The extended Freudenthal magic square via tensor categories

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(Based on joint work with
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Symplectic triple systems

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, containing a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{F})$ that admits a $\mathbb{Z}/2$-grading of the form:

$$\mathfrak{g} = \big(\mathfrak{sl}_2(\mathbb{F}) \oplus \mathfrak{d}\big) \oplus \big(\mathbb{F}^2 \otimes T\big).$$

In this case, T becomes a so-called symplectic triple system, and the bracket of odd elements works as follows:

$$[u \otimes x, v \otimes y] = (x \mid y)\gamma_{u,v} + \langle u \mid v \rangle d_{x,y}$$

for all $u, v \in \mathbb{F}^2$ and $x, y \in T$, for a skew-symmetric bilinear form $(\cdot \mid \cdot)$ on T and a symmetric bilinear map $T \times T \to \mathfrak{d}$, $(x, y) \mapsto d_{x,y}$; where $\langle u \mid v \rangle$ is, up to scalars, the unique $\mathfrak{sl}_2(\mathbb{F})$-invariant bilinear form on $\mathbb{F}^2$, and $\gamma_{u,v} = \langle u \mid \cdot \rangle v + \langle v \mid \cdot \rangle u$.

Symplectic triple systems and Lie superalgebras

It was realized (E. 2006) that, in case the characteristic of $\mathbb{F}$ is 3, then the $\mathbb{Z}/2$-graded vector space $\mathfrak{d} \oplus T$, with bracket given by the bracket in $\mathfrak{d}$, the action of $\mathfrak{d}$ in T, and by $[x, y] = d_{x,y}$ for $x, y \in T$, endows $\mathfrak{d} \oplus T$ with a structure of Lie superalgebra.

This led to the construction of a family of new simple contragredient Lie superalgebras specific of characteristic 3.

A surprising generalization

Arun S. Kannan has considered recently (2022) a much more general and surprising way of passing from Lie algebras to Lie superalgebras.

Kannan considered, over fields of characteristic 3, exceptional simple Lie algebras endowed with a nilpotent derivation d with $d^3 = 0$.
In the situation in the previous slide, one may take d equal to the adjoint action of $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{F})$.

This allows to view the Lie algebra as a Lie algebra in the category Rep $\boldsymbol{\alpha}_3$ of representations of the affine group scheme

$$\boldsymbol{\alpha}_3 : R \mapsto \{r \in R \mid r^3 = 0\}$$

(that is, the kernel of the Frobenius endomorphism of the additive group scheme $\mathbb{G}_a$).

The semisimplification of Rep $\boldsymbol{\alpha}_3$ is the Verlinde category Ver_3, which is equivalent to the category of vector superspaces, obtaining in this way a path from Lie algebras to Lie superalgebras.

Outline

1. Symmetric tensor categories
2. Semisimplification
3. From algebras to superalgebras in characteristic 3
4. From octonions to composition superalgebras
5. The extended Freudenthal magic square

The extended Freudenthal magic square via tensor categories

Monoidal categories

A monoidal category is a category $\mathfrak{C}$ with a bifunctor

$$\otimes : \mathfrak{C} \times \mathfrak{C} \to \mathfrak{C}$$

such that:

- There is a unit object $\mathbf{1}$ with natural isomorphisms (unitors)

$$X \otimes \mathbf{1} \simeq X \simeq \mathbf{1} \otimes X.$$

- There are natural isomorphisms (associators)

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z).$$

- Natural coherence conditions for the unitors and associators hold.

Monoidal functors

A functor $F : \mathfrak{C} \to \mathfrak{D}$ between monoidal categories is a monoidal functor if $F(\mathbf{1}) \simeq \mathbf{1}$ and there are natural isomorphisms

$$J_{X,Y} : F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$$

with natural coherence conditions with associators.

Symmetric monoidal categories

A braiding in a monoidal category $\mathfrak{C}$ is a natural isomorphism

$$c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$$

satisfying natural compatibility conditions with unitors and associators.

