Quaternions and Octonions

Alberto Elduque

Instituto Universitario de Matemáticas y Aplicaciones
Universidad de Zaragoza

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Outline

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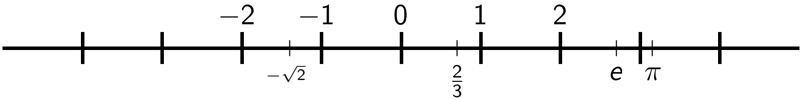
Real numbers

$$\mathbb{R} = \{\text{real numbers}\}$$

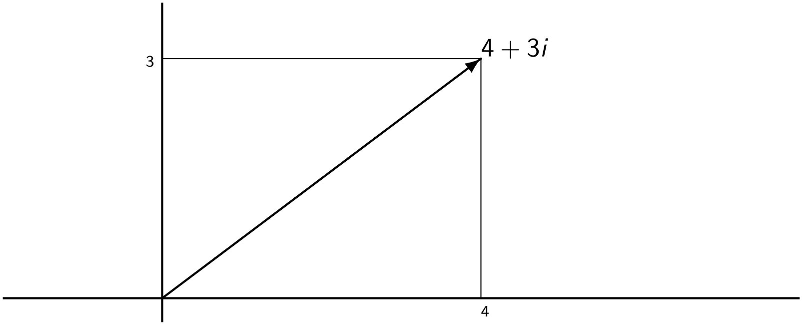
Real numbers are used in measurements.



But we cannot solve equations as simple as $X^2 + 1 = 0$!

Complex numbers

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} \, (\simeq \mathbb{R}^2)$$



$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Complex numbers: properties

Exercise

$$|z_1 z_2| = |z_1||z_2|$$

($|.|$ denotes the usual length.)

Exercise

Rotation of angle α in $\mathbb{R}^2 \leftrightarrow$ multiplication by $e^{i\alpha} = \cos\alpha + i\sin\alpha$.

$$SO(2) \simeq \{z \in \mathbb{C} : |z| = 1\} \simeq S^1$$

Outline

A three-dimensional algebra?

Hamilton tried to find a multiplication, analogous to the multiplication of complex numbers, but in dimension 3, that should respect the "law of moduli": $|z_1 z_2| = |z_1||z_2|$:

$$(a + bi + cj)(a' + b'i + c'j) = ???$$

$$(\text{assuming } i^2 = -1 = j^2)$$

Problem: $ij, \quad ji?$

After years of struggle, he found the solution on October 16, 1843.

A spark flashed forth

Letter from Sir W. R. Hamilton to his son Rev. Archibald H. Hamilton, dated August 5 1865:

MY DEAR ARCHIBALD -
(1) I had been wishing for an occasion of corresponding a little with you on QUATERNIONS: and such now presents itself, by your mentioning in your note of yesterday, received this morning, that you "have been reflecting on several points connected with them" (the quaternions), "particularly on the Multiplication of Vectors."

(2) No more important, or indeed fundamental question, in the whole Theory of Quaternions, can be proposed than that which thus inquires What is such MULTIPLICATION? What are its Rules, its Objects, its Results? What Analogies exist between it and other Operations, which have received the same general Name? And finally, what is (if any) its Utility?

A spark flashed forth

(3) If I may be allowed to speak of myself in connexion with the subject, I might do so in a way which would bring you in, by referring to an ante-quaternionic time, when you were a mere child, but had caught from me the conception of a Vector, as represented by a Triplet: and indeed I happen to be able to put the finger of memory upon the year and month - October, 1843 - when having recently returned from visits to Cork and Parsonstown, connected with a meeting of the British Association, the desire to discover the laws of the multiplication referred to regained with me a certain strength and earnestness, which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "Well, Papa, can you multiply triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them."

A spark flashed forth

(4) But on the 16th day of the same month - which happened to be a Monday, and a Council day of the Royal Irish Academy - I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery.

A spark flashed forth

Nor could I resist the impulse -unphilosophical as it may have been- to cut with a knife on a stone of Brougham Bridge[1], as we passed it, the fundamental formula with the symbols, i, j, k; namely,

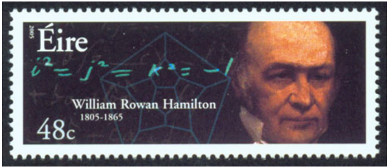
$$i^2 = j^2 = k^2 = ijk = -1$$

which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (October 16th, 1843), which records the fact, that I then asked for and obtained leave to read a Paper on Quaternions, at the First General Meeting of the session: which reading took place accordingly, on Monday the 13th of the November following. With this quaternion of paragraphs I close this letter I.; but I hope to follow it up very shortly with another.
Your affectionate father, WILLIAM ROWAN HAMILTON.

[1]The actual name of this bridge is Broome, not Brougham

$\mathbb{H}$

$$\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k,$$
$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$



Hamilton and his quaternions

Some properties of $\mathbb{H}$

- $|q_1 q_2| = |q_1||q_2| \ \forall q_1, q_2 \in \mathbb{H}$
  $(|a + bi + cj + dk|^2 = a^2 + b^2 + c^2 + d^2)$

- $\mathbb{H}$ is an associative division algebra (but it is not commutative).
  Therefore $S^3 \simeq \{q \in \mathbb{H} : |q| = 1\}$ is a (Lie) group.
  (This implies the parallelizability of S^3.)

- $\mathbb{H}_0 = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \simeq \mathbb{R}^3, \quad \mathbb{H} = \mathbb{R} \oplus \mathbb{H}_0$, and $\forall u, v \in \mathbb{H}_0$:

$$uv = -u \cdot v + u \times v$$

  (where $u \cdot v$ and $u \times v$ denote the usual scalar and cross products).

Some properties of $\mathbb{H}$

- $\forall q = a1 + u \in \mathbb{H}$, $q^2 = (a^2 - u \cdot u) + 2au$, so
$$q^2 - (2a)q + |q|^2 = 0 \qquad (\mathbb{H} \text{ is quadratic.})$$
- The map $q = a + u \mapsto \bar{q} = a - u$ is an involution, with $q + \bar{q} = 2a$ and $q\bar{q} = \bar{q}q = |q|^2$.
- $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j \simeq \mathbb{C}^2$ is a two-dimensional vector space over $\mathbb{C}$. Multiplication is given by:
$$(p_1 + p_2 j)(q_1 + q_2 j) = (p_1 q_1 - \bar{q}_2 p_2) + (q_2 p_1 + p_2 \bar{q}_1)j$$

Outline

Rotations in three-dimensional space

$$q \in \mathbb{H},\ |q| = 1 \ \Rightarrow \exists \alpha \in [0,\pi],\ u \in \mathbb{H}_0,\ |u| = 1$$
$$\text{such that } q = (\cos\alpha)1 + (\sin\alpha)u$$

Take $v \in \mathbb{H}_0$ of norm 1 and orthogonal to u, so that $\{u, v, u \times v\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3 = \mathbb{H}_0$.

Consider the linear map:

$$\begin{aligned} \varphi_q : \mathbb{H}_0 &\longrightarrow \mathbb{H}_0, \\ x &\mapsto qxq^{-1} = qx\bar{q}. \end{aligned}$$

Coordinate matrix of φ_q

$$\varphi_q(u) = quq^{-1} = u \qquad (\text{as } uq = qu),$$

$$\begin{aligned}\varphi_q(v) &= \big((\cos\alpha)1 + (\sin\alpha)u\big)v((\cos\alpha)1 - (\sin\alpha)u) \\ &= \big((\cos\alpha)v + (\sin\alpha)u \times v\big)((\cos\alpha)1 - (\sin\alpha)u) \\ &= (\cos^2\alpha)v + 2(\cos\alpha\sin\alpha)u \times v - (\sin^2\alpha)(u \times v) \times u \\ &= (\cos 2\alpha)v + (\sin 2\alpha)u \times v,\end{aligned}$$

$$\varphi_q(u \times v) = ... = -(\sin 2\alpha)v + (\cos 2\alpha)u \times v.$$

Coordinate matrix of φ_q

Thus the coordinate matrix of φ_q relative to the basis $\{u, v, u \times v\}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\alpha & -\sin 2\alpha \\ 0 & \sin 2\alpha & \cos 2\alpha \end{pmatrix}$$

φ_q is a rotation around the axis $\mathbb{R}^+ u$ of angle 2α

$SO(3)$

The map

$$\varphi : S^3 \simeq \{q \in \mathbb{H} : |q| = 1\} \longrightarrow SO(3), \quad q \mapsto \varphi_q$$

is a surjective (Lie) group homomorphism with $\ker \varphi = \{\pm 1\}$:

$$S^3/_{\{\pm 1\}} \simeq SO(3)$$

(S^3 is the universal cover of $SO(3)$)

Rotations in three-dimensional space

Rotations in $\mathbb{R}^3 \quad \longleftrightarrow \quad$ Conjugation in $\mathbb{H}_0$ by norm 1 quaternions "modulo ± 1"

It is quite easy now to compose rotations in three-dimensional space!

It is enough to multiply norm 1 quaternions! $(\varphi_p \circ \varphi_q = \varphi_{pq})$

Now one can deduce easily the formulas by Olinde Rodrigues (1840) for the composition of rotations.

Rotations in $\mathbb{R}^4$

- $\forall p \in \mathbb{H}$ with $|p| = 1$, the left (resp. right) multiplication L_p (resp. R_p) by p is an isometry, due to the multiplicativity of the norm.

