New simple Lie superalgebras in characteristic 3

Alberto Elduque
Universidad de Zaragoza

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§1. Symplectic triple systems

$\big(V, \langle .|. \rangle\big)$ two dimensional vector space with a nonzero alternating bilinear form.

$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ $\mathbb{Z}_2$-graded Lie algebra with

$$\begin{cases} \mathfrak{g}_{\bar{0}} = \mathfrak{sp}(V) \oplus \mathfrak{s} & \text{(direct sum of ideals)}, \\ \mathfrak{g}_{\bar{1}} = V \otimes T & \text{(as a module for } \mathfrak{g}_{\bar{0}}\text{)}, \end{cases}$$

where T is a module for $\mathfrak{s}$.

$\mathfrak{sp}(V)$-invariance gives

$$[a \otimes x, b \otimes y] = (x|y)\gamma_{a,b} + \langle a|b \rangle d_{x,y}$$

where

$$\begin{cases} (.|.) : T \times T \to k & \text{alternating bilinear form}, \\ d : T \times T \to \mathfrak{s} & \text{symmetric bilinear map}, \\ \gamma_{a,b} = \langle a|.\rangle b + \langle b|.\rangle a\,. \end{cases}$$

Define

$$[xyz] = d_{x,y}(z).$$

Then, for any $x, y, z, u, v, w \in T$:

$[xyz] = [yxz]$

$[xyz] - [xzy] = (x|z)y - (x|y)z + 2(y|z)x$

$[xy[uvw]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]]$

$([xyu]|v) + (u|[xyv]) = 0$

$\Big(T, [...], (.|.)\Big)$ is called a *symplectic triple system* (**Yamaguti**, 1975).

The converse holds with $\mathfrak{s} = \mathrm{inder}(T) = d_{T,T}$.

Examples:

- *Symplectic case*:

$$\mathfrak{sp}(V \perp W) = \Big(\mathfrak{sp}(V) \oplus \mathfrak{sp}(W)\Big) \oplus \Big(V \otimes W\Big).$$

- *Orthogonal case*:

$$\begin{aligned} &\mathfrak{so}\Big((V \otimes V) \perp W\Big) \\ &= \Big(\mathfrak{so}(V \otimes V) \oplus \mathfrak{so}(W)\Big) \oplus \Big(V \otimes V \otimes W\Big), \\ &= \Big(\mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \oplus \mathfrak{so}(W)\Big) \oplus \Big(V \otimes V \otimes W\Big). \end{aligned}$$

- *Special case*:

$$\mathfrak{sl}(V \oplus W) = \Big(\mathfrak{sp}(V) \oplus \mathfrak{gl}(W)\Big) \oplus \Big(V \otimes (W \oplus W^*)\Big).$$

- G_2 *case*:

$$\mathfrak{g}_2 = \Big(\mathfrak{sp}(V) \oplus \mathfrak{sl}_2(k)\Big) \oplus \Big(V \otimes W\Big)$$

  (dim $W = 4$).

- F_4, E_6, E_7, E_8: The symplectic triple systems here are related to the exceptional Freudenthal triple systems.

$(.|.)$ nondegenerate alternating bilinear form on a vector space T,

xyz, $[xyz]$ two triple products on T related by
$xyz = [xyz] - (x \mid z)y - (y \mid z)x$.

Then $\big(T, [...], (.|.)\big)$ is a symplectic triple system if and only if either $xyz = 0$ for any $x, y, z \in T$, or $\big(T, xyz, (.|.)\big)$ is a *Freudenthal triple system*:

xyz is symmetric in its arguments,

$(x|yzt)$ is symmetric in its arguments,

$$(xyy)xz + (yxx)yz + (xyy|z)x + (yxx|z)y \\ + (x|z)xyy + (y|z)yxx = 0,$$

Meyberg's classification (1968) of the simple Freudenthal triple systems shows that the examples above cover all the simple symplectic triple systems if $\operatorname{char} k \neq 2, 3$.

Brown's classification (1984) of Freudenthal triple systems in characteristic 3 gives two more possibilities for simple symplectic triple systems:

- $\dim T_{2,\epsilon} = 2$ $(0 \neq \epsilon \in k)$:

  $$[aab] = [aba] = [baa] = \epsilon a,$$
  $$[abb] = [bab] = [bba] = -\epsilon b$$

  for a symplectic basis $\{a, b\}$.

