Gradings and S-structures on Lie algebras

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Outline

Gradings

Definition

Given a group G, a G-grading on a (finite-dimensional nonassociative) algebra $\mathcal{A}$ is a decomposition

$$\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g,$$

such that $\mathcal{A}_{g_1}\mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_1 g_2}$ for all $g_1, g_2 \in G$.

Noteworthy example: Cartan grading

$$\mathcal{L} = \mathcal{H} \oplus \Big(\bigoplus_{\alpha \in \Phi} \mathcal{L}_\alpha\Big)$$

(root space decomposition of a semisimple complex Lie algebra).

This is a grading by $\mathbb{Z}^n$, $n = \operatorname{rank} \mathcal{L}$.

Affine group schemes

Affine group scheme: Representable functor $\mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$.

$$\mathbf{G} \simeq \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{G}], .), \quad \mathbb{F}[\mathbf{G}] \text{ Hopf algebra.}$$

The generic element of $\mathbf{G}$ is

$$\mathrm{id}_{\mathbb{F}[\mathbf{G}]} \in \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}[\mathbf{G}], \mathbb{F}[\mathbf{G}]) \simeq \mathbf{G}(\mathbb{F}[\mathbf{G}]).$$

Example: general linear group

$$\mathbf{GL}_n : R \to \mathrm{GL}_n(R) := \{\text{invertible } n \times n\text{-matrices over } R\}$$

$$\mathbb{F}[\mathbf{GL}_n] = \mathbb{F}[X_{ij}, T]/(\det(X_{ij})T - 1) = \mathbb{F}[x_{ij}, t]$$

The generic element is the generic matrix (x_{ij}).

Gradings and diagonalizable group schemes

Given an abelian group G, any G-grading $\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ determines a homomorphism of affine group schemes

$$\eta_\Gamma : \mathbf{D}(G) \longrightarrow \mathbf{Aut}(\mathcal{A})$$

where $\mathbf{D}(G)$ is the diagonalizable group scheme represented by the group algebra $\mathbb{F}G$, with its natural structure of Hopf algebra defined by $\Delta(g) = g \otimes g$ for all $g \in G$.

For any $R \in \mathrm{Alg}_{\mathbb{F}}$,

$$\begin{aligned} (\eta_\Gamma)_R : \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, R) &\longrightarrow \mathrm{Aut}_R(\mathcal{A} \otimes_{\mathbb{F}} R) \\ f &\mapsto \Big(x \otimes r \mapsto x \otimes f(g) r\Big), \end{aligned}$$

for all $g \in G$, $x \in \mathcal{A}_g$, $r \in R$, and $f \in \mathrm{Hom}_{\mathrm{Alg}_{\mathbb{F}}}(\mathbb{F}G, R)$.

Gradings and diagonalizable group schemes

Conversely, a homomorphism

$$\eta : \mathbf{D}(G) \to \mathbf{Aut}(\mathcal{A})$$

defines a generic automorphism ρ of $\mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G$ (as an algebra over $\mathbb{F}G$), given by the image under η of the generic element of $\mathbf{D}(G)$.

Then $\mathcal{A}$ becomes G-graded: $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, with

$$\mathcal{A}_g = \{a \in \mathcal{A} \mid \rho(a \otimes 1) = a \otimes g\}.$$

(The set of *eigenvectors* in $\mathcal{A}$ with *eigenvalue* g for the generic automorphism.)

Vinberg considered a large extension of this idea in his paper

E.B. Vinberg,
Non-abelian gradings of Lie algebras.
50th Seminar "Sophus Lie", 19–38,
Banach Center Publ., **113**,
Polish Acad. Sci. Inst. Math., Warsaw, 2017.

He substituted the diagonalizable group schemes by arbitrary reductive algebraic groups.

$\mathbf{S}$-structures

Definition

Let $\mathbf{S}$ be a reductive algebraic group and let $\mathcal{L}$ be a Lie algebra.
An $\mathbf{S}$-structure on $\mathcal{L}$ is a homomorphism

$$\Psi : \mathbf{S} \to \mathbf{Aut}(\mathcal{L}).$$

This definition is too general, so some restrictions on $\mathcal{L}$ as a module for the reductive group $\mathbf{S}$ must be imposed.

