From the Albert algebra to Kac's ten-dimensional Jordan superalgebra

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On the occasion of
Saïd Benayadi's 60th birthday

A bizarre result

In 2005, Kevin McCrimmon considered the Grassmann envelope of Kac's ten-dimensional simple Jordan superalgebra K_{10} and obtained, in his own words, *the bizarre result that in characteristic 5 (but not otherwise), it is the Jordan algebra over a shaped cubic form over Γ_0*. This means that K_{10} satisfies the super version of the Cayley-Hamilton equation of degree 3.

This bizarre result led to the discovery of a new exceptional simple Lie superalgebra: $\mathfrak{el}(5;5)$, specific of characteristic 5.

It turns out that this bizarre result is a direct consequence of the fact that, in characteristic 5, K_{10} can be obtained from the *Albert algebra* using a semisimplification process in suitable tensor categories.

Outline

From the Albert algebra to Kac's ten-dimensional Jordan superalgebra

1. Symmetric tensor categories
2. Semisimplification
3. From algebras to superalgebras
4. From the Albert algebra to K_{10}

Monoidal categories

A monoidal category is a category $\mathfrak{C}$ with a bifunctor

$$\otimes\colon \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$$

such that:

- There is a unit object $\mathbf{1}$ with natural isomorphisms (unitors)

$$X \otimes \mathbf{1} \simeq X \simeq \mathbf{1} \otimes X.$$

- There are natural isomorphisms (associators)

$$a_{X,Y,Z}\colon (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z).$$

- Natural coherence conditions for the unitors and associators hold.

Monoidal functors

A functor $F\colon \mathfrak{C} \to \mathfrak{D}$ between monoidal categories is a monoidal functor if $F(\mathbf{1}) \simeq \mathbf{1}$ and there are natural isomorphisms

$$J_{X,Y}\colon F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$$

with natural coherence conditions with associators.

Symmetric monoidal categories

A braiding in a monoidal category $\mathfrak{C}$ is a natural isomorphism

$$c_{X,Y} \colon X \otimes Y \longrightarrow Y \otimes X$$

satisfying natural compatibility conditions with unitors and associators.

A symmetric monoidal category is a monoidal category endowed with a symmetric braiding: $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$.

Rigid symmetric monoidal categories

A symmetric monoidal category is rigid if every object X has a dual object X^* with

- an evaluation $\mathrm{ev}_X\colon X^* \otimes X \to \mathbf{1}$,
- a coevaluation $\mathrm{coev}_X\colon \mathbf{1} \to X \otimes X^*$,

such that the following compositions are the identity morphisms:

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$
$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

(Unitor and associator morphisms are omitted.)

Symmetric tensor categories

A symmetric tensor category $\mathfrak{C}$ over a field $\mathbb{F}$ is a rigid symmetric monoidal category with the following extra properties:

- It is abelian and even more: it is $\mathbb{F}$-linear and $\otimes$ is 'bilinear'.
- It is locally finite: objects have 'finite length' and morphism spaces are finite-dimensional.
- $\mathrm{End}_{\mathfrak{C}}(\mathbf{1}) = \mathbb{F}\,\mathrm{id}_{\mathbf{1}}$.

Examples

$\mathsf{Vec}_{\mathbb{F}}$: The category of finite-dimensional vector spaces.

$\mathsf{Rep}H$: The category of finite-dimensional representations of a triangular Hopf algebra.

$\mathsf{Rep}\,G$: The category of finite-dimensional representations of an affine group scheme.

$\mathsf{sVec}_{\mathbb{F}}$: The category of finite-dimensional vector superspaces.

Algebras in a symmetric tensor category

An algebra in a symmetric tensor category $\mathfrak{C}$ is an object $\mathcal{A}$ endowed with a morphism $\mu\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$.

The algebra $(\mathcal{A},\mu)$ is

- commutative if $\mu\circ c_{\mathcal{A},\mathcal{A}}=\mu$,
- associative if $\mu\circ(\mu\otimes\mathrm{id}_{\mathcal{A}})=\mu\circ(\mathrm{id}_{\mathcal{A}}\otimes\mu)$ (associator morphisms are omitted),
- Lie if it is anticommutative: $\mu\circ c_{\mathcal{A},\mathcal{A}}=-\mu$, and

$$\mu\circ(\mu\otimes\mathrm{id}_{\mathcal{A}})\circ(\mathrm{id}_{\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}}+c_{\mathcal{A}\otimes\mathcal{A},\mathcal{A}}+c_{\mathcal{A},\mathcal{A}\otimes\mathcal{A}})=0,$$

- Jordan if ....
- ....

Superalgebras are algebras in $\mathrm{sVec}_{\mathbb{F}}$.