  The associated $\mathbb{Z}_2$-graded Lie algebras are the 10-dimensional simple Lie algebras of **Kostrikin** (1970): $\mathfrak{g}(T_{2,\epsilon}) = L(\epsilon)$.

- $\dim T_8 = 8$, such that $\text{inder}(T_8) = L(1)$.

  The associated $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{g}(T_8)$ is a 29-dimensional simple Lie algebra defined by **Brown** (1982).

§2. Symplectic triple systems in characteristic 3

Theorem. *Let* $\Bigl(T, [...], (.|.)\Bigr)$ *be a symplectic triple system over a field of characteristic* 3*. Define the superalgebra* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(T) = \tilde{\mathfrak{g}}_{\bar{0}} \oplus \tilde{\mathfrak{g}}_{\bar{1}}$*, with:*

$$\tilde{\mathfrak{g}}_{\bar{0}} = \text{inder}(T), \qquad \tilde{\mathfrak{g}}_{\bar{1}} = T,$$

and superanticommutative multiplication given by the multiplication on $\tilde{\mathfrak{g}}_{\bar{0}}$*, the natural action of* $\tilde{\mathfrak{g}}_{\bar{0}}$ *on* $\tilde{\mathfrak{g}}_{\bar{1}}$*, and by*

$$[x, y] = d_{x,y} = [xy.], \text{ for any } x, y \in T.$$

Then $\tilde{\mathfrak{g}}(T)$ *is a Lie superalgebra. Moreover,* T *is simple if and only if so is* $\tilde{\mathfrak{g}}(T)$*.*

$$
\begin{cases}
\text{Symplectic case:} & \tilde{\mathfrak{g}} \simeq \mathfrak{osp}(1, 2r) \\
\text{Special case:} & \tilde{\mathfrak{g}} \simeq \mathfrak{psl}(1, r) \\
\text{Orthogonal case:} & \tilde{\mathfrak{g}} \simeq \mathfrak{osp}(r, 2) \\
T_{2,\epsilon}\text{:} & \tilde{\mathfrak{g}} \simeq \mathfrak{osp}(1, 2)
\end{cases}
$$

So, up to now, only well-known simple Lie superalgebras are obtained.

But:

Theorem. *Let k be an algebraically closed field of characteristic* 3*. Then there are simple finite dimensional Lie superalgebras $\tilde{\mathfrak{g}}$ over k satisfying:*

(i) $\dim \tilde{\mathfrak{g}} = 18\,(= 10+8)$, $\tilde{\mathfrak{g}}_{\bar{0}}$ *is the Kostrikin Lie algebra $L(1)$ and $\tilde{\mathfrak{g}}_{\bar{1}}$ is an* 8*-dimensional irreducible module.*
This is obtained from the symplectic triple system T_8.

(ii) $\dim \tilde{\mathfrak{g}} = 35\,(= 21 + 14)$, $\tilde{\mathfrak{g}}_{\bar{0}}$ *is the symplectic Lie algebra $\mathfrak{sp}_6(k)$ and $\tilde{\mathfrak{g}}_{\bar{1}}$ is a* 14*-dimensional irreducible module for $\tilde{\mathfrak{g}}_{\bar{0}}$.*
This comes from the exceptional symplectic triple system related to F_4.

(iii) $\dim \tilde{\mathfrak{g}} = 54\,(= 34 + 20)$, $\tilde{\mathfrak{g}}_{\bar{0}}$ is the projective special Lie algebra $\mathfrak{psl}_6(k)$ and $\tilde{\mathfrak{g}}_{\bar{1}}$ is a 20-dimensional irreducible module for $\tilde{\mathfrak{g}}_{\bar{0}}$. This comes from the exceptional symplectic triple system related to E_6.

(iv) $\dim \tilde{\mathfrak{g}} = 98\,(= 66 + 32)$, $\tilde{\mathfrak{g}}_{\bar{0}}$ is the orthogonal Lie algebra $\mathfrak{so}_{12}(k)$ and $\tilde{\mathfrak{g}}_{\bar{1}}$ is a 32-dimensional irreducible module for $\tilde{\mathfrak{g}}_{\bar{0}}$ (spin module).
This comes from the exceptional symplectic triple system related to E_7.

(v) $\dim \tilde{\mathfrak{g}} = 189\,(= 133 + 56)$, $\tilde{\mathfrak{g}}_{\bar{0}}$ is the simple Lie algebra of type E_7 and $\tilde{\mathfrak{g}}_{\bar{1}}$ is a 56-dimensional irreducible module for $\tilde{\mathfrak{g}}_{\bar{0}}$. This comes from the exceptional symplectic triple system related to E_8.