- For $p = (\cos\alpha)1 + (\sin\alpha)u$, ($\alpha \in [0, \pi]$, $u \in \mathbb{H}_0$, $|u| = 1$), we have $p^2 - 2(\cos\alpha)p + 1 = 0$, so the minimal polynomial of the multiplication by p is either $X \pm 1$ for $p = \mp 1$, or the irreducible polynomial $X^2 - 2(\cos\alpha)X + 1$ otherwise.

- Hence the characteristic polynomial of the multiplication by p is always
$$\left(X^2 - 2(\cos\alpha)X + 1\right)^2$$
and, in particular, the determinant of the multiplication by p is 1.

Multiplication by norm 1 quaternions are rotations in $\mathbb{H} \simeq \mathbb{R}^4$.

Rotations in $\mathbb{R}^4$

- If ψ is a rotation in $\mathbb{R}^4 \simeq \mathbb{H}$, $a = \psi(1)$ is a norm 1 quaternion, and
$$L_{\bar{a}} \circ \psi(1) = \bar{a}a = |a|^2 = 1,$$
so $L_{\bar{a}} \circ \psi$ is actually a rotation in $\mathbb{R}^3 \simeq \mathbb{H}_0$.

- Therefore, there is a norm 1 quaternion $q \in \mathbb{H}$ such that
$$\bar{a}\psi(x) = qxq^{-1}$$
for any $x \in \mathbb{H}$. That is:
$$\psi(x) = (aq)xq^{-1} \quad \forall x \in \mathbb{H}.$$

$SO(4)$

The map

$$\begin{aligned}\Psi : S^3 \times S^3 &\longrightarrow SO(4), \\ (p, q) &\mapsto \psi_{p,q} \ (x \mapsto pxq^{-1})\end{aligned}$$

is a surjective (Lie) group homomorphism with $\ker \Psi = \{\pm(1,1)\}$.

$$S^3 \times S^3/_{\{\pm(1,1)\}} \simeq SO(4)$$

(From here we get $\quad SO(3) \times SO(3) \simeq PSO(4)$)

Rotations in $\mathbb{R}^4$

It is quite easy to compose rotations in four-dimensional space!

It is enough to multiply pairs of norm 1 quaternions!
$(\psi_{p_1,q_1} \circ \psi_{p_2,q_2} = \psi_{p_1 p_2, q_1 q_2})$

Exercise

What kind of rotation is $\psi_{p,q}$ for $p + \bar{p} = 2\cos\alpha$ and $q + \bar{q} = 2\cos\beta$?

Solution: A "double rotation" with angles $\alpha + \beta$ and $\alpha - \beta$.

Outline

Octonions (1843-1845)

> *There is still something in the system which gravels me. I have not yet any clear views as to the extent to which we are at liberty arbitrarily to create imaginaries, and to endow them with supernatural properties.*
>
> *If with your alchemy you can make three pounds of gold, why should you stop there?*

(Letter from John T. Graves to Hamilton, dated October 26, 1843!)

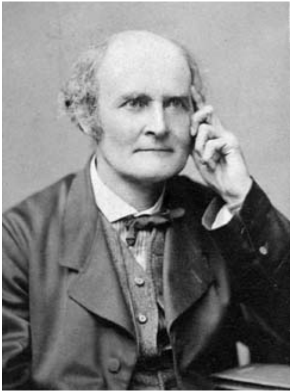
Octonions (1843-1845)

The algebra of quaternions is obtained by doubling suitably the field of complex numbers:

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j.$$

Doubling again we get the octonions (Graves – Cayley):

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l.$$



Arthur Cayley

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l = \mathbb{R}\langle 1, i, j, k, l, il, jl, kl \rangle$$

with multiplication

$$(p_1 + p_2 l)(q_1 + q_2 l) = (p_1 q_1 - \bar{q}_2 p_2) + (q_2 p_1 + p_2 \bar{q}_1) l$$

and norm:

$$|p_1 + p_2 l|^2 = |p_1|^2 + |p_2|^2$$

These are the same formulas that allow us to pass from $\mathbb{C}$ to $\mathbb{H}$!

Some algebraic properties

- $|xy| = |x||y|$, $\forall x, y \in \mathbb{O}$.
- $\mathbb{O}$ is a division algebra, it is neither commutative nor associative!

  But it is *alternative*: any two elements generate an associative subalgebra.

  Theorem (Zorn 1933): The only finite-dimensional real alternative division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

  The only such associative algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (Frobenius 1877).
- $S^7 \simeq \{x \in \mathbb{O} : |x| = 1\}$ is not a group (associativity fails), but it constitutes the most important example of a *Moufang loop*.
- $\mathbb{O}_0 = \mathbb{R}\langle i, j, k, l, il, jl, kl\rangle$. $\forall u, v \in \mathbb{O}_0$:

  $$uv = -u \cdot v + u \times v.$$

  (Cross product in $\mathbb{R}^7$!: $(u \times v) \times v = (u \cdot v)v - (v \cdot v)u$.)
- $\mathbb{O}$ is *quadratic*: $\forall x = a1 + u \in \mathbb{O}$, $x^2 - 2ax + |x|^2 = 0$.

Some geometric properties

- The groups $Spin_7$ and $Spin_8$ (universal covers of $SO(7)$ and $SO(8)$) can be described easily in terms of octonions.
- $\mathbb{O}$ division algebra $\Rightarrow S^7$ parallelizable.
  S^1, S^3 and S^7 are the only parallelizable spheres (Milnor and Kervaire).
- $S^6 \simeq \{x \in \mathbb{O}_0 : |x| = 1\}$ is endowed with an *almost complex structure*, inherited from the multiplication of octonions.
  S^2 and S^6 are the only spheres with such structures (Adams).
- *Non-desarguesian projective plane* $\mathbb{OP}^2$.
- The only spheres that can be described as homogeneous spaces of nonclassical groups are $S^6 = \operatorname{Aut}\mathbb{O}/SU(3)$, $S^7 = Spin_7/\operatorname{Aut}\mathbb{O}$ and $S^{15} = Spin_9/Spin_7$.

$\mathbb{O}$ is certainly a beautiful mathematical object!

The saying that God is the mathematician, so that, even with meager experimental support, a mathematically beautiful theory will ultimately have a greater chance of being correct, has been attributed to Dirac. Octonion algebra may surely be called a beautiful mathematical entity.

It is possible that this and other non-associative algebras (other than Lie algebras) may play some essential future role in the ultimate theory, yet to be discovered.

Susumu Okubo

Thank you!

Composition algebras

Alberto Elduque

August-September 2021

Outline

Composition algebras

Definition

A **composition algebra** is an algebra (over a field $\mathbb{F}$) endowed with a **nonsingular**, **multiplicative** quadratic form $\mathrm{n} : \mathcal{C} \to \mathbb{F}$, called the **norm**.

Definition

A **Hurwitz algebra** is a unital composition algebra.

Composition algebras

- $\mathsf{n} : \mathcal{C} \to \mathbb{F}$ is a **quadratic form**, so $\mathsf{n}(\alpha x) = \alpha^2 \mathsf{n}(x)$ for any $\alpha \in \mathbb{F}$ and $x \in \mathcal{C}$, and the map $\mathcal{C} \times \mathcal{C} \to \mathbb{F}$ defined by

$$\mathsf{n}(x, y) := \mathsf{n}(x + y) - \mathsf{n}(x) - \mathsf{n}(y)$$

  is a bilinear form, called the **polar form**.

- n is **nonsingular**: either $\{x \in \mathcal{C} \mid \mathsf{n}(x, \mathcal{C}) = 0\}$ is 0, or the characteristic of $\mathbb{F}$ is 2 and $\{x \in \mathcal{C} \mid \mathsf{n}(x, \mathcal{C}) = 0\} = \mathbb{F}z$ with $\mathsf{n}(z) \neq 0$.

- n is **multiplicative**: $\mathsf{n}(xy) = \mathsf{n}(x)\mathsf{n}(y)$ for all $x, y \in \mathcal{C}$.

Linearizations of the multiplicative property

Substitute y by $y+z$ in $\mathsf{n}(xy)=\mathsf{n}(x)\mathsf{n}(y)$ to get

$$\mathsf{n}\big(x(y+z)\big)=\mathsf{n}(xy+xz)=\mathsf{n}(xy,xz)+\mathsf{n}(xy)+\mathsf{n}(xz),$$

$$\begin{aligned}\mathsf{n}(x)\mathsf{n}(y+z)&=\mathsf{n}(x)\Big(\mathsf{n}(y,z)+\mathsf{n}(y)+\mathsf{n}(z)\Big)\\&=\mathsf{n}(x)\mathsf{n}(y,z)+\mathsf{n}(x)\mathsf{n}(y)+\mathsf{n}(x)\mathsf{n}(z).\end{aligned}$$

thus obtaining

$$\mathsf{n}(xy,xz)=\mathsf{n}(x)\mathsf{n}(y,z)$$

Substitute now x by $x+t$ to get

$$\mathsf{n}(xy,tz)+\mathsf{n}(ty,xz)=\mathsf{n}(x,t)\mathsf{n}(y,z)$$

Hurwitz algebras

Goal

To prove that Hurwitz algebras are the analogues of the classical algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.
In particular, their dimension is restricted to 1, 2, 4, or 8.

First properties of Hurwitz algebras

Theorem

Let $\mathcal{C}$ be a Hurwitz algebra.