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Traces in symmetric tensor categories

Given a morphism $f \in \mathrm{End}_{\mathfrak{C}}(X)$ in a symmetric tensor category, its trace $\mathrm{tr}_X(f)$ is the following element in $\mathrm{End}_{\mathfrak{C}}(\mathbf{1}) \simeq \mathbb{F}$:

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}_{X^*}} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{\mathrm{ev}_\mathrm{X}} \mathbf{1}$$

The dimension of an object X is $\dim_{\mathfrak{C}}(X) := \mathrm{tr}_X(\mathrm{id}_X)$.

Negligible morphisms

A morphism $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ in a symmetric tensor category is said to be negligible if

$$\mathrm{tr}_Y(f \circ g) = 0 \quad \text{for all} \quad g \in \mathrm{Hom}_{\mathfrak{C}}(Y, X).$$

Denote by $\mathcal{N}(X, Y)$ be the subspace of negligible morphims in $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$.

The subspaces $\mathcal{N}(X, Y)$ form a tensor ideal.

Semisimplification of a symmetric tensor category

This means that we can define a new category $\mathfrak{C}^{ss}$ with the same objects as $\mathfrak{C}$, but with morphisms given by the quotient with the subspace of negligible morphisms:

$$\mathrm{Hom}_{\mathfrak{C}^{ss}}(X,Y) := \mathrm{Hom}_{\mathfrak{C}}(X,Y)/\mathcal{N}(X,Y).$$

$\mathfrak{C}^{ss}$ is called the semisimplification of $\mathfrak{C}$.

The natural functor $S\colon \mathfrak{C} \to \mathfrak{C}^{ss}$ which is the identity on objects, and sends any morphism f to its class $[f]$ modulo negligible morphisms is a braided, monoidal, $\mathbb{F}$-linear functor.

Semisimplification

The semisimplification $\mathfrak{C}^{ss}$ is semisimple: any object is a direct sum of finitely many simple objects.

The simple objects in $\mathfrak{C}^{ss}$ correspond to the indecomposable objects in $\mathfrak{C}$ of nonzero dimension.

Any indecomposable object in $\mathfrak{C}$ with $\dim_{\mathfrak{C}}(X) = 0$ becomes isomorphic to the zero object in $\mathfrak{C}^{ss}$.

Verlinde category

Definition

Let $\mathbb{F}$ be a field of characteristic $p > 0$ and let $\mathsf{Rep}\,\mathsf{C}_p$ be the category of finite-dimensional representations of the cyclic group of order p (or of the associated constant group scheme).
This is a symmetric tensor category and its semisimplification is called the Verlinde category Ver_p.

The Verlinde category Ver_p also appears as the semisimplification of

$$\mathsf{Rep}\,\boldsymbol{\alpha}_p \cong \mathsf{Rep}\,\mathbb{F}[t]/(t^p).$$

An algebra in $\mathsf{Rep}\,\boldsymbol{\alpha}_p$ is just an algebra with a nilpotent derivation d such that $d^p = 0$.

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Semisimplification of $\operatorname{Rep} \mathsf{C}_p$

$\operatorname{char} \mathbb{F} = p$. Fix a generator σ of C_p.

The indecomposable objects in $\operatorname{Rep} \mathsf{C}_p$ are, up to isomorphism, the modules

$$L_i = \operatorname{span}\{v_0, \ldots, v_{i-1}\}$$

for $i = 1, \ldots, p$, with

$$\sigma(v_j) = v_j + v_{j+1},\ j = 0, \ldots, i-2, \quad \sigma(v_{i-1}) = v_{i-1}.$$

Any object $\mathcal{A}$ in $\operatorname{Rep} \mathsf{C}_p$ decomposes, nonuniquely, as

$$\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_p,$$

where $\mathcal{A}_i$ is a direct sum of copies of L_i, $i = 1, 2, \ldots, p$.

Semisimplification of $\mathsf{Rep}\,\mathsf{C}_p$. Properties

- $L_1, \ldots, L_{p-1}$ are simple objects in Ver_p, while L_p is isomorphic to 0.
- Ver_p is semisimple: any object is isomorphic to a direct sum of copies of $L_1, \ldots, L_{p-1}$.
- $\mathrm{End}_{\mathsf{Ver}_p}(L_i) = \mathbb{F}[\mathrm{id}_{L_i}] \neq 0$ for $i = 1, \ldots, p-1$, $\mathrm{End}_{\mathsf{Ver}_p}(L_p) = 0$, and $\mathrm{Hom}_{\mathsf{Ver}_p}(L_i, L_j) = 0$ for $1 \leq i \neq j \leq p-1$.
- $L_1 \otimes L_i$ and $L_i \otimes L_1$ are isomorphic to L_i, for $i = 1, \ldots, p$, both in $\mathsf{Rep}\,\mathsf{C}_p$ and in Ver_p.
- $L_{p-1} \otimes L_{p-1}$ is isomorphic to L_1 in Ver_p.