A symmetric monoidal category is a monoidal category endowed with a symmetric braiding: $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$.

Rigid symmetric monoidal categories

A symmetric monoidal category is rigid if every object X has a dual object X^* with

- an evaluation $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$,
- a coevaluation $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$,

such that the following compositions are the identity morphisms:

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^* \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$
$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

(Unitor and associator morphisms are omitted.)

Symmetric tensor categories

A symmetric tensor category $\mathfrak{C}$ over a field $\mathbb{F}$ is a rigid symmetric monoidal category with the following extra properties:

- It is abelian and even more: it is $\mathbb{F}$-linear and $\otimes$ is 'bilinear'.
- It is locally finite: objects have 'finite length' and morphism spaces are finite-dimensional.
- $\mathrm{End}_{\mathfrak{C}}(\mathbf{1}) = \mathbb{F}\,\mathrm{id}_{\mathbf{1}}$.

Examples

$\mathsf{Vec}_{\mathbb{F}}$: The category of finite-dimensional vector spaces.

$\mathsf{Rep}H$: The category of finite-dimensional representations of a triangular Hopf algebra.

$\mathsf{Rep}\,G$: The category of finite-dimensional representations of an affine group scheme.

$\mathsf{sVec}_{\mathbb{F}}$: The category of finite-dimensional vector superspaces.

Algebras in a symmetric tensor category

An algebra in a symmetric tensor category $\mathfrak{C}$ is an object $\mathcal{A}$ endowed with a morphism $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$.

The algebra $(\mathcal{A}, \mu)$ is

- commutative if $\mu \circ c_{\mathcal{A},\mathcal{A}} = \mu$,
- associative if $\mu \circ (\mu \otimes \mathrm{id}_{\mathcal{A}}) = \mu \circ (\mathrm{id}_{\mathcal{A}} \otimes \mu)$ (associator morphisms are omitted),
- Lie if it is anticommutative: $\mu \circ c_{\mathcal{A},\mathcal{A}} = -\mu$, and

$$\mu \circ (\mu \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}} + c_{\mathcal{A}\otimes\mathcal{A},\mathcal{A}} + c_{\mathcal{A},\mathcal{A}\otimes\mathcal{A}}) = 0,$$

- Jordan if ....
- ....

Superalgebras are algebras in $\mathrm{sVec}_{\mathbb{F}}$.

The extended Freudenthal magic square via tensor categories

Traces in symmetric tensor categories

Given a morphism $f \in \mathrm{End}_{\mathfrak{C}}(X)$ in a symmetric tensor category, its trace $\mathrm{tr}_X(f)$ is the following element in $\mathrm{End}_{\mathfrak{C}}(\mathbf{1}) \simeq \mathbb{F}$:

$$\mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}_{X^*}} X \otimes X^* \xrightarrow{c_{X,X^*}} X^* \otimes X \xrightarrow{\mathrm{ev}_X} \mathbf{1}$$

The dimension of an object X is $\dim_{\mathfrak{C}}(X) := \mathrm{tr}_X(\mathrm{id}_X)$.

Negligible morphisms

A morphism $f \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ in a symmetric tensor category is said to be negligible if

$$\mathrm{tr}_Y(f \circ g) = 0 \quad \text{for all} \quad g \in \mathrm{Hom}_{\mathfrak{C}}(Y, X).$$

Denote by $\mathcal{N}(X, Y)$ be the subspace of negligible morphims in $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$.

The subspaces $\mathcal{N}(X, Y)$ form a tensor ideal.

Semisimplification of a symmetric tensor category

This means that we can define a new category $\mathfrak{C}^{ss}$ with the same objects as $\mathfrak{C}$, but with morphisms given by the quotient with the subspace of negligible morphisms:

$$\mathrm{Hom}_{\mathfrak{C}^{ss}}(X, Y) := \mathrm{Hom}_{\mathfrak{C}}(X, Y)/\mathcal{N}(X, Y).$$

$\mathfrak{C}^{ss}$ is called the semisimplification of $\mathfrak{C}$.