1. *Either the polar form* $\mathsf{n}(.,.)$ *is nondegenerate, or the characteristic of* $\mathbb{F}$ *is* 2 *and* $\mathcal{C} = \mathbb{F}1$.
2. *The map* $x \mapsto \overline{x} := \mathsf{n}(x,1)1 - x$ *is an involution (i.e.,* $\overline{\overline{x}} = x$ *and* $\overline{xy} = \overline{y}\,\overline{x}$ *for all* x, y*).*
3. $\mathsf{n}(xy, z) = \mathsf{n}(y, \overline{x}z) = \mathsf{n}(x, z\overline{y})$ *for all* x, y, z.
4. $\mathcal{C}$ *is* ***alternative****:* $x(xy) = x^2y$, $(yx)x = yx^2$ *for all* x, y.
5. $\mathcal{C}$ *satisfies the* ***Cayley-Hamilton equation****:*

$$x^2 - \mathsf{n}(x,1)x + \mathsf{n}(x)1 = 0$$

for all x.

First properties of Hurwitz algebras. Proof

Because of the linearizations above, with $t = 1$ we get

$$\mathsf{n}(xy, z) + \mathsf{n}(y, xz) = \mathsf{n}(x, 1)\mathsf{n}(y, z)$$

which can be rewritten as

$$\mathsf{n}(xy, z) = \mathsf{n}\big(y, (\mathsf{n}(x, 1)1 - x)z\big) = \mathsf{n}(y, \overline{x}z)$$

and symmetrically we obtain $\mathsf{n}(xy, z) = \mathsf{n}(x, z\overline{y})$.

As n is nonsingular, if $\mathsf{n}(.,.)$ is degenerate, then the characteristic of $\mathbb{F}$ is 2 and $\mathcal{C}^\perp := \{x \in \mathcal{C} \mid \mathsf{n}(x, \mathcal{C}) = 0\} = \mathbb{F}z$ with $\mathsf{n}(z) \neq 0$.
For $x, y \in \mathcal{C}$, $\mathsf{n}(xz, y) = \mathsf{n}(z, \overline{x}y) = 0$. Hence $xz = f(x)z$ for a linear form $f : \mathcal{C} \to \mathbb{F}$.
But $\mathsf{n}(xz) = \mathsf{n}(x)\mathsf{n}(z)$ and $\mathsf{n}(xz) = f(x)^2\mathsf{n}(z)$, so we conclude $\mathsf{n}(x) = f(x)^2$, and hence

$$\mathsf{n}(x, y) = f(x + y)^2 - f(x)^2 - f(y)^2 = 2f(x)f(y) = 0.$$

Therefore $\mathcal{C} = \mathcal{C}^\perp$ and $\mathcal{C}$ has dimension 1: $\mathcal{C} = \mathbb{F}1$. (In this case all the properties follow trivially.)

First properties of Hurwitz algebras. Proof

Now, $x \mapsto \overline{x}$ is an involutive isometry (exercise!), and hence, assuming that $\mathsf{n}(.,.)$ is nondegenerate,

$$\mathsf{n}(\overline{xy}, z) = \mathsf{n}(xy, \overline{z}) = \mathsf{n}(x, \overline{z}\,\overline{y}) = \mathsf{n}(zx, \overline{y}) = \mathsf{n}(z, \overline{y}\,\overline{x}) = \mathsf{n}(\overline{y}\,\overline{x}, z)$$

The nondegeneracy of $\mathsf{n}(.,.)$ gives $\overline{xy} = \overline{y}\,\overline{x}$.

Also, for all x, y, z,

$$\mathsf{n}\big(\mathsf{n}(x)y, z\big) = \mathsf{n}(x)\mathsf{n}(y, z) = \mathsf{n}(xy, xz) = \mathsf{n}(\overline{x}(xy), z)$$

so $\overline{x}(xy) = \mathsf{n}(x)y$. (And, in the same vein, we get $(yx)\overline{x} = \mathsf{n}(x)y$.)
With $y = 1$, we obtain $\overline{x}x = \mathsf{n}(x)1$, and this is equivalent to the Cayley-Hamilton equation.
But $\overline{x}(xy) = \mathsf{n}(x, 1)xy - x(xy)$, and
$\mathsf{n}(x)y = (\overline{x}x)y = \mathsf{n}(x, 1)xy - x^2y$. Thus we obtain $x^2y = x(xy)$ ...

Cayley-Dickson doubling process

Let $\mathcal{C}$ be a Hurwitz algebra, and assume that $\mathcal{Q}$ is a proper unital subalgebra of $\mathcal{C}$ such that the restriction of $\mathsf{n}(.,.)$ to $\mathcal{Q}$ is nondegenerate.

Goal

To show that $\mathcal{C}$ contains also a subalgebra obtained by doubling $\mathcal{Q}$, in a way similar to the construction of $\mathbb{H}$ from two copies of $\mathbb{C}$, or the construction of $\mathbb{O}$ from two copies of $\mathbb{H}$.

Cayley-Dickson doubling process

By nondegeneracy of n, $\mathcal{C} = \mathcal{Q} \oplus \mathcal{Q}^{\perp}$. Pick $u \in \mathcal{Q}^{\perp}$ with $\mathsf{n}(u) \neq 0$, and let $\alpha = -\mathsf{n}(u)$.

As $1 \in \mathcal{Q}$, $\mathsf{n}(u, 1) = 0$ and hence $\overline{u} = -u$ and $u^2 = \alpha 1$ by the Cayley-Hamilton equation. Also $(xu)u = \alpha x$ for all x, so the right multiplication by u (denoted R_u) is bijective.

Cayley-Dickson doubling process. Some properties

Lemma

Under the conditions above, the subspaces $\mathcal{Q}$ and $\mathcal{Q}u$ are orthogonal (i.e., $\mathcal{Q}u \subseteq \mathcal{Q}^\perp$), and the following properties hold for all $x, y \in \mathcal{Q}$.

- $xu = u\overline{x}$,
- $x(yu) = (yx)u$,
- $(yu)x = (y\overline{x})u$,
- $(xu)(yu) = \alpha\overline{y}x$.

Cayley-Dickson doubling process. Proof

For any $x, y \in \mathcal{Q}$, $\mathsf{n}(x, yu) = \mathsf{n}(\overline{y}x, u) \in \mathsf{n}(\mathcal{Q}, u) = 0$, so $\mathcal{Q}u$ is a subspace orthogonal to $\mathcal{Q}$.

From $\mathsf{n}(xu, 1) = \mathsf{n}(u, \overline{x}) \in \mathsf{n}(u, \mathcal{Q}) = 0$, it follows that $\overline{xu} = -xu$. But $\overline{xu} = \overline{u}\,\overline{x} = -u\overline{x}$, whence $xu = u\overline{x}$.

Now $x(yu) = -x(\overline{yu}) = -x(\overline{u}\,\overline{y})$ and this is equal to $u(\overline{x}\,\overline{y}) = u(\overline{yx}) = (yx)u$.

In a similar vein $(yu)x = -(y\overline{u})x = (y\overline{x})u$, and $(xu)(yu) = -(\overline{xu})(yu) = \overline{y}\big((xu)u\big) = \overline{y}(xu^2) = \alpha\overline{y}x$.

Cayley-Dickson doubling process

Therefore, the subspace $\mathcal{Q} \oplus \mathcal{Q}u$ is also a subalgebra, and the restriction of $\mathsf{n}(.,.)$ to it is nondegenerate. The multiplication and norm are given by:

$$(a + bu)(c + du) = (ac + \alpha\bar{d}b) + (da + b\bar{c})u,$$

$$\mathsf{n}(a + bu) = \mathsf{n}(a) - \alpha\mathsf{n}(b),$$

for all $a, b, c, d \in \mathcal{Q}$.

This is exactly what happens with the construction of $\mathbb{O}$ from $\mathbb{H}$!!

Cayley-Dickson doubling process

Moreover,

$$\mathsf{n}\big((a+bu)(c+du)\big) = \mathsf{n}(ac+\alpha\bar{d}b) - \alpha\mathsf{n}(da+b\bar{c}),$$

while on the other hand

$$\begin{aligned}\mathsf{n}(a+bu)\mathsf{n}(c+du) &= \big(\mathsf{n}(a)-\alpha\mathsf{n}(b)\big)\big(\mathsf{n}(c)-\alpha\mathsf{n}(d)\big)\\ &= \mathsf{n}(a)\mathsf{n}(c)+\alpha^2\mathsf{n}(b)\mathsf{n}(d)-\alpha\big(\mathsf{n}(d)\mathsf{n}(a)+\mathsf{n}(b)\mathsf{n}(c)\big)\\ &= \mathsf{n}(ac)+\alpha^2\mathsf{n}(\bar{d}b)-\alpha\big(\mathsf{n}(da)+\mathsf{n}(b\bar{c})\big).\end{aligned}$$

We conclude that $\mathsf{n}(ac,\alpha\bar{d}b)-\alpha\mathsf{n}(da,b\bar{c})=0$, or $\mathsf{n}\big(d(ac),b\big)=\mathsf{n}\big((da)c,b\big)$.

The nondegeneracy of the restriction of $\mathsf{n}(.,.)$ to $\mathcal{Q}$ implies then that $\mathcal{Q}$ is associative!
In particular, any proper subalgebra of $\mathcal{C}$ with nondegenerate restricted norm is associative.

Cayley-Dickson doubling process

Conversely, given an associative Hurwitz algebra $\mathcal{Q}$ with nondegenerate $\mathrm{n}(.,.)$, and a nonzero scalar $\alpha \in \mathbb{F}$, consider the direct sum of two copies of $\mathcal{Q}$: $\mathcal{C} = \mathcal{Q} \oplus \mathcal{Q}u$, with multiplication and norm given before.

