# Non-group gradings on simple Lie algebras

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# Outline

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# Gradings on Lie algebras

The paper

J. Patera and H. Zassenhaus
*Gradings on Lie Algebras, I.*
Linear Algebra Appl. **112** (1989), 87–159,

represented the beginning of a systematic study of gradings on Lie algebras.

Many of the definitions we use nowadays are given there.

# Set-gradings and (semi)group gradings

## Definition

A **set-grading** on a Lie algebra $\mathcal{L}$ is a decomposition:

$$\mathcal{L} = \bigoplus_{s \in S} \mathcal{L}_s$$

where $\mathcal{L}_s \neq 0$ for any $s$ in the grading set $S$, such that for any $s, s' \in S$, either $[\mathcal{L}_s, \mathcal{L}_{s'}] = 0$ or there exists $s'' \in S$ such that $0 \neq [\mathcal{L}_s, \mathcal{L}_{s'}] \subseteq \mathcal{L}_{s''}$.

## Definition

Given a (semi)group $G$, a $G$-grading on $\mathcal{L}$ is a decomposition $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$, such that $[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] \subseteq \mathcal{L}_{g_1 g_2}$ for all $g_1, g_2 \in G$.

The support of $\Gamma$ is the subset $\mathrm{Supp}\,(\Gamma) := \{g \in G \mid \mathcal{L}_g \neq 0\}$.

# Universal group

Given the set grading $\Gamma : \mathcal{L} = \bigoplus_{s\in S} \mathcal{L}_s$, its **universal group** is the group defined by generators (the elements of $S$) and relations as follows:

$$U(\Gamma) := \left\langle S \mid s_1 s_2 = s_3 \ \forall s_1, s_2, s_3 \in S \text{ s.t. } 0 \neq [\mathcal{L}_{s_1}, \mathcal{L}_{s_2}] \subseteq \mathcal{L}_{s_3} \right\rangle.$$

There is a natural map $\iota : S \to U(\Gamma)$ taking $s$ to its coset modulo the relations, and $\Gamma$ can be realized as a group grading if and only if $\iota$ is one-to-one.

$U(\Gamma)$ is abelian if $\mathcal{L}$ is simple.

# Diagonal group

The **diagonal group** of $\Gamma$ is the group

$$\mathrm{Diag}(\Gamma) := \{\varphi \in \mathrm{Aut}(\mathcal{L}) \mid \varphi|_{\mathcal{L}_s} \in \mathbb{F}^{\times}\mathrm{id}\ \forall s \in S\}$$

Any $\varphi \in \mathrm{Diag}(\Gamma)$ gives a map $\chi : S \to \mathbb{F}^{\times}$ by the equation $\varphi|_{\mathcal{L}_s} = \chi(s)\mathrm{id}$.

This map induces a a character with values in $\mathbb{F}$:

$$\chi : U(\Gamma) \to \mathbb{F}^{\times}.$$

and conversely, any character $\chi$ determines a unique element in $\mathrm{Diag}(\Gamma)$.

$$\mathrm{Diag}(\Gamma) \simeq \mathrm{Hom}\big(U(\Gamma), \mathbb{F}^{\times}\big).$$

# Incorrect assertion

Theorem 1.(d) of Patera-Zassenhaus' paper incorrectly asserts that, given any set grading on a Lie algebra $\Gamma : \mathcal{L} = \bigoplus_{s\in S} \mathcal{L}_s$, there is an abelian semigroup and a one-to-one map $\iota : S \to G$, so that $\Gamma$ is realized as a $G$-grading through $\iota$.

# Counterexamples

A. Elduque
*A Lie grading which is not a semigroup grading*.
Linear Algebra Appl. **418** (2006), no. 1, 312–314.

A. Elduque
*More non-semigroup Lie gradings*.
Linear Algebra Appl. **431** (2009), no. 9, 1603–1606.

- $\mathcal{L} = (\mathbb{F}a \oplus \mathbb{F}u) \oplus \mathbb{F}v \oplus \mathbb{F}w$, with $[a,u] = u$, $[a,v] = w$, $[a,w] = v$, all other brackets being zero.
- $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathfrak{su}_2 = (\mathbb{F}h + \mathbb{F}x + \mathbb{F}y) \oplus (\mathbb{F}e_1 + \mathbb{F}e_2 + \mathbb{F}e_3)$ (semisimple), with $[h,x] = 2x$, $[h,y] = -2y$, $[x,y] = h$, $[e_i, e_{i+1}] = e_{i+2}$ (indices modulo $3$), with grading

$$\mathcal{L} = (\mathbb{F}h + \mathbb{F}e_1) \oplus \mathbb{F}x \oplus \mathbb{F}y \oplus \mathbb{F}e_2 \oplus \mathbb{F}e_3.$$

# What about simple algebras?

In the paper of 2006 the following question was raised:

*Can any set grading on a finite-dimensional simple Lie algebra over $\mathbb{C}$ be realized as a group grading?*

### Remark

Cristina Draper proved that if $G$ is a semigroup and $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ is a $G$-grading on a simple Lie algebra $\mathcal{L}$, then $\mathrm{Supp}\,(\Gamma)$ generates an abelian subgroup of $G$ so, in particular, $\Gamma$ can be realized as a group grading.