The natural functor $S : \mathfrak{C} \to \mathfrak{C}^{ss}$ which is the identity on objects, and sends any morphism to its class modulo negligible morphisms is a braided, monoidal, $\mathbb{F}$-linear functor.

Semisimplification

The semisimplification $\mathfrak{C}^{ss}$ is semisimple: any object is a direct sum of finitely many simple objects.

The simple objects in $\mathfrak{C}^{ss}$ correspond to the indecomposable objects in $\mathfrak{C}$ of nonzero dimension.

Any object in $\mathfrak{C}$ with $\dim_{\mathfrak{C}}(X) = 0$ becomes isomorphic to the zero object in $\mathfrak{C}^{ss}$.

Verlinde category

Definition

Let $\mathbb{F}$ be a field of characteristic $p > 0$ and let $\mathsf{Rep}\,\mathsf{C}_p$ be the category of finite-dimensional representations of the cyclic group of order p (or of the associated constant group scheme).
This is a symmetric tensor category and its semisimplification is called the Verlinde category Ver_p.

The Verlinde category Ver_p also appears as the semisimplification of $\mathsf{Rep}\,\boldsymbol{\alpha}_p$.

The extended Freudenthal magic square via tensor categories

Semisimplification of $\mathrm{Rep}\,\mathsf{C}_3$ $(\mathrm{char}\,\mathbb{F} = 3)$

Fix a generator σ of C_3.

The indecomposable objects in $\mathrm{Rep}\,\mathsf{C}_3$ are, up to isomorphism,

$$V_0 = \mathbb{F}, \qquad V_1 = \mathbb{F}v_0 + \mathbb{F}v_1, \qquad V_2 = \mathbb{F}w_0 + \mathbb{F}w_1 + \mathbb{F}w_2,$$

where the action of σ is trivial on V_0, and

$$\sigma(v_0) = v_0 + v_1,\ \sigma(v_1) = v_1;\ \sigma(w_0) = w_0 + w_1,\ \sigma(w_1) = w_1 + w_2,\ \sigma(w_2) = w_2.$$

Any object $\mathcal{A}$ in $\mathrm{Rep}\,\mathsf{C}_3$ decomposes, nonuniquely, as

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2,$$

where $\mathcal{A}_i$ is a direct sum of copies of V_i, $i = 0, 1, 2$.

Semisimplification of $\mathrm{Rep}\,\mathsf{C}_3$ $(\mathrm{char}\,\mathbb{F} = 3)$. Properties

- $\mathrm{End}_{\mathsf{Ver}_3}(V_i) = \mathbb{F}[\mathrm{id}_{V_i}] \neq 0$ for $i = 0, 1$, $\mathrm{End}_{\mathsf{Ver}_3}(V_2) = 0$, $\mathrm{Hom}_{\mathsf{Ver}_3}(V_i, V_j) = 0$ for $i \neq j$.

- V_0 and V_1 are simple objects in Ver_3, while V_2 is isomorphic to 0.

- Ver_3 is semisimple: any object is isomorphic to a direct sum of copies of V_0 and V_1.

- $V_0 \otimes V_i$ and $V_i \otimes V_0$ are isomorphic to V_i, for $i = 0, 1$, both in $\mathrm{Rep}\,\mathsf{C}_3$ and in Ver_3; while $V_1 \otimes V_1$ is isomorphic to V_0 in Ver_3.

