A modified signed Brauer algebra as centralizer algebra of the unitary group

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Motivation

A. Gray and L.M. Hervella: "The Sixteen Classes of Almost Hermitian Manifolds", Ann. Mat. Pura Appl. 1980:

(M, g, J) almost Hermitian manifold, $\dim M = 2n$

Riemannian metrig g, connection ∇,

almost complex structure J,

Kähler form $F(X, Y) = g(JX, Y)$.

The tensor $\alpha = \nabla F$ satisfies $\forall X, Y, Z \in \chi(M)$:

$$\alpha(X, Y, Z) = -\alpha(X, Z, Y) = -\alpha(X, JY, JZ)$$

so for any point $m \in M$, $(\nabla F)_m$ belongs to

$$W_m = \{\alpha \in M_m^* \otimes M_m^* \otimes M_m^* : \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, Jy, Jz)\}.$$

Decomposition of W_m into irreducible modules under the action of $U(n)$?

- **Problem:**

  V **complex vector space,** $\dim_{\mathbb{C}} V = n$,

  $h : V \times V \to \mathbb{C}$ **hermitian inner product,**

Decompose $\bigotimes_{\mathbb{R}}^{r} V$ **into irreducible** $U(V,h)$**-modules.**

- **Associated problem:**

$$\mathrm{End}_{U(V,h)}(\otimes_{\mathbb{R}}^{r} V) = ?$$

  (centralizer algebra)

Remark:

$$\mathbb{C} \otimes_{\mathbb{R}} V \cong V \oplus V^*$$

$$1 \otimes v \mapsto \big(v, h(-, v)\big)$$

Therefore, by extending scalars,

$$\mathbb{C} \otimes_{\mathbb{R}} (\otimes_{\mathbb{R}}^{r} V) = \bigoplus_{p=0}^{r} \binom{r}{p} (\otimes_{\mathbb{C}}^{p} V) \otimes_{\mathbb{C}} (\otimes_{\mathbb{C}}^{r-p} V^*)$$

Can apply Benkart et al.: "Tensor Product Representations of General Linear Groups and Their Connections with Brauer Algebras", J. Algebra 1994.

$$\mathbf{End}_{U(V,h)}(\otimes_{\mathbb{R}}^{r} V) = ?$$

1. **Invariants:**

$$f : V \times \overset{r}{\cdots} \times V \to \mathbb{R}$$

multilinear and $U(V,h)$-invariant.

2. **Centralizer algebra:**

$$\begin{aligned} \mathbf{End}_{U(V,h)}(\otimes_{\mathbb{R}}^{r} V) &\cong \Big((\otimes_{\mathbb{R}}^{r} V) \otimes_{\mathbb{R}} (\otimes_{\mathbb{R}}^{r} V_{\mathbb{R}}^{*}) \Big)^{U(V,h)} \\ &\cong \Big(\otimes_{\mathbb{R}}^{2r} V_{\mathbb{R}}^{*} \Big)^{U(V,h)} \end{aligned}$$

3. **Combinatorial description.**

4. **Decomposition into irreducibles.**

1. Invariants:

$$J : V \to V, \quad v \mapsto iv,$$

$$\begin{cases} h(v,w) = \langle v \mid w\rangle + i\langle v \mid Jw\rangle \\ \langle\,|\,\rangle \text{ inner product.} \end{cases}$$

$\forall l = 1, \dots r$:

$$J_l \; : \; \otimes^r_{\mathbb{R}} V \longrightarrow \otimes^r_{\mathbb{R}} V$$

$$v_1 \otimes \cdots \otimes v_r \mapsto v_1 \otimes \cdots \otimes Jv_l \otimes \cdots \otimes v_r$$

$$J_l \in \mathrm{End}_{U(V,h)}(\otimes^r_{\mathbb{R}} V).$$

$$\mathcal{J} = \mathrm{alg}_{\mathbb{R}}\langle J_l' s\rangle \cong \otimes^r_{\mathbb{R}}\mathbb{C} \hookrightarrow \mathrm{End}_{U(V,h)}(\otimes^r_{\mathbb{R}} V)$$

$\forall I \subseteq \{2, \dots, r\}$:

$$e_I = \frac{1}{2^{r-1}} \prod_{l \in I} (1 - J_1 J_l) \prod_{1 \neq l \notin I} (1 + J_1 J_l)$$

is a primitive idempotent of $\mathcal{J}$, $1 = \sum_I e_I$ and

$$\mathcal{J} = \bigoplus_{I \subseteq \{2,\dots,r\}} \mathbb{C} e_I$$

$$(\otimes_{\mathbb{R}}^r V)e_I = \{x \in \otimes_{\mathbb{R}}^r V : xJ_l = xJ_1 \ \forall l \in I,$$
$$xJ_l = -xJ_1 \ \forall 1 \neq l \notin I\}$$