S-structures

Vinberg himself considered in his paper two different situations:

- A nontrivial $\mathbf{SL}_2$-structure on a Lie algebra $\mathcal{L}$ is called very short if $\mathcal{L}$ decomposes, as a module for $\mathbf{SL}_2$, as a sum of copies of the adjoint module and of the trivial one-dimensional module.

  Collecting isomorphic submodules, a very short $\mathbf{SL}_2$-structure gives an isotypic decomposition of the form

  $$\mathcal{L} = (\mathfrak{sl}_2 \otimes \mathcal{J}) \oplus \mathcal{D}$$

  and it turns out that the Lie bracket on $\mathcal{L}$ induces a Jordan product on $\mathcal{J}$. The subalgebra $\mathcal{D}$ acts by derivations on $\mathcal{J}$. (All this goes back to Tits.)

S-structures

- A nontrivial $\mathbf{SL}_3$-structure on a Lie algebra $\mathcal{L}$ is called short if $\mathcal{L}$ decomposes as the direct sum of one copy of the adjoint representation, copies of its natural three-dimensional module and of its dual, and copies of the trivial representation, so that the isotypic decomposition is:

$$\mathcal{L} = \mathfrak{sl}_3 \oplus (V \otimes \mathcal{J}) \oplus (V^* \otimes \mathcal{J}') \oplus \mathcal{D}.$$

For simple $\mathcal{L}$, the subspaces $\mathcal{J}$ and $\mathcal{J}'$ inherit a structure of a cubic Jordan algebra. (A more general situation was considered by Benkart and E.)

Gradings by root systems

Some interesting $\mathbf{S}$-structures have already appeared in the literature.

Definition (Berman-Moody)

A Lie algebra $\mathcal{L}$ is **graded by the reduced root system** Φ if:

1. $\mathcal{L}$ contains a finite-dimensional simple Lie algebra $\mathfrak{s} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{s}_\alpha\right)$, whose root system is Φ relative to the Cartan subalgebra $\mathfrak{h} = \mathfrak{s}_0$,
2. $\mathcal{L} = \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathcal{L}(\alpha)$, where $\mathcal{L}(\alpha) = \{X \in \mathcal{L} \mid [H, X] = \alpha(H)X \ \forall H \in \mathfrak{h}\}$, and
3. $\mathcal{L}(0) = \sum_{\alpha\in\Phi}[\mathcal{L}(\alpha), \mathcal{L}(-\alpha)]$.

The subalgebra $\mathfrak{s}$ is said to be a grading subalgebra of $\mathcal{L}$.

Simply laced case (Berman-Moody)

In the simply laced case, with the exception of type A_1, $\mathfrak{s}$ is a simple Lie algebra of type A, D, or E, and $\mathcal{L}$ decomposes, as a module for its grading subalgebra $\mathfrak{s}$, as a direct sum of copies of the adjoint module and of the trivial one-dimensional module.

Hence we get an isotypic decomposition:

$$\mathcal{L} = (\mathfrak{s} \otimes \mathcal{A}) \oplus \mathcal{D}.$$

The Lie bracket in $\mathcal{L}$ provides a multiplication on $\mathcal{A}$, which is commutative and associative for types D and E, associative for A_r $(r \geq 3)$, and alternative for A_2.

If $\mathbf{S}$ is the simply connected group with Lie algebra $\mathfrak{s}$, the action of $\mathfrak{s}$ integrates to an $\mathbf{S}$-structure on $\mathcal{L}$.

Non simply laced case (Benkart-Zelmanov)

The remaining cases, with Φ reduced, were described by Benkart and Zelmanov, inspired by Tits construction of the Lie algebras in Freudenthal's Magic Square by means of composition algebras and cubic Jordan algebras.

In the non simply-laced case (types B, C, F_4, and G_2), the isotypic decomposition also includes copies of the irreducible module for $\mathfrak{s}$ whose highest weight is the highest short root:

$$\mathcal{L} = (\mathfrak{s} \otimes \mathcal{A}) \oplus (W \otimes \mathcal{B}) \oplus \mathcal{D}.$$

The sum $\mathfrak{a} = \mathcal{A} \oplus \mathcal{B}$ is a unital algebra (the coordinate algebra) on which $\mathcal{D}$ acts by derivations, and its type depends on Φ.