- The braiding in Ver_3, for objects X, Y, is given by $[c_{X,Y}]$, where $c_{X,Y}$ is the (swap) braiding in $\mathrm{Rep}\,\mathsf{C}_3$. Then, identifying $V_0 \otimes V_0 \simeq V_0$, $V_0 \otimes V_1 \simeq V_1 \simeq V_1 \otimes V_0$, and $V_1 \otimes V_1 \simeq V_0$, we have

$$[c_{V_0,V_0}] = [\mathrm{id}_{V_0}], \quad [c_{V_0,V_1}] = [\mathrm{id}_{V_1}] = [c_{V_1,V_0}], \quad [c_{V_1,V_1}] = -[\mathrm{id}_{V_0}].$$

$\mathsf{Ver}_3 \simeq \mathsf{sVec}_{\mathbb{F}}$

The categories $\mathsf{sVec}_{\mathbb{F}}$ and Ver_3 are equivalent through the ($\mathbb{F}$-linear braided monoidal) functor

$$\begin{aligned} F : \mathsf{sVec}_{\mathbb{F}} &\longrightarrow \mathsf{Ver}_3 \\ X_{\bar 0} \oplus X_{\bar 1} &\mapsto X_{\bar 0} \oplus (X_{\bar 1} \otimes V_1) \\ f_{\bar 0} \oplus f_{\bar 1} &\mapsto [f_{\bar 0} \oplus (f_{\bar 1} \otimes \mathrm{id}_{V_1})], \end{aligned}$$

F is a monoidal functor with natural isomorphism

$$J : F(\cdot) \otimes F(\cdot) \to F(\cdot \otimes \cdot)$$

given by $J_{X,Y} = [j_{X,Y}]$, where $j_{X,Y}$ is the following morphism in $\mathsf{Rep}\,\mathsf{C}_3$:

$\mathsf{Ver}_3 \simeq \mathsf{sVec}_{\mathbb{F}}$

$$
\begin{aligned}
j_{X,Y} : \Big(X_{\bar 0} \oplus (X_{\bar 1} \otimes V_1)\Big) \otimes \Big(Y_{\bar 0} \oplus (Y_{\bar 1} \otimes V_1)\Big) \longrightarrow & \\
\big((X_{\bar 0} \otimes Y_{\bar 0}) \oplus (X_{\bar 1} \otimes Y_{\bar 1})\big) \oplus \Big(\big((X_{\bar 0} \otimes Y_{\bar 1}) \oplus (X_{\bar 1} \otimes Y_{\bar 0})\big) \otimes V_1\Big) & \\
x_{\bar 0} \otimes y_{\bar 0} \mapsto x_{\bar 0} \otimes y_{\bar 0}, & \\
x_{\bar 0} \otimes (y_{\bar 1} \otimes v) \mapsto (x_{\bar 0} \otimes y_{\bar 1}) \otimes v, & \\
(x_{\bar 1} \otimes v) \otimes y_{\bar 0} \mapsto (x_{\bar 1} \otimes y_{\bar 0}) \otimes v, & \\
(x_{\bar 1} \otimes u) \otimes (y_{\bar 1} \otimes v) \mapsto \lambda(u \otimes v) x_{\bar 1} \otimes y_{\bar 1}, &
\end{aligned}
$$

where λ sends symmetric tensors to 0, and $v_0 \otimes v_1$ to 1.

From algebras in Rep C_3 to superalgebras

If $(\mathcal{A}, \mu)$ is an algebra in Rep C_3 (i.e., **an algebra endowed with an automorphism of order** 3), then $(\mathcal{A}, [\mu])$ is an algebra in Ver_3 and, up to isomorphism, there is a unique superalgebra $(A = A_{\bar{0}} \oplus A_{\bar{1}}, m)$ such that $F(A)$, with the product given by

$$F(A) \otimes F(A) \xrightarrow{J_{A,A}} F(A \otimes A) \xrightarrow{F(m)} F(A)$$

is isomorphic to $(\mathcal{A}, [\mu])$.