# Non-group gradings on simple Lie algebras

# Pure gradings

$\mathcal{L}$: finite-dimensional semisimple Lie algebra over $\mathbb{F} = \overline{\mathbb{F}}$, $\mathrm{char}(\mathbb{F}) = 0$.

**Definition (Hesselink 1982)**

A grading $\Gamma$ by the abelian group $G$: $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$, is said to be **pure** if there exists a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ and an element $g \in G$, $g \neq e$, such that $\mathcal{H}$ is contained in $\mathcal{L}_g$.

# Pure gradings: diagonal group

## Proposition

*Let $G$ be an abelian group and let $\Gamma$ be a pure $G$-grading of $\mathcal{L}$. Then there is an (order $2$) automorphism $\sigma \in \mathrm{Aut}(\mathcal{L})$ with $\sigma|_{\mathcal{H}} = -\mathrm{id}$, such that $\mathrm{Diag}(\Gamma)$ is the cartesian product of $\mathrm{Diag}(\Gamma) \cap T_2$ and the subgroup generated by $\sigma$:*

$$\mathrm{Diag}(\Gamma) = (\mathrm{Diag}(\Gamma) \cap T_2) \times \langle \sigma \rangle,$$

*where $T_2$ is the $2$-torsion part of the torus fixing elementwise $\mathcal{H}$.*
*In particular, $\mathrm{Diag}(\Gamma)$ is a finite $2$-elementary group.*

# What pure gradings look like?

- Fix a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$ and an automorphism $\sigma \in \mathrm{Aut}(\mathcal{L})$ such that $\sigma|_{\mathcal{H}} = -\mathrm{id}$ (unique, up to conjugation).
- Denote by $Q$ the root lattice $Q := \mathbb{Z}\Phi = \mathbb{Z}\Delta$. The torus $T$ is naturally isomorphic to the group of characters of $Q$, and this restricts to a group isomorphism $T_2 \simeq \mathrm{Hom}(Q/2Q, \{\pm 1\})$.
- Any $\chi \in \mathrm{Hom}(Q/2Q, \{\pm 1\})$ corresponds to the automorphism $\tau_\chi$ whose restriction to $\mathcal{L}_\alpha$ is $\chi(\alpha + 2Q)\mathrm{id}$ for all $\alpha \in \Phi$. In particular, any element of $T_2$ acts as $\pm\mathrm{id}$ on the two-dimensional subspace $\mathcal{L}_\alpha + \mathcal{L}_{-\alpha}$ for any $\alpha \in \Phi^+$.
- This gives a bijection:

$$\begin{aligned} \{\text{subgroups of } T_2\} &\longrightarrow \{\text{subgroups of } Q \text{ containing } 2Q\} \\ S \quad &\mapsto \quad E \text{ such that } E/2Q = \bigcap_{\tau_\chi \in S} \ker \chi\,. \end{aligned}$$

In the reverse direction, a subgroup $E$ with $2Q \leq E \leq Q$ corresponds to the subgroup $T_E := \{\tau_\chi \mid \chi(E/2Q) = 1\}$.

# What pure gradings look like?

- For any positive root $\alpha \in \Phi^+$, pick a nonzero element $x_\alpha \in \mathcal{L}_\alpha$. Let $E$ be a subgroup with $2Q \leq E \leq Q$ and denote by $\overline{Q}_E$ the 2-elementary group $Q/E \times \mathbb{Z}/2$.
- Define the $\overline{Q}_E$-grading $\Gamma_E$ on $\mathcal{L}$ as follows:

$$\mathcal{L}_{(q+E,\bar{0})} = \bigoplus_{\alpha\in\Phi^+\cap(q+E)} \mathbb{F}\left(x_\alpha + \sigma(x_\alpha)\right),$$

$$\mathcal{L}_{(q+E,\bar{1})} = \begin{cases} \bigoplus_{\alpha\in\Phi^+\cap(q+E)} \mathbb{F}\left(x_\alpha - \sigma(x_\alpha)\right) & \text{if } q+E \neq E, \\ \mathcal{H} \oplus \left(\bigoplus_{\alpha\in\Phi^+\cap(q+E)} \mathbb{F}\left(x_\alpha - \sigma(x_\alpha)\right)\right) & \text{if } q+E = E. \end{cases}$$

This is what pure gradings look like!