The previous arguments show that $\mathcal{C}$ is again a Hurwitz algebra, which is said to be obtained by the **Cayley-Dickson doubling process** from $\mathcal{Q}$ and α.
This algebra is denoted by $\mathfrak{CD}(\mathcal{Q}, \alpha)$.

Remark

$\mathfrak{CD}(\mathcal{Q}, \alpha)$ is associative if and only if $\mathcal{Q}$ is commutative. This follows from $x(yu) = (yx)u$. If the algebra is associative this equals $(xy)u$, and it forces $xy = yx$ for al $x, y \in \mathcal{Q}$.
The converse is left as an exercise!

Generalized Hurwitz Theorem

Theorem (Generalized Hurwitz Theorem)

Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following:

1. *The ground field $\mathbb{F}$.*
2. *A two-dimensional separable commutative and associative algebra: $\mathcal{K} = \mathbb{F}1 \oplus \mathbb{F}v$, with $v^2 = v + \mu 1$, $\mu \in \mathbb{F}$ with $4\mu + 1 \neq 0$, and $\mathsf{n}(\epsilon + \delta v) = \epsilon^2 - \mu\delta^2 + \epsilon\delta$, for $\epsilon, \delta \in \mathbb{F}$.*
3. *A quaternion algebra $\mathcal{Q} = \mathfrak{CD}(\mathcal{K}, \beta)$ for $\mathcal{K}$ as in 2 and $0 \neq \beta \in \mathbb{F}$.*
4. *A Cayley (or octonion) algebra $\mathcal{C} = \mathfrak{CD}(\mathcal{Q}, \gamma)$, for $\mathcal{Q}$ as in 3 and $0 \neq \gamma \in \mathbb{F}$.*

In particular, the dimension of a Hurwitz algebra is finite and restricted to $1, 2, 4$ or 8.

Generalized Hurwitz Theorem. Proof

- The only Hurwitz algebra of dimension 1 is the ground field.
- If $\mathcal{C}$ is a Hurwitz algebra and $\dim_{\mathbb{F}} \mathcal{C} > 1$, there is an element $v \in \mathcal{C} \setminus \mathbb{F}1$ such that $\mathsf{n}(v, 1) = 1$ and $\mathsf{n}|_{\mathbb{F}1+\mathbb{F}v}$ is nondegenerate. The Cayley-Hamilton equation shows that $v^2 - v + \mathsf{n}(v)1 = 0$, so $v^2 = v + \mu 1$, with $\mu = -\mathsf{n}(v)$. The nondegeneracy condition is equivalent to the condition $4\mu + 1 \neq 0$. Then $\mathcal{K} = \mathbb{F}1 + \mathbb{F}v$ is a Hurwitz subalgebra of $\mathcal{C}$ and, if $\dim_{\mathbb{F}} \mathcal{C} = 2$, we are done.

Generalized Hurwitz Theorem. Proof

- If $\dim_{\mathbb{F}} \mathcal{C} > 2$ we may take an element $u \in \mathcal{K}^{\perp}$ with $\mathsf{n}(u) = -\beta \neq 0$, and hence the subspace $\mathcal{Q} = \mathcal{K} \oplus \mathcal{K}u$ is a subalgebra of $\mathcal{C}$ isomorphic to $\mathfrak{CD}(\mathcal{K}, \beta)$. By the previous remark, $\mathcal{Q}$ is associative (as $\mathcal{K}$ is commutative), but it fails to be commutative, as $vu = u\bar{v} \neq uv$. If $\dim_{\mathbb{F}} \mathcal{C} = 4$, we are done.

- Finally, if $\dim_{\mathbb{F}} \mathcal{C} > 4$, we may take an element $u' \in \mathcal{Q}^{\perp}$ with $\mathsf{n}(u') = -\gamma \neq 0$, and hence the subspace $\mathcal{Q} \oplus \mathcal{Q}u'$ is a subalgebra of $\mathcal{C}$ isomorphic to $\mathfrak{CD}(\mathcal{Q}, \gamma)$, which is not associative by the previous remark, so it is necessarily the whole $\mathcal{C}$.

Generalized Hurwitz Theorem. Characteristic not two

Note that if $\operatorname{char} \mathbb{F} \neq 2$, the restriction of $\mathsf{n}(.,.)$ to $\mathbb{F}1$ is nondegenerate, so we could have used the same argument in all steps.

Corollary

Every Hurwitz algebra over a field $\mathbb{F}$ of characteristic not 2 is isomorphic to one of the following:

1. *The ground field $\mathbb{F}$.*
2. *A two-dimensional algebra $\mathcal{K} = \mathfrak{CD}(\mathbb{F}, \alpha)$ for a nonzero scalar α.*
3. *A quaternion algebra $\mathcal{Q} = \mathfrak{CD}(\mathcal{K}, \beta)$ for $\mathcal{K}$ as in 2 and $0 \neq \beta \in \mathbb{F}$.*
4. *A Cayley (or octonion) algebra $\mathcal{C} = \mathfrak{CD}(\mathcal{Q}, \gamma)$, for $\mathcal{Q}$ as in 3 and $0 \neq \gamma \in \mathbb{F}$.*

Generalized Hurwitz Theorem. Real case

Over the real field $\mathbb{R}$, the scalars α, β and γ in the Corollary can be taken to be ± 1. Note that $\mathbb{C} \cong \mathfrak{CD}(\mathbb{R}, -1)$, $\mathbb{H} \cong \mathfrak{CD}(\mathbb{C}, -1)$ and $\mathbb{O} \cong \mathfrak{CD}(\mathbb{H}, -1)$.

Remark

Hurwitz (1898) only considered the real case with a positive definite norm. Over the years this was extended in several ways. The actual version of the Generalized Hurwitz Theorem seems to appear for the first time in a paper by Jacobson (1958) (if $\operatorname{char} \mathbb{F} \neq 2$) and van der Blij and Springer (1959).

Isomorphism problem

Proposition

Two Hurwitz algebras over a field are isomorphic if and only if their norms are isometric.

Proof

Any isomorphism of Hurwitz algebras is, in particular, an isometry of the corresponding norms, due to the Cayley-Hamilton equation. The converse follows from Witt's Cancellation Theorem.

1, 2, 4, 8, or infinity

Corollary

Let $\mathcal{C}$ be a finite-dimensional composition algebra. Then its dimension is either 1, 2, 4 or 8.

Proof

Let $a \in \mathcal{C}$ be an element of nonzero norm. Then $u = \frac{1}{\mathsf{n}(a)} a^2$ satisfies $\mathsf{n}(u) = 1$. Using the so called Kaplansky's trick, consider the new multiplication

$$x \diamond y = R_u^{-1}(x) L_u^{-1}(y).$$

We still have $\mathsf{n}(x \diamond y) = \mathsf{n}(x)\mathsf{n}(y)$, so $(\mathcal{C}, \diamond, \mathsf{n})$ is a composition algebra too. But $u^2 \diamond x = u L_u^{-1}(x) = x = x \diamond u^2$ for all x, so the element u^2 is the unity of $(\mathcal{C}, \diamond)$ and $(\mathcal{C}, \diamond)$ is a Hurwitz algebra, and hence $\dim_{\mathbb{F}} \mathcal{C}$ is restricted to $1, 2, 4$ or 8.

1, 2, 4, 8, or infinity

However, contrary to the thoughts expressed by Kaplansky in 1953, there are examples of infinite-dimensional composition algebras.

For example (Urbanik and Wright 1960), let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection (for instance, $\varphi(n,m) = 2^{n-1}(2m-1)$), and let $\mathcal{A}$ be a vector space over a field $\mathbb{F}$ of characteristic not 2 with a countable basis $\{u_n : n \in \mathbb{N}\}$. Define a multiplication and a norm on $\mathcal{A}$ by

$$u_n u_m = u_{\varphi(n,m)}, \qquad \mathsf{n}(u_n, u_m) = 2\delta_{n,m}.$$

Then $\mathcal{A}$ is a composition algebra.

There are examples of infinite-dimensional composition algebras of arbitrary infinite dimension, which are even left unital.
(E.–Pérez-Izquierdo 1997.)

1. Hurwitz algebras
2. Isotropic Hurwitz algebras
3. Symmetric composition algebras
4. Triality

Isotropic Hurwitz algebras

Assume now that the norm of a Hurwitz algebra $\mathcal{C}$ represents 0. That is, there is a nonzero element $a \in \mathcal{C}$ such that $\mathsf{n}(a) = 0$. This is always the case if $\dim_{\mathbb{F}} \mathcal{C} \geq 2$ and $\mathbb{F}$ is algebraically closed.

Goal

To check that, for each possible dimension 2, 4, or 8, there is just one such Hurwitz algebra, up to isomorphism.

Existence of orthogonal idempotents

Take $b \in \mathcal{C}$ such that $\mathsf{n}(a, \bar{b}) = 1$, so that $\mathsf{n}(ab, 1) = 1$. Also $\mathsf{n}(ab) = \mathsf{n}(a)\mathsf{n}(b) = 0$. By the Cayley-Hamilton equation, the nonzero element $e_1 := ab$ satisfies $e_1^2 = e_1$, that is, e_1 is an idempotent.

Consider too the idempotent $e_2 := 1 - e_1 = \overline{e_1}$, and the subalgebra $\mathcal{K} = \mathbb{F}e_1 \oplus \mathbb{F}e_2 \, (\cong \mathbb{F} \times \mathbb{F})$. $(1 = e_1 + e_2)$.