Nonreduced root systems

Finally, the case of the nonreduced root systems BC_r (Allison, Benkart, Gao, Smirnov) gives isotypic decompositions with four components, with two exceptions of five components:

- BC_1-graded Lie algebras with grading subalgebra of type D_1 (which reduces to 5-gradings), and
- BC_2-graded Lie algebras with grading subalgebra of type $D_2 = A_1 \times A_1$.

Short $\mathbf{SL}_2$-structures

Definition

An $\mathbf{SL}_2$-structure $\Psi : \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$ on a Lie algebra $\mathcal{L}$ is said to be short if $\mathcal{L}$ decomposes, as a module for $\mathbf{SL}_2$ via Ψ, into a direct sum of copies of the adjoint, natural, and trivial modules.

Therefore, we get an isotypic decomposition:

$$\mathcal{L} = \big(\mathfrak{sl}(V) \otimes \mathcal{J}\big) \oplus \big(V \otimes \mathcal{T}\big) \oplus \mathcal{D},$$

for vector spaces $\mathcal{J}$, $\mathcal{T}$, and $\mathcal{D}$, where V is the natural two-dimensional representation of $\mathbf{SL}_2 \simeq \mathbf{SL}(V)$. The action of $\mathbf{SL}_2$ is given by the adjoint action of $\mathbf{SL}_2$ on $\mathfrak{sl}(V)$, its natural action on V, and the trivial action on $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{D}$.

The subspace $\mathcal{D}$, being the subspace of fixed elements by $\mathbf{SL}_2$, is a subalgebra of $\mathcal{L}$.

Isotypic decomposition

$$\mathcal{L} = \big(\mathfrak{sl}(V) \otimes \mathcal{J}\big) \oplus \big(V \otimes \mathcal{T}\big) \oplus \mathcal{D},$$

The $\mathfrak{sl}(V)$-invariance of the Lie bracket in our Lie algebra $\mathcal{L}$ gives, for any $f, g \in \mathfrak{sl}(V)$, $u, v \in V$ and $D \in \mathcal{D}$, the following conditions:

$$\begin{aligned}
[f \otimes a, g \otimes b] &= [f, g] \otimes a \cdot b + 2\,\mathrm{tr}(fg) D_{a,b},\\
[f \otimes a, u \otimes x] &= f(u) \otimes a \bullet x,\\
[u \otimes x, v \otimes y] &= \gamma_{u,v} \otimes \langle x \mid y \rangle + (u \mid v) d_{x,y},\\
[D, f \otimes a] &= f \otimes D(a),\\
[D, u \otimes x] &= u \otimes D(x),
\end{aligned}$$

for suitable bilinear maps.

J-ternary algebras of Allison

Definition

Let $\mathcal{J}$ be a unital Jordan algebra with multiplication $a \cdot b$ for $a, b \in \mathcal{J}$. Let $\mathcal{T}$ be a unital special Jordan module for $\mathcal{J}$ with action $a \bullet x$ for $a \in \mathcal{J}$ and $x \in \mathcal{T}$. Assume $\langle . | . \rangle : \mathcal{T} \times \mathcal{T} \to \mathcal{J}$ is a skew-symmetric bilinear map and $(.,.,.) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a trilinear product on $\mathcal{T}$. Then the pair $(\mathcal{J}, \mathcal{T})$ is called a **J-ternary algebra** if the following axioms hold for any $a \in \mathcal{J}$ and $x, y, z, w, v \in \mathcal{T}$:

(JT1) $a \cdot \langle x|y \rangle = \frac{1}{2}(\langle a \bullet x|y \rangle + \langle x|a \bullet y \rangle)$,

(JT2) $a \bullet (x, y, z) = (a \bullet x, y, z) - (x, a \bullet y, z) + (x, y, a \bullet z)$,

(JT3) $(x, y, z) = (z, y, x) - \langle x|z \rangle \bullet y$,

(JT4) $(x, y, z) = (y, x, z) + \langle x|y \rangle \bullet z$,

(JT5) $\langle (x, y, z)|w \rangle + \langle z|(x, y, w) \rangle = \langle x|\langle z|w \rangle \bullet y \rangle$,

(JT6) $(x, y, (z, w, v)) =$
$$((x, y, z), w, v) + (z, (y, x, w), v) + (z, w, (x, y, v)).$$

J-ternary algebras: prototypical example

Let $(\mathcal{A}, *)$ be an associative algebra with involution, and let $\mathcal{T}$ be a left $\mathcal{A}$-module endowed with a skew-hermitian form $\mathsf{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$:

- h is $\mathbb{F}$-bilinear,
- $\mathsf{h}(ax, y) = a\mathsf{h}(x, y)$
- $\mathsf{h}(x, y) = -\mathsf{h}(y, x)^*$

for any $a \in \mathcal{A}$ and $x, y \in \mathcal{T}$.

