Quaternions and Octonions

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Outline

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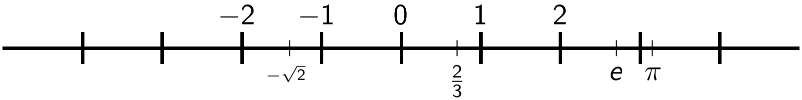
Real numbers

$$\mathbb{R} = \{\text{real numbers}\}$$

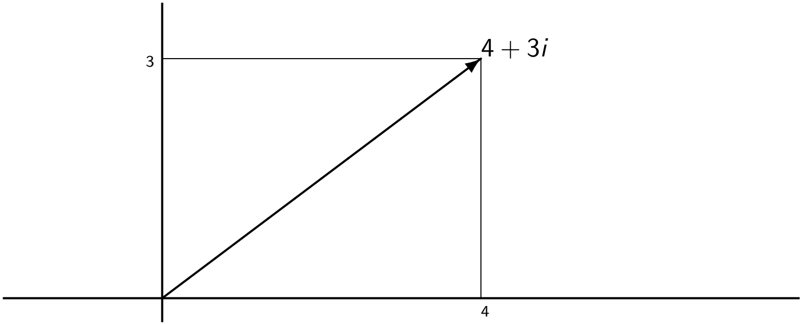
Real numbers are used in measurements.



But we cannot solve equations as simple as $X^2 + 1 = 0$!

Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} \, (\simeq \mathbb{R}^2)$$



$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Complex numbers: properties

Exercise

$$|z_1 z_2| = |z_1||z_2|$$

($|.|$ denotes the usual length.)

Exercise

Rotation of angle α in $\mathbb{R}^2 \leftrightarrow$ multiplication by $e^{i\alpha} = \cos\alpha + i\sin\alpha$.

$$SO(2) \simeq \{z \in \mathbb{C} : |z| = 1\} \simeq S^1$$

Outline

A three-dimensional algebra?

Hamilton tried to find a multiplication, analogous to the multiplication of complex numbers, but in dimension 3, that should respect the "law of moduli": $|z_1 z_2| = |z_1||z_2|$:

$$(a + bi + cj)(a' + b'i + c'j) = ???$$

$$\text{(assuming } i^2 = -1 = j^2\text{)}$$

Problem: $ij, \quad ji$?

After years of struggle, he found the solution on October 16, 1843.

A spark flashed forth

Letter from Sir W. R. Hamilton to his son Rev. Archibald H. Hamilton, dated August 5 1865:

MY DEAR ARCHIBALD -
(1) I had been wishing for an occasion of corresponding a little with you on QUATERNIONS: and such now presents itself, by your mentioning in your note of yesterday, received this morning, that you "have been reflecting on several points connected with them" (the quaternions), "particularly on the Multiplication of Vectors."

(2) No more important, or indeed fundamental question, in the whole Theory of Quaternions, can be proposed than that which thus inquires What is such MULTIPLICATION? What are its Rules, its Objects, its Results? What Analogies exist between it and other Operations, which have received the same general Name? And finally, what is (if any) its Utility?

A spark flashed forth

(3) If I may be allowed to speak of myself in connexion with the subject, I might do so in a way which would bring you in, by referring to an ante-quaternionic time, when you were a mere child, but had caught from me the conception of a Vector, as represented by a Triplet: and indeed I happen to be able to put the finger of memory upon the year and month - October, 1843 - when having recently returned from visits to Cork and Parsonstown, connected with a meeting of the British Association, the desire to discover the laws of the multiplication referred to regained with me a certain strength and earnestness, which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "Well, Papa, can you multiply triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them."

A spark flashed forth

(4) But on the 16th day of the same month - which happened to be a Monday, and a Council day of the Royal Irish Academy - I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery.