With

$$\begin{cases} V_1 = V & \\ V_l = V & \text{if } l \in I \\ V_l = V^* & \text{if } 1 \neq l \notin I \end{cases}$$

$$(\otimes_{\mathbb{R}}^r V)e_I \ \cong \ V_1 \otimes_{\mathbb{C}} V_2 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} V_r$$

$$(v_1 \otimes \cdots \otimes v_r)e_I \mapsto w_1 \otimes \cdots \otimes w_r$$

$$\begin{cases} w_l = v_l & \text{if } l = 1 \text{ or } l \in I \\ w_l = h(-, v_l) & \text{if } 1 \neq l \notin I \end{cases}$$

This is an isomorphism of $U(V, h)$-modules and of complex vector spaces (action on the first slots).

1. Invariants.

Now, the theory of invariants for the action of $U(V,h)$ on $V_1 \otimes_{\mathbb{C}} V_2 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} V_r$ give:

Theorem:

$f : V \times \overset{r}{\cdots} \times V \to \mathbb{R}$ multilinear and $U(V,h)$-invariant. Then either $f = 0$ or $r = 2m$ and f is a linear combination of

$$\left\langle v_{\sigma(1)} \mid J^{\delta_1} v_{\sigma(2)} \right\rangle \cdots \left\langle v_{\sigma(2m-1)} \mid J^{\delta_m} v_{\sigma(2m)} \right\rangle$$

$(\sigma \in S_r,\ \delta_1, \ldots, \delta_m \in \{0,1\})$.

For $r \leq \dim_{\mathbb{C}} V$, this is proved by N. Iwahori (1958), and it is asserted in full generality in Gray-Hervella's paper.

2. Centralizer algebra:

$$f : V \times V \times V \times V \times V \times V \longrightarrow \mathbb{R}$$

$$(v_1, v_2, v_3, v_4, v_5, v_6) \mapsto$$

$$\langle v_1 \mid Jv_3 \rangle \langle v_2 \mid Jv_4 \rangle \langle v_5 \mid Jv_6 \rangle$$

$\{e_a\}_{a=1}^{2n}$ orthonormal basis of V,

$\{\epsilon_a\}_{a=1}^{2n}$ dual basis of $V_{\mathbb{R}}^*$

$$f \simeq \sum_{a,b,c=1}^{2n} \epsilon_a \otimes \epsilon_b \otimes \epsilon_a \circ J \otimes \epsilon_b \circ J \otimes \epsilon_c \otimes \epsilon_c \circ J$$

$$\in (\otimes_{\mathbb{R}}^6 V_{\mathbb{R}}^*)^{U(V,h)}$$

$$\simeq (-1)^2 \sum_{a,b,c=1}^{2n} \epsilon_a \otimes \epsilon_b \otimes \epsilon_a \circ J \otimes Je_b \otimes e_c \otimes Je_c$$

$$\in ((\otimes_{\mathbb{R}}^3 V_{\mathbb{R}}^*) \otimes_{\mathbb{R}} (\otimes_{\mathbb{R}}^3 V))^{U(V,h)}$$

$$\simeq \Bigg(v_1 \otimes v_2 \otimes v_3 \mapsto$$

$$\sum_{a,b,c=1}^{2n} \epsilon_a(v_1)\epsilon_b(v_2)\epsilon_a(Jv_3)Je_b \otimes e_c \otimes Je_c \Bigg)$$

$$= \Bigg(v_1 \otimes v_2 \otimes v_3 \mapsto$$

$$\sum_{c=1}^{2n} \langle v_1 \mid Jv_3 \rangle (Jv_2) \otimes e_c \otimes Je_c \Bigg)$$

$$= J_2 J_3 c_{13} (1\,2) J_3 \in \mathrm{End}_{U(V,h)}(\otimes_{\mathbb{R}}^3 V)$$

2. Centralizer algebra.

where

$$(v_1 \otimes v_2 \otimes v_3) c_{13} = \langle v_1 \mid v_3 \rangle \sum_{a=1}^{2n} e_a \otimes v_2 \otimes e_a$$

$$\left.\begin{matrix} J_l \\ c_{pq} \\ \rho(S_r) \end{matrix}\right\} \text{ they all belong to } \mathrm{End}_{U(V,h)}(\otimes_{\mathbb{R}}^r V)$$