Then $\Big(\mathcal{J} = \mathcal{H}(\mathcal{A}, *), \mathcal{T}\Big)$ is a J-ternary algebra with $a \bullet x = ax$, and

- $\langle x \mid y \rangle = \mathsf{h}(x, y) - \mathsf{h}(y, x)$,
- $(x, y, z) = \mathsf{h}(x, y)z + \mathsf{h}(z, x)y + \mathsf{h}(z, y)x$,

for all $a \in \mathcal{J}$ and $x, y \in \mathcal{T}$.

J-ternary algebras and structurable algebras

Let $(\mathcal{A}, \text{-})$ be a structurable algebra, and let s be a skew-symmetric element ($\bar{s} = -s$) such that the left multiplication L_s is bijective. Let $\mathcal{S}$ be the subspace of skew-symmetric elements.

Then the pair $(\mathcal{S}, \mathcal{A})$ is a J-ternary algebra with the following operations:

- $a \cdot b = \frac{1}{2}\big(a(sb) + b(sa)\big), \quad a \bullet x = a(sx)$,
- $\langle x \mid y \rangle = x\bar{y} - y\bar{x}$,
- $(x, y, z) = -V_{x,sy}(z)$,

for $a, b \in \mathcal{S}$ and $x, y, z \in \mathcal{A}$.

Short $\mathbf{SL}_2$-structures and J-ternary algebras

Theorem

Let $\mathcal{L}$ be a Lie algebra endowed with an inner short $\mathbf{SL}_2$-structure with isotypic decomposition

$$\mathcal{L} = \big(\mathfrak{sl}(V) \otimes \mathcal{J}\big) \oplus \big(V \otimes \mathcal{T}\big) \oplus \mathcal{D}.$$

Then the pair $(\mathcal{J}, \mathcal{T})$ is a J-ternary algebra with the triple product on $\mathcal{T}$ given by

$$(x, y, z) = \frac{1}{2}\Big(-d_{x,y}(z) + \langle x|y\rangle \bullet z\Big).$$

for all $x, y, z \in \mathcal{T}$.

Theorem (continued)

Conversely, if $(\mathcal{J}, \mathcal{T})$ is a J-ternary algebra, then the vector space

$$\mathcal{L}(\mathcal{J}, \mathcal{T}) := (\mathfrak{sl}(V) \otimes \mathcal{J}) \oplus (V \otimes \mathcal{T}) \oplus \mathcal{D},$$

is a Lie algebra with a short $\mathbf{SL}_2$-structure, where $\mathcal{D}$ is the subalgebra of transformations of $\mathcal{J} \oplus \mathcal{T}$ spanned by the following maps:

$$\begin{aligned}
D_{a,b}(c) &= a \cdot (b \cdot c) - b \cdot (a \cdot c), \\
D_{a,b}(x) &= \frac{1}{4}(a \bullet (b \bullet x) - b \bullet (a \bullet x)), \\
d_{x,y}(a) &= \langle a \bullet x | y \rangle - \langle x | a \bullet y \rangle, \\
d_{x,y}(z) &= \langle x | y \rangle \bullet z - 2(x, y, z),
\end{aligned}$$

for $a, b, c \in \mathcal{J}$ and $x, y, z \in \mathcal{T}$.

Short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures

Definition

An $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structure $\Psi : \mathbf{SL}_2 \times \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$ on a Lie algebra $\mathcal{L}$ is said to be short if $\mathcal{L}$ decomposes, as a module for $\mathbf{SL}_2 \times \mathbf{SL}_2$ via Ψ, into a direct sum of copies of the following modules:

- the adjoint module for each of the two copies of $\mathbf{SL}_2$,
- the natural two-dimensional modules V_1 and V_0 for each of the two copies of $\mathbf{SL}_2$,
- the tensor product $V_1 \otimes V_0$, and
- the trivial one-dimensional module.