A spark flashed forth

Nor could I resist the impulse -unphilosophical as it may have been- to cut with a knife on a stone of Brougham Bridge[1], as we passed it, the fundamental formula with the symbols, i, j, k; namely,

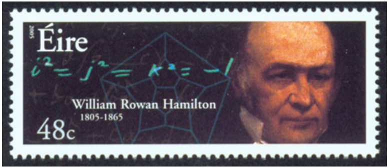
$$i^2 = j^2 = k^2 = ijk = -1$$

which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (October 16th, 1843), which records the fact, that I then asked for and obtained leave to read a Paper on Quaternions, at the First General Meeting of the session: which reading took place accordingly, on Monday the 13th of the November following.
With this quaternion of paragraphs I close this letter I.; but I hope to follow it up very shortly with another.
Your affectionate father, WILLIAM ROWAN HAMILTON.

[1] The actual name of this bridge is Broome, not Brougham

$\mathbb{H}$

$$\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k,$$
$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$



Hamilton and his quaternions

Some properties of $\mathbb{H}$

- $|q_1 q_2| = |q_1||q_2| \ \forall q_1, q_2 \in \mathbb{H}$
  $(|a + bi + cj + dk|^2 = a^2 + b^2 + c^2 + d^2)$

- $\mathbb{H}$ is an associative division algebra (but it is not commutative).
  Therefore $S^3 \simeq \{q \in \mathbb{H} : |q| = 1\}$ is a (Lie) group.
  (This implies the parallelizability of S^3.)

- $\mathbb{H}_0 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \simeq \mathbb{R}^3, \quad \mathbb{H} = \mathbb{R} \oplus \mathbb{H}_0$, and $\forall u, v \in \mathbb{H}_0$:

$$uv = -u \cdot v + u \times v$$

  (where $u \cdot v$ and $u \times v$ denote the usual scalar and cross products).

Some properties of $\mathbb{H}$

- $\forall q = a1 + u \in \mathbb{H}$, $q^2 = (a^2 - u \cdot u) + 2au$, so
$$q^2 - (2a)q + |q|^2 = 0 \qquad (\mathbb{H} \text{ is quadratic.})$$
- The map $q = a + u \mapsto \bar{q} = a - u$ is an involution, with $q + \bar{q} = 2a$ and $q\bar{q} = \bar{q}q = |q|^2$.
- $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j \simeq \mathbb{C}^2$ is a two-dimensional vector space over $\mathbb{C}$. Multiplication is given by:
$$(p_1 + p_2 j)(q_1 + q_2 j) = (p_1 q_1 - \bar{q}_2 p_2) + (q_2 p_1 + p_2 \bar{q}_1)j$$

Outline

Rotations in three-dimensional space

$$q \in \mathbb{H},\ |q| = 1 \ \Rightarrow \exists \alpha \in [0, \pi],\ u \in \mathbb{H}_0,\ |u| = 1$$
$$\text{such that } q = (\cos\alpha)1 + (\sin\alpha)u$$

Take $v \in \mathbb{H}_0$ of norm 1 and orthogonal to u, so that $\{u, v, u \times v\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3 = \mathbb{H}_0$.

Consider the linear map:

$$\varphi_q : \mathbb{H}_0 \longrightarrow \mathbb{H}_0,$$
$$x \mapsto qxq^{-1} = qx\bar{q}.$$

Coordinate matrix of φ_q

$$\varphi_q(u) = quq^{-1} = u \qquad (\text{as } uq = qu),$$

$$\begin{aligned}\varphi_q(v) &= \big((\cos\alpha)1 + (\sin\alpha)u\big)v((\cos\alpha)1 - (\sin\alpha)u) \\ &= \big((\cos\alpha)v + (\sin\alpha)u \times v\big)((\cos\alpha)1 - (\sin\alpha)u) \\ &= (\cos^2\alpha)v + 2(\cos\alpha\sin\alpha)u \times v - (\sin^2\alpha)(u \times v) \times u \\ &= (\cos 2\alpha)v + (\sin 2\alpha)u \times v,\end{aligned}$$

$$\varphi_q(u \times v) = \ldots = -(\sin 2\alpha)v + (\cos 2\alpha)u \times v.$$