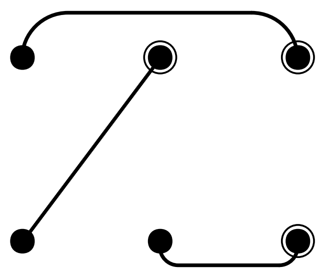
and

$$\boxed{End_{U(V,h)}(\otimes_{\mathbb{R}}^r V) = \mathrm{alg}_{\mathbb{R}} \left\{ \rho(S_r),\ J_l\text{'s},\ c_{pq}\text{'s} \right\}}$$

2. Centralizer algebra.

3. Combinatorial description:

$$\boldsymbol{v_1 \otimes v_2 \otimes v_3 \mapsto \langle v_1 \mid J v_3 \rangle \sum_{a=1}^{2n} J v_2 \otimes e_a \otimes J e_a}$$



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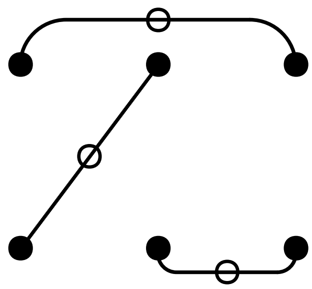
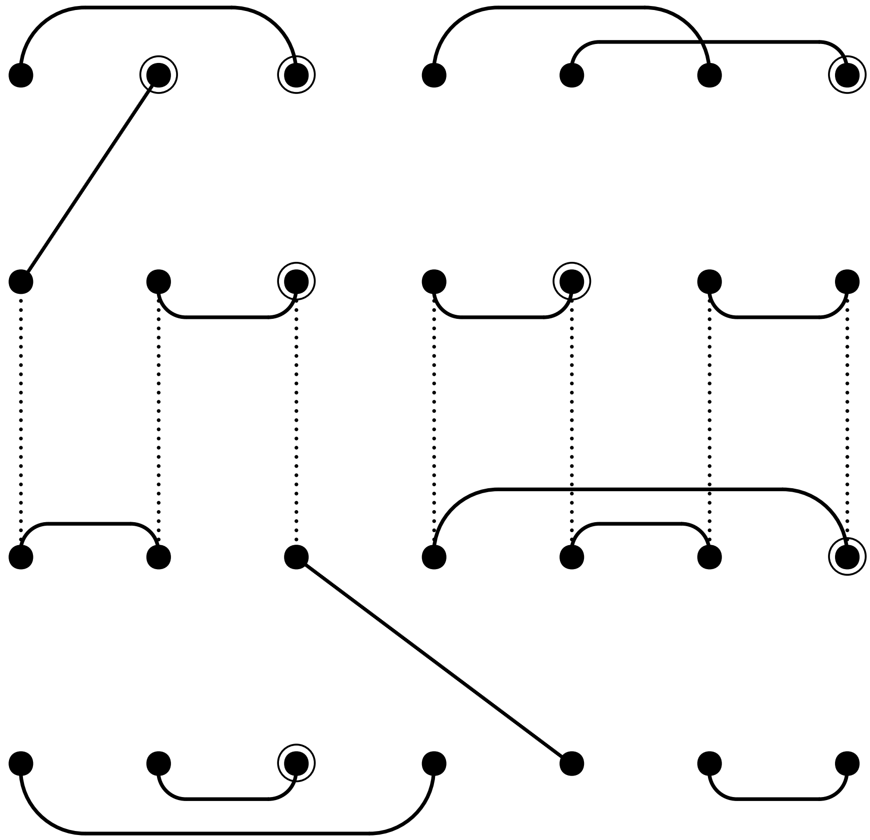
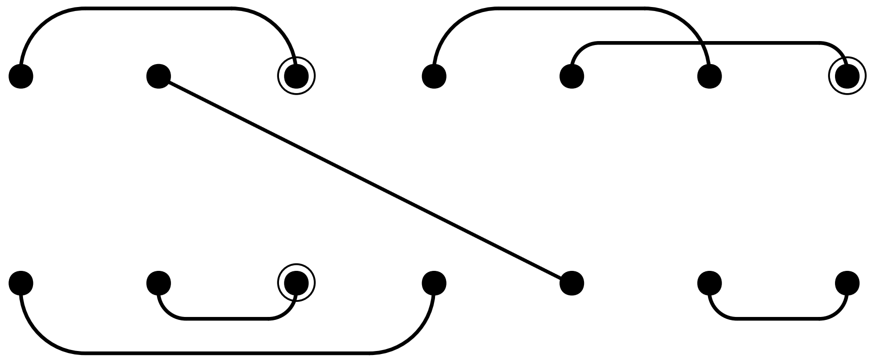