Isotypic decomposition

$$\mathcal{L} = \big(\mathfrak{sl}(V_1) \otimes \mathcal{J}_1\big) \oplus \big(\mathfrak{sl}(V_0) \otimes \mathcal{J}_0\big) \oplus \big((V_1 \otimes V_0) \otimes \mathcal{J}_{\frac{1}{2}}\big) \\ \oplus \big(V_1 \otimes \mathcal{T}_1\big) \oplus \big(V_0 \otimes \mathcal{T}_0\big) \oplus \mathcal{S}.$$

It is not a good idea to try to expand naively the Lie bracket on $\mathcal{L}$ to deduce the associated bilinear maps among the components $\mathcal{J}_1$, $\mathcal{J}_0$, $\mathcal{J}_{\frac{1}{2}}$, $\mathcal{T}_1$, $\mathcal{T}_0$, and $\mathcal{S}$.

Trick!

Consider the diagonal subgroup of $\mathbf{SL}_2 \times \mathbf{SL}_2$. The composition

$$\mathbf{SL}_2 \xrightarrow{\Delta} \mathbf{SL}_2 \times \mathbf{SL}_2 \xrightarrow{\Psi} \mathbf{Aut}(\mathcal{L}),$$

where Δ is the diagonal embedding $g \mapsto (g, g)$, gives an $\mathbf{SL}_2$-structure on $\mathcal{L}$.

This structure is short!!

Short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures and J-ternary algebras

Theorem

Let $\mathcal{L}$ be a Lie algebra endowed with an inner short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structure $\Psi : \mathbf{SL}_2 \times \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$. Let $(\mathcal{J}, \mathcal{T})$ be the J-ternary algebra that coordinatizes the short $\mathbf{SL}_2$-structure $\Psi \circ \Delta$, with isotypic decomposition:

$$\mathcal{L} = \big(\mathfrak{sl}(V) \otimes \mathcal{J}\big) \oplus \big(V \otimes \mathcal{T}\big) \oplus \mathcal{D},$$

where V is the two-dimensional natural module for $\mathbf{SL}_2$.
Then the unital Jordan algebra $\mathcal{J}$ contains an idempotent $e = e^{\cdot 2} \neq 0, 1$ such that the subalgebra

$$\big(\mathfrak{sl}(V) \otimes e\big) \oplus \big(\mathfrak{sl}(V) \otimes (1 - e)\big).$$

is the (image of) the Lie algebra of $\mathbf{SL}_2 \times \mathbf{SL}_2$.

Short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures and J-ternary algebras

Theorem (continued)

Conversely, if $\mathcal{L}$ is a Lie algebra endowed with an inner short $\mathbf{SL}_2$-structure and coordinate J-ternary algebra $(\mathcal{J}, \mathcal{T})$ such that the unital Jordan algebra $\mathcal{J}$ contains a nontrivial idempotent e, then $\mathcal{L}$ is endowed with a short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structure whose isotypic decomposition is the following:

$$\mathcal{L} = (\mathfrak{sl}(V) \otimes \mathcal{J}_1) \oplus (\mathfrak{sl}(V) \otimes \mathcal{J}_0) \oplus \Big((\mathfrak{sl}(V) \otimes \mathcal{J}_{\frac{1}{2}}) \oplus D_{e,\mathcal{J}_{\frac{1}{2}}} \Big) \oplus (V \otimes \mathcal{T}_1) \oplus (V \otimes \mathcal{T}_0) \oplus \mathcal{S},$$

where $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_{\frac{1}{2}} \oplus \mathcal{J}_0$ is the Peirce decomposition of $\mathcal{J}$ relative to the idempotent e, and $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_0$ is the induced decomposition on $\mathcal{T}$

Short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures on simple Lie algebras

Classical: The short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures on the classical simple Lie algebras are coordinatized by 'prototypical' J-ternary algebras.

Exceptional: The short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures on the exceptional simple Lie algebras are coordinatized by the J-ternary algebras obtained from the structurable algebras $\mathcal{C}_1 \otimes \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are unital composition algebras and $\dim_{\mathbb{F}} \mathcal{C}_1 = 8$.

References

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Thank you!