For any $x \in \mathcal{K}^\perp$,

$$xe_1 + \overline{xe_1} = \mathsf{n}(xe_1, 1)1 = \mathsf{n}(x, \overline{e_1})1 = \mathsf{n}(x, e_2)1 = 0$$

and, as $\bar{x} = -x$ and $\overline{e_1} = e_2$, we conclude that $xe_1 = e_2x$, and in the same way, $xe_2 = e_1x$, for any $x \in \mathcal{K}^\perp$.

Peirce decomposition

But $x = 1x = e_1x + e_2x$, and
$e_2(e_1x) = (1 - e_1)(e_1x) = 0 = e_1(e_2x)$.
It follows that $\mathcal{K}^\perp$ splits as

$$\mathcal{K}^\perp = \mathcal{U} \oplus \mathcal{V}$$

with

$$\mathcal{U} = \{x \in \mathcal{C} : e_1x = x = xe_2,\ e_2x = 0 = xe_1\},$$
$$\mathcal{V} = \{x \in \mathcal{C} : e_2x = x = xe_1,\ e_1x = 0 = xe_2\}.$$

Peirce decomposition

For any $u \in \mathcal{U}$, $\mathsf{n}(u) = \mathsf{n}(e_1 u) = \mathsf{n}(e_1)\mathsf{n}(u) = 0$, so $\mathcal{U}$, and $\mathcal{V}$ too, are totally isotropic subspaces of $\mathcal{K}^\perp$ paired by the norm.

In particular, $\dim_{\mathbb{F}} \mathcal{U} = \dim_{\mathbb{F}} \mathcal{V}$, and this common value is either 0, 1 or 3, depending on $\dim_{\mathbb{F}} \mathcal{C}$ being 2, 4 or 8.

The case of $\dim_{\mathbb{F}} \mathcal{U} = 0$ is trivial, and the case of $\dim_{\mathbb{F}} \mathcal{U} = 1$ is quite easy (and subsumed in the arguments below). Hence, let us assume that $\mathcal{C}$ is a Cayley algebra (dimension 8), so $\dim_{\mathbb{F}} \mathcal{U} = \dim_{\mathbb{F}} \mathcal{V} = 3$.

For any $u_1, u_2 \in \mathcal{U}$ and $v \in \mathcal{V}$, we get

$$\mathsf{n}(u_1 u_2, \mathcal{K}) \subseteq \mathsf{n}(u_1, \mathcal{K} u_2) \subseteq \mathsf{n}(\mathcal{U}, \mathcal{U}) = 0,$$
$$\mathsf{n}(u_1 u_2, v) = \mathsf{n}(u_1 u_2, e_2 v) = -\mathsf{n}(e_2 u_2, u_1 v) + \mathsf{n}(u_1, e_2)\mathsf{n}(u_2, v) = 0.$$

Hence $\mathcal{U}^2$ is orthogonal to both $\mathcal{K}$ and $\mathcal{V}$, so it must be contained in $\mathcal{V}$. Also $\mathcal{V}^2 \subseteq \mathcal{U}$.

Peirce decomposition

Besides,

$$\mathsf{n}(\mathcal{U},\mathcal{U}\mathcal{V}) \subseteq \mathsf{n}(\mathcal{U}^2,\mathcal{V}) \subseteq \mathsf{n}(\mathcal{V},\mathcal{V}) = 0 = \mathsf{n}(\mathcal{V},\mathcal{U}\mathcal{V}),$$

so that $\mathcal{U}\mathcal{V} \subseteq (\mathcal{U}+\mathcal{V})^\perp = \mathcal{K}$, and also $\mathcal{V}\mathcal{U} \subseteq \mathcal{K}$.

But for any $u \in \mathcal{U}$ and $v \in \mathcal{V}$, we have

$$\mathsf{n}(uv, e_2) = -\mathsf{n}(u, e_2 v) = -\mathsf{n}(u,v), \quad \mathsf{n}(uv, e_1) = -\mathsf{n}(u, e_1 v) = 0,$$

and the analogues for vu. We conclude that

$$uv = -\mathsf{n}(u,v)e_1, \quad vu = -\mathsf{n}(v,u)e_2.$$

Peirce decomposition

Now, for linearly independent elements $u_1, u_2 \in \mathcal{U}$, let $v \in \mathcal{V}$ with $\mathsf{n}(u_1, v) \neq 0 = \mathsf{n}(u_2, v)$. Then the alternative law gives $(u_1u_2)v = -(u_1v)u_2 + u_1(u_2v + vu_2) = -\mathsf{n}(u_1, v)u_2 \neq 0$, so that $u_1u_2 \neq 0$. In particular $\mathcal{U}^2 \neq 0$, and the same happens with $\mathcal{V}$.

Consider the trilinear map:

$$\begin{aligned} \mathcal{U} \times \mathcal{U} \times \mathcal{U} &\longrightarrow \mathbb{F} \\ (x, y, z) &\mapsto \mathsf{n}(xy, z). \end{aligned}$$

This is alternating because $x^2 = 0$ for any $x \in \mathcal{U}$ by the Cayley-Hamilton equation, and $\mathsf{n}(xy, y) = -\mathsf{n}(x, yy) = 0$. It is also nonzero, because $\mathcal{U}^2 \subseteq \mathcal{V}$, so that $\mathsf{n}(\mathcal{U}^2, \mathcal{U}) = \mathsf{n}(\mathcal{U}^2, \mathcal{C}) \neq 0$.

Peirce decomposition

Fix a basis $\{u_1, u_2, u_3\}$ of $\mathcal{U}$ with $\mathsf{n}(u_1u_2, u_3) = 1$ and take $v_1 := u_2u_3$, $v_2 := u_3u_1$, $v_3 := u_1u_2$. Then $\{v_1, v_2, v_3\}$ is the dual basis in $\mathcal{V}$ relative to the norm, and the multiplication of the basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ is completely determined.

For instance,
$v_1v_2 = v_1(u_3u_1) = -u_3(v_1u_1) = -u_3(-\mathsf{n}(v_1, u_1)e_2) = u_3$, ...

Multiplication table

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

Split Cayley algebra

The Cayley algebra with this multiplication table is called the **split Cayley algebra** and denoted by $\mathcal{C}_s(\mathbb{F})$. The subalgebra spanned by e_1, e_2, u_1, v_1 is isomorphic to the algebra $\mathrm{Mat}_2(\mathbb{F})$ of 2×2 matrices.

Theorem

There are, up to isomorphism, only three Hurwitz algebras with isotropic norm: $\mathbb{F} \times \mathbb{F}$, $\mathrm{Mat}_2(\mathbb{F})$, *and* $\mathcal{C}_s(\mathbb{F})$.

Real Hurwitz algebras

Corollary

The real Hurwitz algebras are, up to isomorphism, the following algebras:

- *the classical division algebras* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, *and* $\mathbb{O}$, *and*
- *the algebras* $\mathbb{R} \times \mathbb{R}$, $\mathrm{Mat}_2(\mathbb{R})$, *and* $\mathcal{C}_s(\mathbb{R})$.

Proof

It is enough to take into account that a nondegenerate quadratic form over $\mathbb{R}$ is either isotropic or definite. Hence the norm of a real Hurwitz algebra is either isotropic or positive definite (as $\mathsf{n}(1) = 1$).

Symmetric composition algebras

Definition

A composition algebra $(\mathcal{S}, *, \mathsf{n})$ is said to be a **symmetric composition algebra** if

$$\mathsf{n}(x * y, z) = \mathsf{n}(x, y * z)$$

for all $x, y, z \in \mathcal{S}$.

Theorem

Let $(\mathcal{S}, *, \mathsf{n})$ be a composition algebra. The following conditions are equivalent:

(a) $(\mathcal{S}, *, \mathsf{n})$ *is symmetric.*

(b) *For any $x, y \in \mathcal{S}$, $(x * y) * x = x * (y * x) = \mathsf{n}(x)y$.*

The dimension of any symmetric composition algebra is finite, and hence restricted to $1, 2, 4$, or 8.

Symmetric composition algebras

Sketch of proof

If $(\mathcal{S}, *, \mathsf{n})$ is symmetric, then for any $x, y, z \in \mathcal{S}$,

$$\mathsf{n}\big((x * y) * x, z\big) = \mathsf{n}(x * y, x * z) = \mathsf{n}(x)\mathsf{n}(y, z) = \mathsf{n}\big(\mathsf{n}(x)y, z\big)$$

so that $(x * y) * x - \mathsf{n}(x)y \in \mathcal{S}^{\perp}$. Also

$$\begin{aligned}
&\mathsf{n}\big((x * y) * x - \mathsf{n}(x)y\big) \\
&\quad = \mathsf{n}\big((x * y) * x\big) + \mathsf{n}(x)^2\mathsf{n}(y) - \mathsf{n}(x)\mathsf{n}\big((x * y) * x, y\big) \\
&\quad = 2\mathsf{n}(x)^2\mathsf{n}(y) - \mathsf{n}(x)\mathsf{n}(x * y, x * y) = 0,
\end{aligned}$$

whence (b), since n is nonsingular.

Symmetric composition algebras

Conversely, take $x, y, z \in \mathcal{S}$ with $\mathsf{n}(y) \neq 0$, so that L_y and R_y are bijective, and hence there is an element $z' \in \mathcal{S}$ with $z = z' * y$. Then:

$$\begin{aligned}\mathsf{n}(x * y, z) &= \mathsf{n}(x * y, z' * y) \\ &= \mathsf{n}(x, z')\mathsf{n}(y) = \mathsf{n}\big(x, y * (z' * y)\big) = \mathsf{n}(x, y * z).\end{aligned}$$

This proves (a) assuming $\mathsf{n}(y) \neq 0$, but any isotropic element is the sum of two non isotropic elements, so (a) follows.