Ver_p and $\mathsf{sVec}_{\mathbb{F}}$

The $\mathbb{F}$-linear functor

$$\begin{aligned} F\colon \mathsf{sVec}_{\mathbb{F}} &\longrightarrow \mathsf{Ver}_p \\ X_{\bar 0} \oplus X_{\bar 1} &\mapsto X_{\bar 0} \oplus (X_{\bar 1} \otimes L_{p-1}) \\ f_{\bar 0} \oplus f_{\bar 1} &\mapsto [f_{\bar 0} \oplus (f_{\bar 1} \otimes \mathrm{id}_{L_{p-1}})], \end{aligned}$$

provides an equivalence of symmetric tensor categories between $\mathsf{sVec}_{\mathbb{F}}$ and the full tensor subcategory of Ver_p generated by L_1 and L_{p-1}.

This subcategory is the whole Ver_p if $p = 3$.

From algebras in $\mathsf{Rep}\,\mathsf{C}_p$ to superalgebras

If $(\mathcal{A}, \mu)$ is an algebra in $\mathsf{Rep}\,\mathsf{C}_p$ (i.e., **an algebra endowed with an automorphism of order p**), then $(\mathcal{A}, [\mu])$ is an algebra in Ver_p.

Fix a decomposition $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_p$, where $\mathcal{A}_i$ is a direct sum of copies of L_i for $i = 1, \ldots, p$.

$\mathcal{A}' := \mathcal{A}_1 \oplus \mathcal{A}_{p-1}$ is endowed with a natural multiplication μ' inherited from μ such that

$(\mathcal{A}', [\mu'])$ is a subalgebra of $(\mathcal{A}, [\mu])$ in Ver_p, that lies in the subcategory "$\mathsf{sVec}_{\mathbb{F}}$".

From algebras in Rep C_p to superalgebras

Write $A_{\bar{0}} = \mathcal{A}_1$, and fix a subspace $A_{\bar{1}}$ of $\mathcal{A}_{p-1}$ such that $\mathcal{A}_{p-1} = A_{\bar{1}} \oplus (\sigma - 1)(\mathcal{A}_{p-1})$.

Then, $A := A_{\bar{0}} \oplus A_{\bar{1}}$ is an object in $\mathsf{sVec}_{\mathbb{F}}$, and the image of the morphism in Rep C_p:

$$
\begin{aligned}
\iota_{\mathcal{A}} \colon F(A) = A_{\bar{0}} \oplus (A_{\bar{1}} \otimes L_{p-1}) &\longrightarrow \mathcal{A} \\
a_{\bar{0}} &\mapsto a_{\bar{0}} \in \mathcal{A}_1, \\
a_{\bar{1}} \otimes v_i &\mapsto (\sigma - 1)^i (a_{\bar{1}}) \in \mathcal{A}_{p-1}
\end{aligned}
$$

is $\mathcal{A}' := \mathcal{A}_1 \oplus \mathcal{A}_{p-1}$.

Through the equivalence F, the multiplication in the "superalgebra" $(\mathcal{A}', [\mu'])$ induces a multiplication in the vector superspace $A = A_{\bar{0}} \oplus A_{\bar{1}}$.

What this multiplication looks like?

From algebras in Rep C_p to superalgebras. Recipe

Recall our decomposition

$$\mathcal{A} = A_{\bar{0}} \oplus \mathcal{A}_2 \oplus \cdots \mathcal{A}_{p-2} \oplus A_{\bar{1}} \oplus (\sigma - 1)(\mathcal{A}_{p-1}) \oplus \mathcal{A}_p.$$

Recipe

Take projections relative to the decomposition above, and define a multiplication m ($m(x \otimes y) := x \diamond y$) on $A := A_{\bar{0}} \oplus A_{\bar{1}}$ as follows:

$$x_{\bar{0}} \diamond y_{\bar{0}} = \operatorname{proj}_{A_{\bar{0}}} \mu(x_{\bar{0}} \otimes y_{\bar{0}})$$
$$x_{\bar{0}} \diamond y_{\bar{1}} = \operatorname{proj}_{A_{\bar{1}}} \mu(x_{\bar{0}} \otimes y_{\bar{1}})$$
$$x_{\bar{1}} \diamond y_{\bar{0}} = \operatorname{proj}_{A_{\bar{1}}} \mu(x_{\bar{1}} \otimes y_{\bar{0}})$$
$$x_{\bar{1}} \diamond y_{\bar{1}} = \operatorname{proj}_{A_{\bar{0}}} \mu(x_{\bar{1}} \otimes (\sigma - 1)^{p-2}(y_{\bar{1}}))$$

for all $x_{\bar{0}}, y_{\bar{0}} \in A_{\bar{0}}$ and $x_{\bar{1}}, y_{\bar{1}} \in A_{\bar{1}}$.