To obtain this superalgebra, fix a splitting

$$\mathcal{A} = A_{\bar{0}} \oplus A_{\bar{1}} \oplus (\sigma - \mathrm{id})(A_{\bar{1}}) \oplus \mathcal{A}_2$$

where

- $A_{\bar{0}}$ is a direct sum of copies of V_0,
- $A_{\bar{1}} \oplus (\sigma - \mathrm{id})(A_{\bar{1}})$ is a direct sum of copies of V_1,
- $\mathcal{A}_2$ is a direct sum of copies of V_2, and hence trivial in Ver_3.

From algebras in Rep C_3 to superalgebras

Recipe

Take projections relative to this splitting, and define a multiplication m on $A := A_{\bar{0}} \oplus A_{\bar{1}}$ as follows:

$$
\begin{aligned}
m(x_{\bar{0}} \otimes y_{\bar{0}}) &= \operatorname{proj}_{A_{\bar{0}}} \mu(x_{\bar{0}} \otimes y_{\bar{0}}) \\
m(x_{\bar{0}} \otimes y_{\bar{1}}) &= \operatorname{proj}_{A_{\bar{1}}} \mu(x_{\bar{0}} \otimes y_{\bar{1}}) \\
m(x_{\bar{1}} \otimes y_{\bar{0}}) &= \operatorname{proj}_{A_{\bar{1}}} \mu(x_{\bar{1}} \otimes y_{\bar{0}}) \\
m(x_{\bar{1}} \otimes y_{\bar{1}}) &= \operatorname{proj}_{A_{\bar{0}}} \mu(x_{\bar{1}} \otimes (\sigma - \mathrm{id})(y_{\bar{1}}))
\end{aligned}
$$

Theorem

The superalgebra (A, m) is, up to isomorphism, the unique superalgebra such that $(\mathcal{A}, [\mu])$ and $(F(A), F(m) \circ J_{A,A})$ are isomorphic algebras in Ver_3.

The Verlinde category Ver_p, $p > 3$

If $p > 3$, the Verlinde category Ver_p is no longer equivalent to $\mathsf{sVec}_{\mathbb{F}}$, but contains a full subcategory equivalent to $\mathsf{sVec}_{\mathbb{F}}$, generated by the indecomposable objects in $\mathsf{Rep}\,\mathsf{C}_p$ of dimension 1 and $p-1$.

This was used by Kannan to obtain the simple Lie superalgebra $\mathfrak{el}(5;5)$ by semisimplification of the simple Lie algebra of type E_8.

In characteristic 3, Kannan has obtained all the exceptional simple contragredient Lie superalgebras. These include mostly the superalgebras in the extended Freudenthal magic square (Cunha-E. 2007).

The extended Freudenthal magic square via tensor categories

Composition algebras

A composition algebra over a field $\mathbb{F}$ is a triple $(\mathcal{C}, \mu, \mathbf{n})$, where

- $\mu : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$, $\mu(x \otimes y) = xy$ is the multiplication of $\mathcal{C}$,
- $\mathbf{n} : \mathcal{C} \to \mathbb{F}$ is a nonsingular multiplicative quadratic form, called the norm.

Unital composition algebras (also termed Hurwitz algebras) over a field are the analogues of the classical algebras or real and complex numbers, quaternions, and octonions. In particular their dimension is restricted to 1, 2, 4 or 8.

Hurwitz algebras of dimension 8 are called Cayley algebras or octonion algebras.

Composition algebras in a symmetric tensor category $(\mathrm{char} \neq 2)$

A composition algebra in a symmetric tensor category $\mathfrak{C}$ is an object $\mathcal{A}$ endowed with morphisms $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\mathbf{n} : \mathcal{A} \otimes \mathcal{A} \to \mathbf{1}$, such that the following conditions are satisfied:

Symmetry: $\mathbf{n} \circ c_{\mathcal{A},\mathcal{A}} = \mathbf{n}$, where $c_{\mathcal{A},\mathcal{A}} \in \mathrm{End}_{\mathfrak{C}}(\mathcal{A} \otimes \mathcal{A})$ is the symmetric braiding.