§3. Orthogonal triple systems

Let's superize!

$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ Lie superalgebra with

$$\begin{cases} \mathfrak{g}_{\bar{0}} = \mathfrak{sp}(V) \oplus \mathfrak{s} & \text{(direct sum of ideals)}, \\ \mathfrak{g}_{\bar{1}} = V \otimes T & \text{(as a module for } \mathfrak{g}_{\bar{0}}), \end{cases}$$

where T is a module for $\mathfrak{s}$.

$\mathfrak{sp}(V)$-invariance gives

$$[a \otimes x, b \otimes y] = (x|y)\gamma_{a,b} + \langle a|b\rangle d_{x,y}$$

where

$$\begin{cases} (.|.) : T \times T \to k & \text{symmetric bilinear form}, \\ d : T \times T \to \mathfrak{s} & \text{skew-sym. bilinear map}, \end{cases}$$

Define

$$[xyz] = d_{x,y}(z).$$

Then, for any $x, y, z, u, v, w \in T$:

$[xxz] = 0$

$[xyy] = (x|y)y - (y|y)x$

$[xy[uvw]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]]$

$([xyu]|v) + (u|[xyv]) = 0$

$\Big(T, [\ldots], (.|.)\Big)$ is called an *orthogonal triple system* (**Okubo**, 1993).

The converse holds with $\mathfrak{s} = \text{inder}(T) = d_{T,T}$.

Examples:

- *Special type*:

  $$\mathfrak{psl}(W \oplus V) = \Big(\mathfrak{sp}(V) \oplus \mathfrak{gl}(W)\Big) \oplus \Big(V \otimes (W \oplus W^*)\Big).$$

- *Orthogonal type*:

  $$\mathfrak{osp}(W \perp V) = \Big(\mathfrak{sp}(V) \oplus \mathfrak{so}(W)\Big) \oplus \Big(V \otimes W\Big).$$

- *Symplectic type*:

  $$\begin{aligned}&\mathfrak{osp}\Big((V \otimes V) \perp W\Big)\\ &= \Big(\mathfrak{so}(V \otimes V) \oplus \mathfrak{sp}(W)\Big) \oplus \Big(V \otimes V \otimes W\Big),\\ &= \Big(\mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \oplus \mathfrak{sp}(W)\Big) \oplus \Big(V \otimes V \otimes W\Big).\end{aligned}$$

- D_μ-*type*:

  $D(2,1;\mu)$
  $$= \Big(\mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \oplus \mathfrak{sp}(V)\Big) \oplus \Big(V \otimes V \otimes V\Big).$$

- $G(3)$-*type*:

  $$\mathfrak{g}(3) = \Big(\mathfrak{sp}(V) \oplus \mathfrak{g}_2\Big) \oplus \Big(V \otimes W\Big)$$
  $(\dim W = 7)$.

- $F(4)$-*type*:

  $$\mathfrak{f}_4 = \Big(\mathfrak{sp}(V) \oplus \mathfrak{b}_3\Big) \oplus \Big(V \otimes W\Big)$$
  $(\dim W = 8)$.

There is a close connection between the orthogonal triple systems and the so called $(-1,-1)$ balanced Freudenthal Kantor triple systems (**Yamaguti, Ono**, 1984).

The classification by **E-Kamiya-Okubo** of the simple such systems in characteristic 0 shows that:

The previous examples exhaust the simple orthogonal triple systems over fields of characteristic 0.

All these systems can be defined over fields of characteristic $\neq 2, 3$, but there is no known classification.

§4. Orthogonal triple systems in characteristic 3

- If T is of $G(3)$-type ($\dim T = 7$), then $\operatorname{inder}(T) \simeq \mathfrak{psl}_3$, instead of G_2.

- If T is of $F(4)$-type ($\dim T = 8$), then its symmetric bilinear form $(. \mid .)$ becomes trivial.

- There appears at least a new family of simple orthogonal triple systems:

$J = \mathcal{J}ord(n,1)$ Jordan algebra of a nondegenerate cubic form n with basepoint 1 with $\dim_k J \geq 3$. For any $x \in J$

$$x^3 - t(x)x^2 + s(x)x - n(x)1 = 0,$$

where t is its trace linear form, and $s(x) = t(x^\sharp)$ is the spur quadratic form.