Coordinate matrix of φ_q

Thus the coordinate matrix of φ_q relative to the basis $\{u, v, u \times v\}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\alpha & -\sin 2\alpha \\ 0 & \sin 2\alpha & \cos 2\alpha \end{pmatrix}$$

φ_q is a rotation around the axis $\mathbb{R}^+ u$ of angle 2α

$SO(3)$

The map

$$\begin{aligned} \varphi : S^3 \simeq \{q \in \mathbb{H} : |q| = 1\} &\longrightarrow SO(3), \\ q \quad \mapsto \quad & \varphi_q \end{aligned}$$

is a surjective (Lie) group homomorphism with $\ker \varphi = \{\pm 1\}$:

$$S^3/_{\{\pm 1\}} \simeq SO(3)$$

(S^3 is the universal cover of $SO(3)$)

Rotations in three-dimensional space

Rotations in $\mathbb{R}^3$ $\longleftrightarrow$ Conjugation in $\mathbb{H}_0$ by norm 1 quaternions "modulo ± 1"

It is quite easy now to compose rotations in three-dimensional space!

It is enough to multiply norm 1 quaternions! $(\varphi_p \circ \varphi_q = \varphi_{pq})$

Now one can deduce easily the formulas by Olinde Rodrigues (1840) for the composition of rotations.

Dual quaternions and $SE(3)$

$SE(3)$ denotes the group of proper isometries in three dimensional affine euclidean space.

Consider the algebra of dual numbers

$$\mathbb{R}[\epsilon] = \mathbb{R}1 + \mathbb{R}\epsilon, \qquad \epsilon^2 = 0,$$

and the algebra of dual quaternions

$$D(\mathbb{H}) = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}[\epsilon].$$

In $D(\mathbb{H})$ we have a norm analogous to $|q|^2$ in $\mathbb{H}$, given by

$$\begin{aligned} N(q_1 + q_2\epsilon) &= (q_1 + q_2\epsilon)(\bar{q}_1 + \bar{q}_2\epsilon) \\ &= q_1\bar{q}_1 + (q_1\bar{q}_2 + q_2\bar{q}_1)\epsilon \\ &= |q_1|^2 + \left(q_1\bar{q}_2 + \overline{q_1\bar{q}_2}\right)\epsilon \in \mathbb{R}[\epsilon]. \end{aligned}$$

Unit sphere in dual quaternions

Remark

$q_1 + q_2\epsilon$ is invertible if and only if $N(q_1 + q_2\epsilon)$ is invertible (in $\mathbb{R}[\epsilon]$), if and only if $q_1 \neq 0$.

Denote by $\mathbb{S}$ the 'unit sphere' in $D(\mathbb{H})$:

$$\begin{aligned}\mathbb{S} &= \{Q \in D[\mathbb{H}] : N(Q) = 1\} \\ &= \{q_1 + q_2\epsilon : |q_1| = 1,\ q_1\bar{q}_2 \in \mathbb{H}_0\}.\end{aligned}$$

Proper isometries

For any $q_1 + q_2\epsilon \in \mathbb{S}$ and any $p \in \mathbb{H}_0$:

$$\begin{aligned}(q_1 - q_2\epsilon)(1 + p\epsilon)(\bar{q}_1 + \bar{q}_2\epsilon) \\ &= 1 + (q_1\bar{q}_2 - q_2\bar{q}_1 + q_1 p\bar{q}_1)\epsilon \\ &= 1 + \big(2q_1\bar{q}_2 + q_1 p\bar{q}_1\big)\epsilon.\end{aligned}$$

p is first rotated by q_1, and then translated by $2q_1\bar{q}_2$!!

For any $u \in \mathbb{H}_0$, $2q_1\bar{q}_2 = u$ if and only if $\bar{q}_2 = \frac{1}{2}\bar{q}_1 u$, if and only if $q_2 = -\frac{1}{2}uq_1$, so varying q_1 and q_2 above, any rotation and any translation can be achieved.