Multiplicativity: The following equality of morphisms $\mathcal{A}^{\otimes 4} \to \mathbf{1}$ holds:

$$\mathbf{n} \circ (\mu \otimes \mu) \circ (\mathrm{id} + c_{13}) = (\mathbf{n} \otimes \mathbf{n}) \circ c_{23}$$

Nondegeneracy: The composition

$$\mathcal{A} \xrightarrow{\mathrm{id}_{\mathcal{A}} \otimes \mathrm{coev}_{\mathcal{A}}} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}^* \xrightarrow{\mathbf{n} \otimes \mathrm{id}_{\mathcal{A}^*}} \mathcal{A}^*$$

is an isomorphism.

Order 3 automorphisms of Cayley algebras ($\mathrm{char}\,\mathbb{F} = 3$)

If a Cayley algebra $\mathcal{C}$ over a field of characteristic 3 has an order 3 automorphism, then it is split (isotropic norm), and hence it contains a canonical basis with multiplication:

	e_1	e_2	u_1	u_2	u_3	v_1	v_2	v_3
e_1	e_1	0	u_1	u_2	u_3	0	0	0
e_2	0	e_2	0	0	0	v_1	v_2	v_3
u_1	0	u_1	0	v_3	$-v_2$	$-e_1$	0	0
u_2	0	u_2	$-v_3$	0	v_1	0	$-e_1$	0
u_3	0	u_3	v_2	$-v_1$	0	0	0	$-e_1$
v_1	v_1	0	$-e_2$	0	0	0	u_3	$-u_2$
v_2	v_2	0	0	$-e_2$	0	$-u_3$	0	u_1
v_3	v_3	0	0	0	$-e_2$	u_2	$-u_1$	0

The 'Peirce component' $\mathbb{F}u_1 + \mathbb{F}u_2 + \mathbb{F}u_3$ generates $\mathcal{C}$.

Order 3 automorphisms of Cayley algebras ($\mathrm{char}\,\mathbb{F} = 3$)

Theorem (E. 2018)

Let $(\mathcal{C}, \mu, \mathbf{n})$ be a Cayley algebra over a field $\mathbb{F}$ of characteristic 3, and let σ be an order 3 automorphism of $(\mathcal{C}, \mu, \mathbf{n})$. Then $(\mathcal{C}, \mu, \mathbf{n})$ is the split Cayley algebra and one of the following conditions holds, up to conjugation:

1. $(\sigma - \mathrm{id})^2 = 0$ *and* $\sigma(u_i) = u_i$, $i = 1, 2$, $\sigma(u_3) = u_3 + u_2$.
2. *There is a quadratic étale subalgebra $\mathcal{K}$ of $\mathcal{C}$ fixed elementwise by σ. If $\mathbb{F}$ is algebraically closed, we have $\sigma(u_i) = u_{i+1}$ (indices modulo 3).*
3. $\sigma(u_i) = u_i$, $i = 1, 2$, $\sigma(u_3) = u_3 + v_3 - (e_1 - e_2)$.
4. $\sigma(u_i) = u_i$, $i = 1, 2$, $\sigma(u_3) = u_3 + u_2 + v_3 - (e_1 - e_2)$.

Semisimplification of Cayley algebras ($\mathrm{char}\,\mathbb{F} = 3$)

Each order 3 automorphism of a Cayley algebra allows us to look at it as an algebra in $\mathsf{Rep}\,\mathsf{C}_3$, and hence apply our recipe to get a unital composition superalgebra, obtaining the following possibilities, according to the type of the automorphism in the previous slide:

1. The six-dimensional composition superalgebra $B(4,2)$.
2. A two-dimensional composition algebra.
3. The three-dimensional composition superalgebra $B(1,2)$.
4. Again the three-dimensional composition superalgebra $B(1,2)$.

$B(4,2)$ and $B(1,2)$ were 'discovered' by Shestakov (1997) in his classification of the prime alternative superalgebras.
They are the only 'exceptional' unital composition superalgebras (E.-Okubo 2002).