The algebra (A, m) is an algebra in $\mathsf{sVec}_{\mathbb{F}}$ (a superalgebra).

From algebras in $\operatorname{Rep} \mathsf{C}_p$ to superalgebras. Recipe

Summarizing:

- $(\mathcal{A}, \mu)$ is an algebra in $\operatorname{Rep} \mathsf{C}_p$.
- Split it suitably:
  $\mathcal{A} = A_{\bar{0}} \oplus \mathcal{A}_2 \oplus \cdots \mathcal{A}_{p-2} \oplus A_{\bar{1}} \oplus (\sigma - 1)(\mathcal{A}_{p-1}) \oplus \mathcal{A}_p$.
- $(\mathcal{A}, [\mu])$ contains a nice subalgebra $(\mathcal{A}', [\mu'])$.
- Use our recipe to define a multiplication m on $A = A_{\bar{0}} \oplus A_{\bar{1}}$, which becomes a superalgebra.
- Recall our functor $F\colon \mathsf{sVec}_{\mathbb{F}} \to \mathsf{Ver}_p$.

Theorem

(A, m) is the superalgebra that corresponds to $(\mathcal{A}', [\mu'])$. That is, the algebra in Ver_p obtained by means of

$$F(A) \otimes F(A) \xrightarrow{\cong} F(A \otimes A) \xrightarrow{F(m)} F(A)$$

is isomorphic (through $\iota_{\mathcal{A}}$) to the subalgebra $(\mathcal{A}', [\mu'])$.

From algebras in $\operatorname{Rep} C_p$ to superalgebras

In case $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_{p-1} \oplus \mathcal{A}_p$, this algebra in Ver_p is isomorphic to $(\mathcal{A}, [\mu])$. In this situation, we will say that the algebra $\mathcal{A}$ semisimplifies to the superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, or that A is obtained by semisimplification of $\mathcal{A}$.

From the Albert algebra to Kac's ten-dimensional Jordan superalgebra

From the Albert algebra to K_{10}

A. Elduque, P. Etingof, A.S. Kannan,
From the Albert algebra to Kac's ten-dimensional Jordan superalgebra via tensor categories in characteristic 5.
J. Algebra **666** (2025), 387-414.

The Albert algebra is the algebra of 3×3 Hermitian matrices over the (split) Cayley algebra: $\mathbb{A} = H_3(\mathcal{C})$, with multiplication given by

$$X \cdot Y = \frac{1}{2}(XY + YX).$$

The Albert algebra

$$\mathbb{A} = \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathcal{C}) \oplus \iota_2(\mathcal{C}) \oplus \iota_3(\mathcal{C}),$$

with

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_1(a) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \quad \iota_2(a) = 2\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \quad \iota_3(a) = 2\begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Albert algebra

The group of automorphisms $\mathrm{Aut}(\mathbb{A})$ contains the subgroup

$$\{\varphi \in \mathrm{Aut}(\mathbb{A}) \mid \varphi(E_i) = E_i \ \forall i\} \simeq \mathrm{Spin}(\mathcal{C}).$$

Lemma

Over a field of characteristic 5, $\mathbb{A}$ is endowed with an order 5 automorphism in $\mathrm{Spin}(\mathcal{C})$, such that

- $\iota_1(\mathcal{C})$ *splits as* $3L_1 \oplus L_5$,
- $\iota_i(\mathcal{C})$ *splits as* $2L_4$ *for* $i = 2, 3$,

and hence $\mathbb{A}$ splits as

$$\mathbb{A} = 6L_1 \oplus 4L_4 \oplus L_5.$$

From the Albert algebra to Kac's ten-dimensional Jordan superalgebra

Theorem

In characteristic 5, the Albert algebra $\mathbb{A}$ semisimplifies to Kac's ten-dimensional simple Jordan superalgebra K_{10}.

The semisimplification process explains the bizarre property.

From E_8 to $\mathfrak{el}(5;5)$

The exceptional split simple Lie algebra of type E_8 can be obtained, using a famous construction by Tits, as

$$\mathfrak{Der}(\mathcal{C}) \oplus (\mathcal{C}_0 \otimes \mathbb{A}_0) \otimes \mathfrak{Der}(\mathbb{A}),$$

using the Cayley algebra $\mathcal{C}$ and the Albert algebra $\mathbb{A}$.

The order 5 automorphism of $\mathbb{A}$ extends to an automorphism of E_8, and the outcome is that E_8 semisimplifies to the exceptional Lie superalgebra $\mathfrak{el}(5;5)$, specific of characteristic 5.

Thank you!