Let $J_0 = \{x \in J : t(x) = 0\}$ be the subspace of zero trace elements. Since $\operatorname{char} k = 3$, $t(1) = 0$, so that $k1 \in J_0$. Consider the quotient space $\hat{J} = J_0/k1$. For any $x \in J_0$, let $\hat{x}$ be the class of x modulo $k1$. Define a triple product on $\hat{J}$ by

$$[\hat{x}\hat{y}\hat{z}] = \Big(x(yz) - y(xz)\Big)^{\widehat{}} \ .$$

Then

$\Big(\hat{J}, [...], t(.,.)\Big)$ *is a simple orthogonal triple system.*

Theorem. *Let k be an algebraically closed field of characteristic 3. Then there are simple finite dimensional Lie superalgebras $\mathfrak{g}$ over k satisfying:*

(i) $\dim \mathfrak{g} = 24 \left(= (3+7)+(2\times 7)\right)$, *$\mathfrak{g}_{\bar{0}}$ is the direct sum of $\mathfrak{sl}_2(k)$ and $\mathfrak{psl}_3(k)$ and, as a $\mathfrak{g}_{\bar{0}}$-module, $\mathfrak{g}_{\bar{1}}$ is the tensor product of the natural two dimensional module for $\mathfrak{sl}_2(k)$ and the adjoint module for $\mathfrak{psl}_3(k)$ ($G(3)$-type).*

(ii) $\dim \mathfrak{g} = 37 \left(= 21 + (2\times 8)\right)$, $\mathfrak{g}_{\bar{0}} = \mathfrak{so}_7(k)$ *and, as a $\mathfrak{g}_{\bar{0}}$-module, $\mathfrak{g}_{\bar{1}}$ is the direct sum of two copies of its spin module ($F(4)$-type).*

(iii) $\dim \mathfrak{g} = 50 \left(= (3+21) + (2 \times 13)\right)$, $\mathfrak{g}_{\bar{0}}$ is the direct sum of $\mathfrak{sl}_2(k)$ and $\mathfrak{sp}_6(k)$ and, as a $\mathfrak{g}_{\bar{0}}$-module, $\mathfrak{g}_{\bar{1}}$ is the tensor product of the natural two dimensional module for $\mathfrak{sl}_2(k)$ and of a 13 dimensional irreducible module for $\mathfrak{sp}_6(k)$ ($J = H_3(Q)$).

(iv) $\dim \mathfrak{g} = 105 \left(= (3+52) + (2 \times 25)\right)$, $\mathfrak{g}_{\bar{0}}$ is the direct sum of $\mathfrak{sl}_2(k)$ and of the central simple Lie algebra of type F_4 and, as a $\mathfrak{g}_{\bar{0}}$-module, $\mathfrak{g}_{\bar{1}}$ is the tensor product of the natural two dimensional module for $\mathfrak{sl}_2(k)$ and a 25 dimensional irreducible module for F_4 ($J = H_3(C)$).

Again, in characteristic 3 there is something else:

Theorem. *Let* $\Big(T, [\ldots], (.|.)\Big)$ *be an orthogonal triple system over a field of characteristic* 3. *Define the* $\mathbb{Z}_2$*-graded algebra* $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(T) = \tilde{\mathfrak{g}}_{\bar{0}} \oplus \tilde{\mathfrak{g}}_{\bar{1}}$, *with:*

$$\tilde{\mathfrak{g}}_{\bar{0}} = \mathrm{inder}(T), \qquad \tilde{\mathfrak{g}}_{\bar{1}} = T,$$

and anticommutative multiplication given by the multiplication on $\tilde{\mathfrak{g}}_{\bar{0}}$, *the natural action of* $\tilde{\mathfrak{g}}_{\bar{0}}$ *on* $\tilde{\mathfrak{g}}_{\bar{1}}$, *and by*

$$[x, y] = d_{x,y} = [xy.], \text{ for any } x, y \in T.$$

Then $\tilde{\mathfrak{g}}(T)$ *is a* $\mathbb{Z}_2$*-graded Lie algebra. Moreover,* T *is simple if and only if so is* $\tilde{\mathfrak{g}}(T)$.

The orthogonal triple systems of D_μ-type (respectively $F(4)$-type) provide new models of **Kostrikin** Lie algebras (respectively, of **Brown**'s 29-dimensional simple Lie algebra).