# Back to diagonal groups of pure gradings

For any such $E$, consider the subgroup $E^\circ \subseteq E$ generated by $2Q$ and by

- the roots $\alpha \in \Phi^+$ such that $x_\alpha - \sigma(x_\alpha)$ is in the homogeneous component of $\Gamma_E$ that contains $\mathcal{H}$, and
- the elements $\alpha - \beta$ for $\alpha, \beta \in \Phi^+$ such that $x_\alpha - \sigma(x_\alpha)$ and $x_\beta - \sigma(x_\beta)$ are in the same homogeneous component of $\Gamma_E$.

**Proposition**

*The gradings $\Gamma_E$ and $\Gamma_{E^\circ}$ are equivalent and*

$$\mathrm{Diag}(\Gamma_E) = \mathrm{Diag}(\Gamma_{E^\circ}) = T_{E^\circ} \times \langle \sigma \rangle.$$

$E^\circ$ can be strictly contained in $E$.

# Fixing some notation

$$\mathcal{L} := \mathfrak{so}_{2n} = \left\{ \begin{pmatrix} A & B = -B^T \\ C = -C^T & -A^T \end{pmatrix} \mid A, B, C \in \mathrm{Mat}_n(\mathbb{F}) \right\}.$$

$\mathcal{H} = \{h = \mathrm{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)\}$.

$\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$ (root system).

$\Delta = \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$ (system of simple roots).

$\varepsilon_1$ is the highest weight of the natural module, while $\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$ is the highest weight of one of the two half-spin modules.

The automorphism $\sigma : \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \mapsto \begin{pmatrix} -A^T & C \\ B & A \end{pmatrix}$ satisfies $\sigma|_{\mathcal{H}} = -\mathrm{id}$.

# Pretty example of a pure grading in $\mathfrak{so}_8$

$E = 2W$, with $W$ the weight lattice of the half-spin module:

$$E = 2\left(Q + \mathbb{Z}\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)\right) = 2Q + \mathbb{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4).$$

The grading group of $\Gamma_E$ is $Q/E \times \mathbb{Z}/2 \simeq (\mathbb{Z}/2)^4$.
The positive roots split modulo $E$ into $6$ pairs of orthogonal roots:

$$\begin{array}{lll} \{\varepsilon_1 + \varepsilon_2, \varepsilon_3 + \varepsilon_4\}, & \{\varepsilon_1 + \varepsilon_3, \varepsilon_2 + \varepsilon_4\}, & \{\varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_3\}, \\ \{\varepsilon_1 - \varepsilon_2, \varepsilon_3 - \varepsilon_4\}, & \{\varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_4\}, & \{\varepsilon_1 - \varepsilon_4, \varepsilon_2 - \varepsilon_3\}. \end{array}$$

and the homogeneous spaces are, for each pair $\{\alpha, \beta\}$ above:

$$\begin{aligned} \mathcal{L}_{(\alpha+E,\bar{0})} &= \mathcal{L}_{(\beta+E,\bar{0})} = \mathbb{F}\big(x_\alpha + \sigma(x_\alpha)\big) + \mathbb{F}\big(x_\beta + \sigma(x_\beta)\big) \\ \mathcal{L}_{(\alpha+E,\bar{1})} &= \mathcal{L}_{(\beta+E,\bar{1})} = \mathbb{F}\big(x_\alpha - \sigma(x_\alpha)\big) + \mathbb{F}\big(x_\beta - \sigma(x_\beta)\big) \\ \mathcal{L}_{(0,\bar{1})} &= \mathcal{H} \qquad\qquad\qquad (0 \neq x_\alpha \in \mathcal{L}_\alpha \ \forall \alpha \in \Phi) \end{aligned}$$

Here $E = E^\circ$ and hence the diagonal group of $\Gamma_E$ is $T_E \times \langle\sigma\rangle$, which is isomorphic to $(\mathbb{Z}/2)^4$.

# A non-group grading on $\mathfrak{so}_{26}$

## Idea

To *glue* copies of the *pretty example* in suitable subalgebras of $\mathfrak{so}_{26}$ isomorphic to $\mathfrak{so}_8$.
To select the right subalgebras, the lines in the projective plane $\mathbb{P}^2(\mathbb{F}_3)$ will be used.

We may number the points in the projective plane $\mathbb{P}^2(\mathbb{F}_3)$ from $1$ to $13$, so that the lines are:

$$
\begin{array}{cccc}
\{1,2,3,4\} & \{2,5,8,11\} & \{3,5,9,13\} & \{4,5,10,12\} \\
\{1,5,6,7\} & \{2,6,9,12\} & \{3,6,10,11\} & \{4,6,8,13\} \\
\{1,8,9,10\} & \{2,7,10,13\} & \{3,7,8,12\} & \{4,7,9,11\} \\
\{1,11,12,13\} & & &
\end{array}
$$

## Exercise

Check that any two points lie in a unique line, and any two lines intersect in a unique point.