Finally, Kaplansky's trick shows that $\mathcal{S}$ is finite-dimensional.

Para-Hurwitz algebras

Let $\mathcal{C}$ be a Hurwitz algebra and consider the composition algebra $(\mathcal{C}, \bullet, \mathsf{n})$ with the new product given by

$$x \bullet y := \overline{x}\,\overline{y}.$$

Then

$$\mathsf{n}(x \bullet y, z) = \mathsf{n}(\overline{x}\,\overline{y}, z) = \mathsf{n}(\overline{x}, zy) = \mathsf{n}(x, \overline{zy}) = \mathsf{n}(x, y \bullet z),$$

so that $(\mathcal{C}, \bullet, \mathsf{n})$ is a symmetric composition algebra, called a **para-Hurwitz algebra**.
(Note that $1 \bullet x = x \bullet 1 = \overline{x} = \mathsf{n}(x, 1)1 - x$ for any x: 1 is a **para-unit** of $(\mathcal{C}, \bullet, \mathsf{n})$.)

Okubo algebras

Assume $\operatorname{char}\mathbb{F} \neq 3$ (the case of $\operatorname{char}\mathbb{F} = 3$ requires a different definition), and let $\omega \in \mathbb{F}$ be a primitive cubic root of 1.
Let $\mathcal{A}$ be a central simple associative algebra of degree 3 with trace tr, and let $\mathcal{S} = \mathcal{A}_0 = \{x \in \mathcal{A} : \operatorname{tr}(x) = 0\}$.
For any $x \in \mathcal{S}$ the quadratic form $\frac{1}{2}\operatorname{tr}(x^2)$ makes sense even if $\operatorname{char}\mathbb{F} = 2$ (check this!).

Define now a multiplication and norm on $\mathcal{S}$ by:

$$x * y := \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\operatorname{tr}(xy)1,$$

$$\mathsf{n}(x) := -\frac{1}{2}\operatorname{tr}(x^2),$$

Okubo algebras

For any $x, y \in \mathcal{S}$:

$$\begin{aligned}
(x * y) * x &= \omega(x * y)x - \omega^2 x(x * y) - \frac{\omega - \omega^2}{3}\,\mathrm{tr}\big((x * y)x\big)1 \\
&= \omega^2 xyx - yx^2 - \frac{\omega^2 - 1}{3}\,\mathrm{tr}(xy)x \\
&\qquad - x^2 y + \omega xyx + \frac{1 - \omega}{3}\,\mathrm{tr}(xy)x \\
&\qquad\quad - \frac{\omega - \omega^2}{3}\,\mathrm{tr}\Big((\omega - \omega^2)x^2 y\Big)1 \quad (\mathrm{tr}(x) = 0) \\
&= -(x^2 y + yx^2 + xyx) + \mathrm{tr}(xy)x + \mathrm{tr}(x^2 y)1 \\
&\qquad ((\omega - \omega^2)^2 = -3).
\end{aligned}$$

Okubo algebras

But if $\operatorname{tr}(x) = 0$, then $x^3 - \frac{1}{2}\operatorname{tr}(x^2)x - \det(x)1 = 0$, so

$$x^2y + yx^2 + xyx - \big(\operatorname{tr}(xy)x + \frac{1}{2}\operatorname{tr}(x^2)y\big) \in \mathbb{F}1.$$

Since $(x * y) * x \in \mathcal{A}_0$, we have

$$(x * y) * x = -\frac{1}{2}\operatorname{tr}(x^2)y = x * (y * x).$$

Therefore $(\mathcal{S}, *, \mathsf{n})$ is a symmetric composition algebra, called an **Okubo algebra**.

Okubo algebras

In case $\omega \notin \mathbb{F}$, take $\mathbb{K} = \mathbb{F}[\omega]$ and a central simple associative algebra $\mathcal{A}$ of degree 3 over $\mathbb{K}$ endowed with a $\mathbb{K}/\mathbb{F}$-involution of second kind J.

Then take $\mathcal{S} = K(\mathcal{A}, J)_0 = \{x \in \mathcal{A}_0 : J(x) = -x\}$ (this is a $\mathbb{F}$-subspace) and use the same formulas above to define the multiplication and the norm.

The resulting algebras are termed too **Okubo algebras**.

Okubo algebras. Some remarks

For $\mathbb{F} = \mathbb{R}$, take $\mathcal{A} = \mathrm{Mat}_3(\mathbb{C})$, and then there appears the Okubo algebra $(\mathcal{S}, *, \mathsf{n})$ with $\mathcal{S} = \mathfrak{su}_3 = \{x \in \mathrm{Mat}_3(\mathbb{C}) : \mathrm{tr}(x) = 0,\ x^* = -x\}$ (x^* denotes the conjugate transpose of x). This algebra was termed the algebra of pseudo-octonions by Okubo (1978), who studied these algebras and classified them, under some restrictions, in joint work with Osborn.

The name **Okubo algebras** was given in 1990.

Faulkner (1988) discovered independently Okubo's construction in a more general setting, related to separable alternative algebras of degree 3, and gave the key idea for the classification of the symmetric composition algebras (E.-Myung 1993, $\mathrm{char}\,\mathbb{F} \neq 2, 3$). A different, less elegant, classification was given previously by the same authors (1991) using that Okubo algebras are Lie-admissible.

The name **symmetric composition algebras** was given in *The Book of Involutions*, by Knus et al. (1998).

Okubo algebras. Some remarks

Given an Okubo algebra, note that for any $x, y \in \mathcal{S}$,

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1,$$
$$y * x = \omega yx - \omega^2 xy - \frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1,$$

so that

$$\omega x * y + \omega^2 y * x = (\omega^2 - \omega)xy - (\omega + \omega^2)\frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1,$$

and

$$xy = \frac{\omega}{\omega^2 - \omega} x * y + \frac{\omega^2}{\omega^2 - \omega} y * x + \frac{1}{3}\mathsf{n}(x, y)1,$$

so the product in $\mathcal{A}$ is determined by the product in the Okubo algebra.

Also, as noted by Faulkner, the construction above is valid for separable alternative algebras of degree 3.

Symmetric composition algebras

Theorem (E.-Myung 1993)

Let $\mathbb{F}$ be a field of characteristic not 3.

- *If $\mathbb{F}$ contains a primitive cubic root ω of 1, then the symmetric composition algebras of dimension ≥ 2 are, up to isomorphism, the algebras $(\mathcal{A}_0, *, \mathsf{n})$ for $\mathcal{A}$ a separable alternative algebra of degree 3.*
  Two such symmetric composition algebras are isomorphic if and only if so are the corresponding alternative algebras.
- *If $\mathbb{F}$ does not contain primitive cubic roots of 1, then the symmetric composition algebras of dimension ≥ 2 are, up to isomorphism, the algebras $(K(\mathcal{A}, J)_0, *, \mathsf{n})$ for $\mathcal{A}$ a separable alternative algebra of degree 3.*
  Two such symmetric composition algebras are isomorphic if and only if so are the corresponding alternative algebras, as algebras with involution.

Symmetric composition algebras

Sketch of proof

We can go in the reverse direction of Okubo's construction. Given a symmetric composition algebra $(\mathcal{S}, *, \mathsf{n})$ over a field containing ω, define the algebra $\mathcal{A} = \mathbb{F}1 \oplus \mathcal{S}$ with multiplication determined by the formulas above. Then $\mathcal{A}$ turns out to be a separable alternative algebra of degree 3.

In case $\omega \notin \mathbb{F}$, then we must consider $\mathcal{A} = \mathbb{F}[\omega]1 \oplus \big(\mathbb{F}[\omega] \otimes \mathcal{S}\big)$, with the same formula for the product. In $\mathbb{F}[\omega]$ we have the Galois automorphism $\omega^\tau = \omega^2$. Then the conditions $J(1) = 1$ and $J(s) = -s$ for any $s \in \mathcal{S}$ induce a $\mathbb{F}[\omega]/\mathbb{F}$-involution of the second kind in $\mathcal{A}$.

Symmetric composition algebras

Corollary

Para-Hurwitz and Okubo algebras essentially exhaust, up to isomorphism, the symmetric composition algebras over a field $\mathbb{F}$ of characteristic not 3.

Sketch of proof:

Let ω be a primitive cubic root of 1 in an algebraic closure of $\mathbb{F}$, and let $\mathbb{K} = \mathbb{F}[\omega]$, so that $\mathbb{K} = \mathbb{F}$ if $\omega \in \mathbb{F}$.
A separable alternative algebra of degree 3 over $\mathbb{K}$ is, up to isomorphism, one of the following:

- a central simple associative algebra, and hence we obtain the Okubo algebras,

Symmetric composition algebras

- $\mathcal{A} = \mathbb{K} \times \mathcal{C}$ for a Hurwitz algebra $\mathcal{C}$, in which case $(\mathcal{A}_0, *, \mathrm{n})$ is shown to be isomorphic to the para-Hurwitz algebra attached to $\mathcal{C}$ if $\mathbb{K} = \mathbb{F}$, and $\big(K(\mathcal{A}, J)_0, *, \mathrm{n}\big)$ to the para-Hurwitz algebra attached to $\widehat{\mathcal{C}} = \{x \in \mathcal{C} : J(x) = \overline{x}\}$ if $\mathbb{K} \neq \mathbb{F}$,
- $\mathcal{A} = \mathbb{K} \otimes_{\mathbb{F}} \mathbb{L}$, for a cubic field extension $\mathbb{L}$ of $\mathbb{F}$ (if $\omega \notin \mathbb{F}$, $\mathbb{L} = \{x \in \mathcal{A} : J(x) = x\}$), in which case the symmetric composition algebra is shown to be a twisted form of a two-dimensional para-Hurwitz algebra.