Dual quaternions and $SE(3)$

Identifying the affine euclidean space $\mathbb{R}^3$ with $1 + \mathbb{H}_0\epsilon$, the map

$$\Phi : \mathbb{S} \longrightarrow SE(3)$$
$$q_1 + q_2\epsilon \mapsto \Big(1 + p\epsilon \mapsto (q_1 - q_2\epsilon)(1 + p\epsilon)(\bar{q}_1 + \bar{q}_2\epsilon)\Big)$$

turns out to be a surjective group homomorphism.

$$\begin{aligned}
&\ker \Phi \\
&= \{q_1 + q_2\epsilon \in \mathbb{S} : (q_1 - q_2\epsilon)(1 + p\epsilon) = (1 + p\epsilon)(q_1 + q_2\epsilon)\ \forall p \in \mathbb{H}_0\} \\
&= \{q_1 + q_2\epsilon \in \mathbb{S} : q_1 p - q_2 = p q_1 + q_2\ \forall p \in \mathbb{H}_0\} \\
&= \{q_1 + q_2\epsilon \in \mathbb{S} : q_2 = 0,\ q_1 \in \mathbb{R}1\} = \{\pm 1\}.
\end{aligned}$$

$$\mathbb{S}/\{\pm 1\} \simeq SE(3).$$

Rotations in $\mathbb{R}^4$

- $\forall p \in \mathbb{H}$ with $|p| = 1$, the left (resp. right) multiplication L_p (resp. R_p) by p is an isometry, due to the multiplicativity of the norm.

- For $p = (\cos\alpha)1 + (\sin\alpha)u$, ($\alpha \in [0, \pi]$, $u \in \mathbb{H}_0$, $|u| = 1$), we have $p^2 - 2(\cos\alpha)p + 1 = 0$, so the minimal polynomial of the multiplication by p is either $X \pm 1$ for $p = \mp 1$, or the irreducible polynomial $X^2 - 2(\cos\alpha)X + 1$ otherwise.

- Hence the characteristic polynomial of the multiplication by p is always
$$\left(X^2 - 2(\cos\alpha)X + 1\right)^2$$
and, in particular, the determinant of the multiplication by p is 1.

Multiplication by norm 1 quaternions are rotations in $\mathbb{H} \simeq \mathbb{R}^4$.

Rotations in $\mathbb{R}^4$

- If ψ is a rotation in $\mathbb{R}^4 \simeq \mathbb{H}$, $a = \psi(1)$ is a norm 1 quaternion, and
$$L_{\bar{a}} \circ \psi(1) = \bar{a}a = |a|^2 = 1,$$
so $L_{\bar{a}} \circ \psi$ is actually a rotation in $\mathbb{R}^3 \simeq \mathbb{H}_0$.

- Therefore, there is a norm 1 quaternion $q \in \mathbb{H}$ such that
$$\bar{a}\psi(x) = qxq^{-1}$$
for any $x \in \mathbb{H}$. That is:
$$\psi(x) = (aq)xq^{-1} \quad \forall x \in \mathbb{H}.$$

$SO(4)$

The map

$$\Psi : S^3 \times S^3 \longrightarrow SO(4),$$
$$(p, q) \mapsto \psi_{p,q}\ (x \mapsto pxq^{-1})$$

is a surjective (Lie) group homomorphism with $\ker \Psi = \{\pm(1,1)\}$.

$$S^3 \times S^3/_{\{\pm(1,1)\}} \simeq SO(4)$$

(From here we get $\quad SO(3) \times SO(3) \simeq PSO(4)$)

Rotations in $\mathbb{R}^4$

It is quite easy to compose rotations in four-dimensional space!

It is enough to multiply pairs of norm 1 quaternions!
$(\psi_{p_1,q_1} \circ \psi_{p_2,q_2} = \psi_{p_1 p_2, q_1 q_2})$

Exercise

What kind of rotation is $\psi_{p,q}$ for $p + \bar{p} = 2\cos\alpha$ and $q + \bar{q} = 2\cos\beta$?