= $2n$



3. Combinatorial description.

General rule for multiplication of marked diagrams:

a, b marked diagrams

i) Draw a above b and connect the j^{th} lower vertex of a with the j^{th} upper vertex of b $\forall j = 1, \dots, r$, to get a diagram $a \circ b$. Then ab is a scalar multiple of the marked diagram c whose arcs are the (non-closed) paths of $a \circ b$. The uppermost or rightmost vertex of any arc of c is marked if the number of marked nodes in the arcs of a and b involved in the corresponding path is odd.

ii) If p (resp. l) is a path (resp. loop) of $a \circ b$, move the marks (say s) on it to the uppermost or right most vertex and take

$$\gamma(p) = (-1)^{\#\text{ horizontal moves}}(-1)^{\lfloor \frac{s}{2} \rfloor}$$

$$\gamma(l) = \begin{cases} 0 & \text{if } s \text{ is odd} \\ (-1)^{\#\text{ moves}}(-1)^{\frac{s}{2}} & \text{if } s \text{ is even} \end{cases}$$

iii) Compute $\gamma(a,b) = (2n)^{\#\text{ loops}} \prod_p \gamma(p) \prod_l \gamma(l)$.

iv)

$$\boxed{ab = \gamma(a,b)c}$$

4. Decomposition into irreducibles:

$$\otimes_{\mathbb{R}}^{r} V \;=\; \bigoplus_{I \subseteq \{2,\ldots,r\}} (\otimes_{\mathbb{R}}^{r} V) e_I$$

$$\cong\; \bigoplus_{q=1}^{r} \binom{r}{q} \Big((\otimes_{\mathbb{C}}^{q} V) \otimes_{\mathbb{C}} (\otimes_{\mathbb{C}}^{r-q} V^{*}) \Big)$$

as $U(V,h)$-modules and complex vector spaces.

Now the results of Benkart et al. apply.

Examples:

1) Gray-Hervella (1978):

$$W = \{x \in \otimes^3_{\mathbb{R}} V : x(23) = -x = xJ_2J_3\}$$

$$= \{x \in (\otimes^3_{\mathbb{R}} V)\frac{1}{2}(1 - J_2J_3) : x(23) = -x\}$$

$$\frac{1}{2}(1 - J_2J_3) = e_1 + e_2 \quad \text{orthogonal idempotents}$$

$$e_1 = \frac{1}{4}(1 - J_1J_3)(1 - J_2J_3)$$

$$e_2 = \frac{1}{4}(1 + J_1J_3)(1 - J_2J_3)$$

$$(\otimes^3_{\mathbb{R}} V)\frac{1}{2}(1 - J_2J_3) = (\otimes^3_{\mathbb{R}} V)e_1 \oplus (\otimes^3_{\mathbb{R}} V)e_2$$

$$\cong (V \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V) \oplus (V^* \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V)$$

$$W \cong (V \otimes_{\mathbb{C}} \Lambda^2_{\mathbb{C}} V) \oplus (V^* \otimes_{\mathbb{C}} \Lambda^2_{\mathbb{C}} V)$$

2) Abbena-Garbiero (1988):

$$\boldsymbol{W = \{x \in \otimes_{\mathbb{R}}^3 V : x(23) = -x = -xJ_2J_3\}}$$

$$\boldsymbol{= \{x \in (\otimes_{\mathbb{R}}^3 V)\frac{1}{2}(1 + J_2J_3) : x(23) = -x\}}$$

$$\begin{array}{ccc} (\otimes_{\mathbb{R}}^3 V)\tfrac{1}{2}(1+J_2J_3) & \longleftrightarrow & (V \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V^*) \oplus (V \otimes_{\mathbb{C}} V^* \otimes_{\mathbb{C}} V) \\ {\scriptstyle (23)}\big\downarrow & & \big\downarrow{\scriptstyle \text{"flip"}} \\ (\otimes_{\mathbb{R}}^3 V)\tfrac{1}{2}(1+J_2J_3) & \longleftrightarrow & (V \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V^*) \oplus (V \otimes_{\mathbb{C}} V^* \otimes_{\mathbb{C}} V) \end{array}$$

Therefore,

$$\boldsymbol{W \;\cong\; V \otimes_{\mathbb{C}} V \otimes_{\mathbb{C}} V^*}$$

$$\boldsymbol{(v_1 \otimes v_2 \otimes v_3)\frac{1}{4}\big(1 - (23)\big)\big(1 + J_2J_3\big) \mapsto}$$

$$\boldsymbol{\frac{1}{2}\Big(v_1 \otimes_{\mathbb{C}} v_2 \otimes_{\mathbb{C}} h(-, v_3) - v_1 \otimes_{\mathbb{C}} v_3 \otimes_{\mathbb{C}} h(-, v_2)\Big)}$$

4. Decomposition into irreducibles.