Question

What about characteristic 3?

Symmetric composition algebras

Theorem (Petersson 1969)

Let $\mathbb{F}$ be an algebraically closed field of characteristic $\neq 2, 3$. Then any simple finite-dimensional algebra satisfying

$$(xy)x = x(yx), \quad ((xz)y)(xz) = (x((zy)z))x$$

for any x, y, z is, up to isomorphism, one of the following:

- *The algebra $(\mathcal{B}, \bullet)$, where $\mathcal{B}$ is a Hurwitz algebra and $x \bullet y = \overline{x}\,\overline{y}$ (that is, a para-Hurwitz algebra).*
- *The algebra $(\mathcal{C}_s(\mathbb{F}), *)$, where $\mathcal{C}_s(\mathbb{F})$ is the split Cayley algebra, and $x * y = \varphi(\overline{x})\varphi^2(\overline{y})$, where φ is a precise order 3 automorphism of $\mathcal{C}_s(\mathbb{F})$ given by*

  $$e_i \mapsto e_i,\ i = 1, 2, \quad u_j \mapsto \omega^{j-1}u_j,\ v_j \mapsto \omega^{1-j}v_j,\ j = 1, 2, 3,$$

  where ω is a primitive cubic root of 1.

Petersson algebras

Note that any symmetric composition algebra $(\mathcal{S}, *, \mathrm{n})$ satisfies the conditions in Petersson's result, so the unique, up to isomorphism, Okubo algebra over an algebraically closed field of characteristic $\neq 2, 3$ must be isomorphic to the last algebra in the Theorem above, and this seems to be the first appearance of these algebras in the literature!

Definition

Let $\mathcal{C}$ be a Hurwitz algebra, and let $\varphi \in \mathrm{Aut}(\mathcal{C}, \cdot, \mathrm{n})$ be an automorphism with $\varphi^3 = \mathrm{id}$. The composition algebra $(\mathcal{C}, *, \mathrm{n})$, with

$$x * y = \varphi(\overline{x})\varphi^2(\overline{y})$$

is called a **Petersson algebra**, and denoted by $\mathcal{C}_\varphi$.

Okubo algebras and Petersson algebras

Remark

In case $\varphi = \mathrm{id}$, the Petersson algebra is the para-Hurwitz algebra associated to $(\mathcal{C}, \cdot, \mathsf{n})$.

Consider the following order 3 automorphism φ of $\mathcal{C}_s(\mathbb{F})$ (this is not the one considered by Petersson!):

$$\varphi(e_i) = e_i,\ i = 1, 2, \quad \varphi(u_j) = u_{j+1},\ \varphi(v_j) = v_{j+1} \text{ (indices } j \text{ modulo } 3).$$

With this automorphism we may define Okubo algebras over arbitrary fields.

New definition of Okubo algebras

(valid in all characteristics)

Definition (E.–Pérez-Izquierdo 1996)

Let $\mathbb{F}$ be an arbitrary field. The Petersson algebra $\mathcal{C}_s(\mathbb{F})_\varphi$ is called the **split Okubo algebra** over $\mathbb{F}$.

Its twisted forms (i.e., those composition algebras $(\mathcal{S}, *, \mathrm{n})$ that become isomorphic to the split Okubo algebra after extending scalars to an algebraic closure) are called **Okubo algebras**.

Split Okubo algebra. Multiplication table

$*$	e_1	e_2	u_1	v_1	u_2	v_2	u_3	v_3
e_1	e_2	0	0	$-v_3$	0	$-v_1$	0	$-v_2$
e_2	0	e_1	$-u_3$	0	$-u_1$	0	$-u_2$	0
u_1	$-u_2$	0	v_1	0	$-v_3$	0	0	$-e_1$
v_1	0	$-v_2$	0	u_1	0	$-u_3$	$-e_2$	0
u_2	$-u_3$	0	0	$-e_1$	v_2	0	$-v_1$	0
v_2	0	$-v_3$	$-e_2$	0	0	u_2	0	$-u_1$
u_3	$-u_1$	0	$-v_2$	0	0	$-e_1$	v_3	0
v_3	0	$-v_1$	0	$-u_2$	$-e_2$	0	0	u_3

Okubo algebras. Some remarks

Over fields of characteristic $\neq 3$, our new definition of Okubo algebras coincide with the previous one. Okubo and Osborn gave in 1981 an ad hoc definition of the Okubo algebra over an algebraically closed field of characteristic 3.

The split Okubo algebra does not contain any nonzero element that commutes with every other element, that is, its commutative center is trivial. This is not so for the para-Hurwitz algebras, where the para-unit lies in the commutative center.

Okubo algebras. Characteristic three

Let $\mathbb{F}$ be a field of characteristic 3 and let $0 \neq \alpha, \beta \in \mathbb{F}$. Consider the elements

$$e_1 \otimes \alpha^{1/3}, \quad u_1 \otimes \beta^{1/3}$$

in $\mathcal{C}_s(\mathbb{F}) \otimes_{\mathbb{F}} \overline{\mathbb{F}}$ ($\overline{\mathbb{F}}$ being an algebraic closure of $\mathbb{F}$).

These elements generate, by multiplication and linear combinations over $\mathbb{F}$, a twisted form of the split Okubo algebra $(\mathcal{C}_s(\mathbb{F}), *, \mathsf{n})$. Denote by $\mathcal{O}_{\alpha,\beta}$ this twisted form.

Okubo algebras. Classification in characteristic three

Theorem (E. 1997)

Any symmetric composition algebra $(\mathcal{S}, *, \mathsf{n})$ *over a field* $\mathbb{F}$ *of characteristic* 3 *is either:*

- *A para-Hurwitz algebra. Two such algebras are isomorphic if and only if so are the associated Hurwitz algebras.*
- *A two-dimensional algebra with a basis* $\{u, v\}$ *and multiplication given by*

$$u * u = v, \quad u * v = v * u = u, \quad v * v = \lambda u - v,$$

  for a nonzero scalar $\lambda \in \mathbb{F} \setminus \mathbb{F}^3$*.*
  These algebras do not contain idempotents and are twisted forms of the para-Hurwitz algebras.
  Algebras corresponding to the scalars λ *and* λ' *are isomorphic if and only if* $\mathbb{F}^3\lambda + \mathbb{F}^3(\lambda^2 + 1) = \mathbb{F}^3\lambda' + \mathbb{F}^3((\lambda')^2 + 1)$*.*

Okubo algebras. Classification in characteristic three

Theorem (continued)

- *Isomorphic to* $\mathcal{O}_{\alpha,\beta}$ *for some* $0 \neq \alpha, \beta \in \mathbb{F}$.
  Moreover, $\mathcal{O}_{\alpha,\beta}$ *is isomorphic or antiisomorphic to* $\mathcal{O}_{\gamma,\delta}$ *if and only if*
  $\mathrm{span}_{\mathbb{F}^3}\left\{\alpha^{\pm 1}, \beta^{\pm 1}, \alpha^{\pm 1}\beta^{\pm 1}\right\} = \mathrm{span}_{\mathbb{F}^3}\left\{\gamma^{\pm 1}, \delta^{\pm 1}, \gamma^{\pm 1}\delta^{\pm 1}\right\}$.

Okubo algebras. Classification in characteristic three

The precise statement for the isomorphism condition in the last item is more technical.

A key point in the proof of this Theorem is the study of idempotents on Okubo algebras.
If there are nonzero idempotents, then these algebras are Petersson algebras.
The most difficult case appears in the absence of idempotents.
This is only possible if the ground field $\mathbb{F}$ is not perfect.

Triality

The importance of symmetric composition algebras lies in their connections with the phenomenon of triality in dimension 8, related to the fact that the Dynkin diagram D_4 is the most symmetric one.

Let $(\mathcal{S}, *, \mathsf{n})$ be an eight-dimensional symmetric composition algebra over a field $\mathbb{F}$, that is, $\mathcal{S}$ is either a para-Hurwitz algebra or an Okubo algebra.

Write $L_x(y) = x * y = R_y(x)$. Then

$$L_x R_x = R_x L_x = \mathsf{n}(x)\mathrm{id}$$

for all $x \in \mathcal{S}$ so that, inside $\mathrm{End}_{\mathbb{F}}(\mathcal{S} \oplus \mathcal{S}) \simeq \mathrm{Mat}_2\big(\mathrm{End}_{\mathbb{F}}(\mathcal{S})\big)$, we have

$$\begin{pmatrix} 0 & L_x \\ R_x & 0 \end{pmatrix}^2 = \mathsf{n}(x)\mathrm{id}.$$

Therefore, the map

$$x \mapsto \begin{pmatrix} 0 & L_x \\ R_x & 0 \end{pmatrix}$$

extends to an isomorphism of associative algebras with involution:

$$\Phi : (\mathfrak{Cl}(\mathcal{S}, \mathsf{n}), \tau) \longrightarrow (\mathrm{End}_{\mathbb{F}}(\mathcal{S} \oplus \mathcal{S}), \sigma_{n \perp n})$$

where $\mathfrak{Cl}(\mathcal{S}, \mathsf{n})$ is the Clifford algebra on the quadratic space $(\mathcal{S}, \mathsf{n})$, τ is its canonical involution ($\tau(x) = x$ for any $x \in \mathcal{S}$), and $\sigma_{n \perp n}$ is the orthogonal involution on $\mathrm{End}_{\mathbb{F}}(\mathcal{S} \oplus \mathcal{S})$ induced by the quadratic form where the two copies of $\mathcal{S}$ are orthogonal and the restriction on each copy coincides with the norm. The multiplication in the Clifford algebra will be denoted by $x \cdot y$.

