φ is orthogonal, assume the degree of $\mathcal{R}$ is ≥ 5 and $\neq 8$, and if φ is symplectic, assume it is ≥ 4.
Then the homomorphism $\theta : \mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$ is an isomorphism.

Special linear Lie algebras

Theorem

Let $\mathbb{K}$ be an étale quadratic algebra of dimension 2 over a field $\mathbb{F}$ of characteristic not 2. Let $\mathcal{R}$ be a finite-dimensional central simple associative algebra over $\mathbb{K}$ of degree $n \geq 3$, endowed with an involution φ of the second kind. Let $\mathcal{K} = \mathrm{Skew}(\mathcal{R}, \varphi)$ and $\mathcal{L} = [\mathcal{K}, \mathcal{K}]/([\mathcal{K}, \mathcal{K}] \cap Z(\mathcal{R}))$, which is a simple Lie algebra.
Assume that the characteristic of $\mathbb{F}$ is not 3 if $n = 3$.
Then the natural homomorphism $\theta : \mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$ is an isomorphism.

Gradings in the associative setting

Let $(\mathcal{R}, \varphi)$ be a finite-dimensional G-graded-simple associative algebra with involution, then either:

- $\mathcal{R}$ is not graded-simple: there is a G-graded-simple algebra $\mathcal{S}$, such that $(\mathcal{R}, \varphi) \simeq (\mathcal{S} \times \mathcal{S}^{\mathrm{op}}, \mathrm{ex})$.
- $\mathcal{R}$ is graded-simple: there is a G-graded-division algebra $\mathcal{D}$ endowed with a graded involution φ_0, a right G-graded module $\mathcal{V}$, and a hermitian or skew-hermitian homogeneous form $h : \mathcal{V} \times \mathcal{V} \to \mathcal{D}$, such that $(\mathcal{R}, \varphi) \simeq (\mathrm{End}_{\mathcal{D}}(\mathcal{V}), \varphi_h)$.

The isomorphisms above $\mathbf{Aut}(\mathcal{R}, \varphi) \to \mathbf{Aut}(\mathcal{L})$ allow us to transfer from the associative setting to the Lie algebra setting.

Known results

Patera, Zassenhaus et al. (1989, 1998) obtained a description of the fine gradings on the simple classical Lie algebras over $\mathbb{C}$ (other than D_4) by describing the corresponding quasitori.

The complete classification of the fine gradings, up to equivalence, including D_4, was obtained in 2010.

Over algebraically closed fields of characteristic $\neq 2$, Bahturin et al. (2001–2010) dealt with gradings on matrix algebras (with or without involution) and transferred the results to the classical simple Lie algebras (except D_4, which has a larger automorphism group scheme, because of the triality phenomenon).

Gradings on the classical simple Lie algebras, other than D_4, over algebraically closed fields are 'essentially' obtained by combining Pauli gradings and coarsenings of Cartan gradings.

Known results

Remark

D_4 requires a different treatment (E.-Kochetov).

And so does A_2 in characteristic 3: $\mathbf{Aut}(\mathfrak{psl}_3(\mathbb{F})) \cong \mathbf{Aut}(\text{'octonions'})$.

Over $\mathbb{R}$, gradings on the simple classical Lie algebras have been recently classified, up to isomorphism, in works by Bahturin, E., Kochetov, and Rodrigo-Escudero.

Fine gradings on the simple classical Lie algebras over $\mathbb{R}$, up to equivalence, have been recently classified by E., Kochetov, and Rodrigo-Escudero.

Octonions and G_2

$$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}(\mathbb{O}).$$

There are, up to equivalence, two fine gradings on the octonions (E. 1998, $\operatorname{char}\mathbb{F} \neq 2$):

- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\mathbf{Aut}(\mathbb{O})$.
- A $(\mathbb{Z}/2)^3$-grading that appears naturally while constructing $\mathbb{O}$ from the ground field using the Cayley-Dickson doubling process.

The induced $(\mathbb{Z}/2)^3$-grading on the simple Lie algebra of type G_2 satisfies that $\mathcal{L}_0 = 0$ and $\mathcal{L}_\alpha$ is a Cartan subalgebra of $\mathcal{L}$ for any $0 \neq \alpha \in (\mathbb{Z}/2)^3$.

The situation in characteristic 0 was first settled by Draper and Martín (2006) and, independently, by Bahturin and Tvalavadze (2009).

The Albert algebra and F_4

$\mathbf{Aut}(\mathcal{L}) \cong \mathbf{Aut}(\mathbb{A})$, where $\mathbb{A} = H_3(\mathbb{O})$ is the Albert algebra (exceptional simple Jordan algebra).

There are, up to equivalence, four fine gradings on the Albert algebra –Draper-Martín ($\operatorname{char}\mathbb{F} = 0$, 2009); E.-Kochetov (2012, $\operatorname{char}\mathbb{F} \neq 2$)–:

- The Cartan grading, obtained as the eigenspace decomposition for a maximal torus in $\mathbf{Aut}(\mathbb{A})$.
- A $\mathbb{Z} \times (\mathbb{Z}/2)^3$-grading related to the fine $(\mathbb{Z}/2)^3$-grading on the octonions.
- A $(\mathbb{Z}/2)^5$-grading obtained by combining a natural $(\mathbb{Z}/2)^2$-grading on 3×3 hermitian matrices with the fine grading over $(\mathbb{Z}/2)^3$ of $\mathbb{O}$.
- A $(\mathbb{Z}/3)^3$-grading with $\dim \mathbb{A}_g = 1\ \forall g$ ($\operatorname{char}\mathbb{F} \neq 3$).

The induced $(\mathbb{Z}/3)^3$-grading on the simple Lie algebra of type F_4 satisfies that $\mathcal{L}_0 = 0$ and $\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}$ is a Cartan subalgebra of $\mathcal{L}$ for any $0 \neq \alpha \in (\mathbb{Z}/3)^3$.

The E-series

E_6: Draper-Viruel (2016, over $\mathbb{C}$).

E_7, E_8: Recent work by Jun Yu (2016) classifying conjugacy classes of certain subgroups of the compact Lie groups classifies, in particular, the fine gradings on E_7 and E_8 over $\mathbb{C}$.
This is enough to classify these gradings over arbitrary algebraically closed fields of characteristic 0 (E. 2016).

Open problem:

Fine gradings on simple Lie algebras of type E in the modular case? in the real case?
Gradings on these algebras up to isomorphism?

References

A. Elduque and M. Kochetov.
Gradings on simple Lie algebras.
Mathematical Surveys and Monographs **189**,
American Mathematical Society, 2013.

A. Elduque, M. Kochetov, and A. Rodrigo-Escudero
Gradings on associative algebras with involution and real forms of classical simple Lie algebras.
J. Algebra **590** (2022), 61–138.

Thank you!