Solution: A "double rotation" with angles $\alpha + \beta$ and $\alpha - \beta$.

Outline

Octonions (1843-1845)

> *There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties.*
>
> *If with your alchemy you can make three pounds of gold, why should you stop there?*

(Letter from John T. Graves to Hamilton, dated October 26, 1843!)

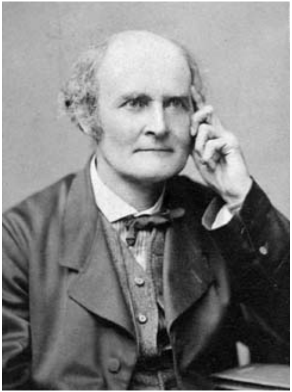
Octonions (1843-1845)

The algebra of quaternions is obtained by doubling suitably the field of complex numbers:

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j.$$

Doubling again we get the octonions (Graves – Cayley):

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l.$$



Arthur Cayley

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l = \mathbb{R}\langle 1, i, j, k, l, il, jl, kl \rangle$$

with multiplication

$$(p_1 + p_2 l)(q_1 + q_2 l) = (p_1 q_1 - \bar{q}_2 p_2) + (q_2 p_1 + p_2 \bar{q}_1) l$$

and norm:

$$|p_1 + p_2 l|^2 = |p_1|^2 + |p_2|^2$$

These are the same formulas that allow us to pass from $\mathbb{C}$ to $\mathbb{H}$!

Some algebraic properties

- $|xy| = |x||y|$, $\forall x, y \in \mathbb{O}$.
- $\mathbb{O}$ is a division algebra, it is neither commutative nor associative!

  But it is *alternative*: any two elements generate an associative subalgebra.

  Theorem (Zorn 1933): The only finite-dimensional real alternative division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

  The only such associative algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (Frobenius 1877).
- $S^7 \simeq \{x \in \mathbb{O} : |x| = 1\}$ is not a group (associativity fails), but it constitutes the most important example of a *Moufang loop*.
- $\mathbb{O}_0 = \mathbb{R}\langle i, j, k, l, il, jl, kl\rangle$. $\forall u, v \in \mathbb{O}_0$:

  $$uv = -u \cdot v + u \times v.$$

  (Cross product in $\mathbb{R}^7$!: $(u \times v) \times v = (u \cdot v)v - (v \cdot v)u$.)
- $\mathbb{O}$ is *quadratic*: $\forall x = a1 + u \in \mathbb{O}$, $x^2 - 2ax + |x|^2 = 0$.

Some geometric properties

- The groups $Spin_7$ and $Spin_8$ (universal covers of $SO(7)$ and $SO(8)$) can be described easily in terms of octonions.
- $\mathbb{O}$ division algebra $\Rightarrow S^7$ parallelizable.
  S^1, S^3 and S^7 are the only parallelizable spheres (Milnor and Kervaire).
- $S^6 \simeq \{x \in \mathbb{O}_0 : |x| = 1\}$ is endowed with an *almost complex structure*, inherited from the multiplication of octonions.
  S^2 and S^6 are the only spheres with such structures (Adams).
- *Non-desarguesian projective plane* $\mathbb{OP}^2$.
- The only spheres that can be described as homogeneous spaces of nonclassical groups are $S^6 = \operatorname{Aut}\mathbb{O}/SU(3)$, $S^7 = Spin_7/\operatorname{Aut}\mathbb{O}$ and $S^{15} = Spin_9/Spin_7$.

$\mathbb{O}$ is certainly a beautiful mathematical object!

The saying that God is the mathematician, so that, even with meager experimental support, a mathematically beautiful theory will ultimately have a greater chance of being correct, has been attributed to Dirac. Octonion algebra may surely be called a beautiful mathematical entity.

It is possible that this and other non-associative algebras (other than Lie algebras) may play some essential future role in the ultimate theory, yet to be discovered.

Susumu Okubo

Thank you!