Natural and half-spin representations

Consider the **spin group**:

$\mathrm{Spin}(\mathcal{S}, n) = \{u \in \mathfrak{Cl}(\mathcal{S}, \mathsf{n})^{\times}_{\bar{0}} : u{\cdot}x{\cdot}u^{-1} \in \mathcal{S},\ u{\cdot}\tau(u) = 1,\ \forall x \in \mathcal{C}\}.$

For any $u \in \mathrm{Spin}(\mathcal{S}, n)$,

$$\Phi(u) = \begin{pmatrix} \rho_u^- & 0 \\ 0 & \rho_u^+ \end{pmatrix}$$

for some $\rho_u^\pm \in \mathrm{O}(\mathcal{S}, n)$ such that

$$\chi_u(x * y) = \rho_u^+(x) * \rho_u^-(y)$$

for any $x, y \in \mathcal{S}$, where $\chi_u(x) = u \cdot x \cdot u^{-1}$ gives the natural representation of $\mathrm{Spin}(\mathcal{S}, \mathsf{n})$, while $\rho_u^{\pm 1}$ give the two half-spin representations.

The formula above links the natural and the two half-spin representations!!

Triality on the spin group

The last condition is equivalent to:

$$\langle \chi_u(x), \rho_u^+(y), \rho_u^-(z)\rangle = \langle x, y, z\rangle$$

for any $x, y, z \in \mathcal{S}$, where $\langle x, y, z\rangle = \mathsf{n}(x, y * z)$, and this has cyclic symmetry:

$$\langle x, y, z\rangle = \langle y, z, x\rangle.$$

Theorem

Let $(\mathcal{S}, *, \mathsf{n})$ be an eight-dimensional symmetric composition algebra. Then:

$$\mathrm{Spin}(\mathcal{S}, \mathsf{n}) \simeq \{(f_0, f_1, f_2) \in \mathrm{O}^+(\mathcal{S}, n)^3 \mid f_0(x * y) = f_1(x) * f_2(y)\ \forall x, y \in \mathcal{S}\}.$$

Moreover, the set of related triples has cyclic symmetry.

Triality on the spin group

The cyclic symmetry on the right hand side induces an outer automorphism of order 3 (trialitarian automorphism) of $\mathrm{Spin}(\mathcal{S}, \mathsf{n})$.

Its fixed subgroup is the group of automorphisms of the symmetric composition algebra $(\mathcal{S}, *, \mathsf{n})$, which is a simple algebraic group of type G_2 in the para-Hurwitz case, and of type A_2 in the Okubo case if $\mathrm{char}\,\mathbb{F} \neq 3$.

The group(-scheme) of automorphisms of an Okubo algebra over a field of characteristic 3 is not smooth!!

Local triality

At the Lie algebra level, assume $\mathrm{char}\,\mathbb{F} \neq 2$, and consider the associated orthogonal Lie algebra

$$\mathfrak{so}(\mathcal{S}, \mathsf{n}) = \{d \in \mathrm{End}_{\mathbb{F}}(\mathcal{S}) : \mathsf{n}\big(d(x), y\big) + \mathsf{n}\big(x, d(y)\big) = 0\ \forall x, y \in \mathcal{S}\}.$$

The **triality Lie algebra** of $(\mathcal{S}, *, \mathsf{n})$ is defined as the following Lie subalgebra of $\mathfrak{so}(\mathcal{S}, \mathsf{n})^3$ (with componentwise bracket):

$$\begin{aligned}\mathfrak{tri}(\mathcal{S}, *, \mathsf{n}) = \{&(d_0, d_1, d_2) \in \mathfrak{so}(\mathcal{S}, \mathsf{n})^3 \mid \\ &d_0(x * y) = d_1(x) * y + x * d_2(y)\ \forall x, y, z \in \mathcal{S}\}.\end{aligned}$$

Local triality

The condition $d_0(x*y) = d_1(x)*y + x*d_2(y)$ for all $x, y \in \mathcal{S}$ is equivalent to the condition

$$\mathsf{n}\big(x*y, d_0(z)\big) + \mathsf{n}\big(d_1(x)*y, z\big) + \mathsf{n}\big(x*d_2(y), z\big) = 0,$$

for any $x, y, z \in \mathcal{S}$.

But $\mathsf{n}(x*y, z) = \mathsf{n}(y*z, x) = \mathsf{n}(z*x, y)$. Therefore, the linear map:

$$\begin{aligned} \theta : \mathfrak{tri}(\mathcal{S}, *, \mathsf{n}) &\longrightarrow \mathfrak{tri}(\mathcal{S}, *, \mathsf{n}) \\ (d_0, d_1, d_2) &\mapsto (d_2, d_0, d_1), \end{aligned}$$

is an automorphism of the Lie algebra $\mathfrak{tri}(\mathcal{S}, *, \mathsf{n})$.

Local triality

Theorem

Let $(\mathcal{S}, *, \mathsf{n})$ be an eight-dimensional symmetric composition algebra over a field of characteristic $\neq 2$. Then:

- **Principle of Local Triality:** *The projection map:*

$$\pi_0 : \mathfrak{tri}(\mathcal{S}, *, \mathsf{n}) \longrightarrow \mathfrak{so}(\mathcal{S}, \mathsf{n})$$
$$(d_0, d_1, d_2) \mapsto d_0$$

  is an isomorphism of Lie algebras.

Local triality

Theorem (continued)

- *For any $x, y \in \mathcal{S}$, the triple*

$$t_{x,y} = \Big(\sigma_{x,y} = \mathsf{n}(x,.)y - \mathsf{n}(y,.)x, \\ \frac{1}{2}\mathsf{n}(x,y)\mathrm{id} - R_x L_y, \frac{1}{2}\mathsf{n}(x,y)\mathrm{id} - L_x R_y\Big)$$

belongs to $\mathfrak{tri}(\mathcal{S}, *, \mathsf{n})$*, and* $\mathfrak{tri}(\mathcal{S}, *, \mathsf{n})$ *is spanned by these elements. Moreover, for any $a, b, x, y \in \mathcal{S}$:*

$$[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}.$$

Freudenthal's Magic Square

Given two symmetric composition algebras $(\mathcal{S}, *, \mathsf{n})$ and $(\mathcal{S}', \star, \mathsf{n}')$, consider the vector space:

$$\mathfrak{g} = \mathfrak{g}(\mathcal{S}, \mathcal{S}') = \big(\mathfrak{tri}(\mathcal{S}) \oplus \mathfrak{tri}(\mathcal{S}')\big) \oplus \Big(\oplus_{i=0}^{2} \iota_i(\mathcal{S} \otimes \mathcal{S}')\Big),$$

where $\iota_i(\mathcal{S} \otimes \mathcal{S}')$ is just a copy of $\mathcal{S} \otimes \mathcal{S}'$ $(i = 0, 1, 2)$ and we write $\mathfrak{tri}(\mathcal{S})$, $\mathfrak{tri}(\mathcal{S}')$ instead of $\mathfrak{tri}(\mathcal{S}, *, \mathsf{n})$ and $\mathfrak{tri}(\mathcal{S}', \star, \mathsf{n}')$ for short.

Freudenthal's Magic Square

Define now an anticommutative bracket on $\mathfrak{g}$ by means of:

- the Lie bracket in $\mathfrak{tri}(\mathcal{S}) \oplus \mathfrak{tri}(\mathcal{S}')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d_0', d_1', d_2'), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d_i'(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' \star y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = \mathsf{n}'(x', y')\theta^i(t_{x,y}) + \mathsf{n}(x, y)\theta'^i(t'_{x',y'}) \in \mathfrak{tri}(\mathcal{S}) \oplus \mathfrak{tri}(\mathcal{S}')$.

Freudenthal's Magic Square

Theorem

Assume $\operatorname{char}\mathbb{F} \neq 2,3$. *With the bracket above,* $\mathfrak{g}(\mathcal{S},\mathcal{S}')$ *is a Lie algebra and, if* $\mathcal{S}_r$ *and* $\mathcal{S}'_s$ *denote symmetric composition algebras of dimension* r *and* s*, then the Lie algebra* $\mathfrak{g}(\mathcal{S}_r,\mathcal{S}'_s)$ *is a (semi)simple Lie algebra whose type is given by Freudenthal's Magic Square:*

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_4$	$\mathcal{S}_8$
$\mathcal{S}'_1$	A_1	A_2	C_3	F_4
$\mathcal{S}'_2$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathcal{S}'_4$	C_3	A_5	D_6	E_7
$\mathcal{S}'_8$	F_4	E_6	E_7	E_8

Freudenthal's Magic Square

Different versions of this result using Hurwitz algebras instead of symmetric composition algebras have appeared over the years.

The advantage of using symmetric composition algebras is that new constructions of the exceptional simple Lie algebras are obtained, and these constructions highlights interesting symmetries due to the different triality automorphisms.

A few changes are needed for characteristic 3. Also, quite interestingly, over fields of characteristic 3 there are nontrivial symmetric composition superalgebras, and these can be plugged into the previous construction to obtain an extended Freudenthal Magic Square that includes some new simple finite dimensional Lie superalgebras (Cunha–E. 2006).

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Thank you!