# A non-group grading on $\mathfrak{so}_{26}$

For any line $\ell = \{i, j, k, l\}$, with $i < j < k < l$, consider the following six pairs of orthogonal positive roots

$$\begin{array}{lll} \{\varepsilon_i + \varepsilon_j, \varepsilon_k + \varepsilon_l\}, & \{\varepsilon_i + \varepsilon_k, \varepsilon_j + \varepsilon_l\}, & \{\varepsilon_i + \varepsilon_l, \varepsilon_j + \varepsilon_k\}, \\ \{\varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l\}, & \{\varepsilon_i - \varepsilon_k, \varepsilon_j - \varepsilon_l\}, & \{\varepsilon_i - \varepsilon_l, \varepsilon_j - \varepsilon_k\}. \end{array}$$

All in all, we have $13 \times 6 = 78$ pairs of orthogonal positive roots, and each $\alpha \in \Phi^+$ appears in exactly one of the pairs.
Then $\mathcal{L}$ decomposes as

$$\mathcal{L} = \mathcal{H} \oplus \Big( \bigoplus_{\{\alpha,\beta\}\ \text{`pair'}} \big(\mathbb{F}(x_\alpha + \sigma(x_\alpha)) + \mathbb{F}(x_\beta + \sigma(x_\beta))\big)\Big)$$
$$\oplus \Big( \bigoplus_{\{\alpha,\beta\}\ \text{`pair'}} \big(\mathbb{F}(x_\alpha - \sigma(x_\alpha)) + \mathbb{F}(x_\beta - \sigma(x_\beta))\big)\Big).$$

$\mathcal{L}$ decomposes into the direct sum of $\mathcal{H}$ and the direct sum of $78 \times 2 = 156$ two-dimensional toral subalgebras.

# A non-group grading on $\mathfrak{so}_{26}$

**Proposition**

*This is a set-grading of $\mathcal{L}$.*

**Sketch of proof**

We must check that the bracket of two homogeneous components is either $0$ or contained in a homogeneous component.

Apart from trivial cases, we are left with two possibilities, depending on the pairs of orthogonal roots: $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$, involved in the definition of the homogeneous components.

These are obtained from two lines $\ell_1, \ell_2$, and there are two possibilities:

- $\ell_1 = \{i, j, k, l\}$, $\ell_2 = \{i, p, q, r\}$ for distinct $i, j, k, l, p, q, r$. Easy!
- $\ell_1 = \ell_2 = \{i, j, k, l\}$. We restrict here to a subalgebra isomorphic to $\mathfrak{so}_8$ and use the pretty example there.

# A non-group grading on $\mathfrak{so}_{26}$

**Theorem**

***This grading is not a group-grading!!***

**Sketch of proof**

- Arguments for pure gradings work here giving $\mathrm{Diag}(\Gamma) = T_E \times \langle \sigma \rangle$.
- For any line $\{i, j, k, l\}$, the element $\varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l$ lies in $E$.

  Sum over all the lines $\{1, i, j, k\}$ to get $4\varepsilon_1 + \sum_{i \neq 1} \varepsilon_i \in E$.
  But $4\varepsilon_1 = 2(\varepsilon_1 + \varepsilon_2) + 2(\varepsilon_1 - \varepsilon_2) \in 2Q \subseteq E$, so $\sum_{i \neq 1} \varepsilon_i \in E$.
- From $\sum_{i \neq 1} \varepsilon_i \in E$, $\sum_{i \neq 2} \varepsilon_i \in E$, get $\varepsilon_1 - \varepsilon_2 \in E$ and, more generally, $\varepsilon_i - \varepsilon_j \in E$ for all $i \neq j$. *($E$ is too big!!)*
- $[Q : E] = 2$, and hence $\mathrm{Diag}(\Gamma) \simeq C_2 \times C_2 \simeq U(\Gamma)$. (Too small!!)

# Some remarks

A careful look at the non-group grading on $\mathfrak{so}_{26}$ shows that the key point is the use of the lines of $\mathbb{P}^2(\mathbb{F}_3)$ to split $\Phi^+$ into a disjoint union of pairs of orthogonal roots.

This projective space can be substituted by a Steiner system of type $S(2,4,n)$ to get, *with the same arguments!*, non-group gradings on $\mathfrak{so}_{2n}$ for $n \equiv 1 \pmod{12}$.

**Question:**

Simpler examples?

Thank you!