The extended Freudenthal magic square via tensor categories

Triality Lie algebra

Let $(\mathcal{C}, \mu, \mathbf{n})$ be a Hurwitz algebra over a field $\mathbb{F}$ of characteristic not 2.

Its triality Lie algebra is

$$\begin{aligned}\mathfrak{tri}(\mathcal{C}, \bullet, \mathbf{n}) := \{(d_0, d_1, d_2) \in \mathfrak{so}(\mathcal{C}, \mathbf{n})^3 \mid \\ d_0(x \bullet y) = d_1(x) \bullet y + x \bullet d_2(y)\ \forall x, y \in \mathcal{C}\}\end{aligned}$$

with $x \bullet y = \overline{x}\,\overline{y}$. $\quad(\overline{x} = \mathbf{n}(x, 1) - x$ is the canonical involution.)

This is a Lie algebra with componentwise Lie bracket, and the cyclic permutation

$$\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$$

is an automorphism (triality automorphism).

Symmetric construction of Freudenthal magic square

The vector space

$$\mathfrak{g}(\mathcal{C},\mathcal{C}') = \big(\mathfrak{tri}(\mathcal{C}) \oplus \mathfrak{tri}(\mathcal{C}')\big) \oplus \Big(\oplus_{i=0}^{2} \iota_i(\mathcal{C}\otimes\mathcal{C}')\Big),$$

where $\iota_i(\mathcal{C}\otimes\mathcal{C}')$ is just a copy of $\mathcal{C}\otimes\mathcal{C}'$ $(i=0,1,2)$ becomes a Lie algebra with:

- the Lie bracket in $\mathfrak{tri}(\mathcal{C}) \oplus \mathfrak{tri}(\mathcal{C}')$ (a Lie subalgebra of $\mathfrak{g}$),
- $\big[(d_0,d_1,d_2),\iota_i(x\otimes x')\big] = \iota_i\big(d_i(x)\otimes x'\big)$,
- $\big[(d_0',d_1',d_2'),\iota_i(x\otimes x')\big] = \iota_i\big(x\otimes d_i'(x')\big)$,
- $\big[\iota_i(x\otimes x'),\iota_{i+1}(y\otimes y')\big] = \iota_{i+2}\big((x\bullet y)\otimes(x'\bullet y')\big)$,
- $\big[\iota_i(x\otimes x'),\iota_i(y\otimes y')\big] = \mathbf{n}'(x',y')\theta^i(t_{x,y}) + \mathbf{n}(x,y)\theta'^i(t'_{x',y'}) \in \mathfrak{tri}(\mathcal{C}) \oplus \mathfrak{tri}(\mathcal{C}')$,
  $\Big(t_{x,y} := \Big(s_{x,y}, \frac{1}{2}\big(r_y l_x - r_x l_y\big), \frac{1}{2}\big(l_y r_x - l_x r_y\big)\Big)$, with
  $s_{x,y} : z \mapsto \mathbf{n}(x,z)y - \mathbf{n}(y,z)x$, $l_x : z \mapsto x\bullet z$, and $r_x : z\mapsto z\bullet x\big)$.

Freudenthal magic square

$\dim \mathcal{C}'$

$\dim \mathcal{C}$

$\mathfrak{g}(\mathcal{C}, \mathcal{C}')$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

(Vinberg, Allison-Faulkner, Barton-Sudbery, Landsberg-Manivel)

Extended Freudenthal magic square in characteristic 3

The previous symmetric construction of Freudenthal magic square works if the Hurwitz algebras are replaced by Hurwitz superalgebras:

$\mathfrak{g}(\mathcal{C},\mathcal{C}')$	$\mathbb{F}$	$\mathcal{K}$	$\mathcal{Q}$	$\mathcal{C}$	$B(1,2)$	$B(4,2)$
$\mathbb{F}$	A_1	$\tilde{A}_2$	C_3	F_4	$6\|8$	$21\|14$
$\mathcal{K}$		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	$11\|14$	$35\|20$
$\mathcal{Q}$			D_6	E_7	$24\|26$	$66\|32$
$\mathcal{C}$				E_8	$55\|50$	$133\|56$
$B(1,2)$					$21\|16$	$36\|40$
$B(4,2)$						$78\|64$

Lie superalgebras in the extended magic square

	$B(1,2)$	$B(4,2)$
$\mathbb{F}$	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
$\mathcal{K}$	$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus ((2) \otimes \mathfrak{psl}_3)$	$\mathfrak{pgl}_6 \oplus (20)$
$\mathcal{Q}$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$\mathfrak{so}_{12} \oplus spin_{12}$
$\mathcal{C}$	$(\mathfrak{sl}_2 \oplus \mathfrak{f}_4) \oplus ((2) \otimes (25))$	$\mathfrak{e}_7 \oplus (56)$
$B(1,2)$	$\mathfrak{so}_7 \oplus 2spin_7$	$\mathfrak{sp}_8 \oplus (40)$
$B(4,2)$	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus spin_{13}$

$\mathfrak{g}(\mathcal{C}, \mathcal{C}')$ as a Lie algebra in $\operatorname{Rep} \mathsf{C}_3$

Any order 3 automorphism of a Cayley algebra $(\mathcal{C}, \mu, \mathbf{n})$ induces an order 3 automorphism of its triality Lie algebra $\mathfrak{tri}(\mathcal{C}, \bullet, \mathbf{n})$ commuting with the triality automorphism.

Therefore, starting with an order 3 automorphism σ of a Cayley algebra $(\mathcal{C}, \mu, \mathbf{n})$ such that its semisimplification is isomorphic to either $B(1,2)$ or $B(4,2)$, there is an order 3 automorphism induced in $\mathfrak{g}(\mathcal{C}, \mathcal{C}')$, where we combine the order 3 automorphism on $\mathcal{C}$ and the identity automorphism in $\mathcal{C}'$. This allows us to consider $\mathfrak{g}(\mathcal{C}, \mathcal{C}')$ as a Lie algebra in $\operatorname{Rep} \mathsf{C}_3$.

The same arguments work if both $\mathcal{C}$ and $\mathcal{C}'$ are Cayley algebras endowed with order 3 automorphisms. We also get an induced order 3 automorphism of $\mathfrak{g}(\mathcal{C}, \mathcal{C}')$. These order 3 automorphisms allow us to see $\mathfrak{g}(\mathcal{C}, \mathcal{C}')$ as a Lie algebra in $\operatorname{Rep} \mathsf{C}_3$.

$\mathfrak{g}(\mathcal{C}, \mathcal{C}')$ as a Lie algebra in $\operatorname{Rep} \mathsf{C}_3$

Theorem

$\mathfrak{g}(\mathcal{C}, \mathcal{C}')^{ss}$ and $\mathfrak{g}(\mathcal{C}^{ss}, \mathcal{C}'^{ss})$ are isomorphic.

Corollary

All the Lie superalgebras in the extended Freudenthal magic square are thus obtained, in a unified way, by semisimplification of Lie algebras in Freudenthal magic square.

Remark

The Lie superalgebras $\mathfrak{g}(B(1,2), B(1,2))$, $\mathfrak{g}(B(1,2), B(4,2))$, and $\mathfrak{g}(B(4,2), B(4.2))$, are all obtained here by semisimplification of the Lie algebra $\mathfrak{g}(\mathcal{C}, \mathcal{C}')$ of type E_8.
Arun Kannan, using $\operatorname{Rep} \boldsymbol{\alpha}_3$ instead of $\operatorname{Rep} \mathsf{C}_3$, obtained $\mathfrak{g}(B(1,2), B(1,2))$ by semisimplification of E_6.

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Thank you!