Graded modules over classical simple Lie algebras

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(joint work with Mikhail Kochetov)

Graded modules. Main questions

Graded Brauer group

Brauer invariant

Solution to the main questions

Brauer invariants for the representations of the classical simple Lie algebras

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By complete reducibility, W is a direct sum of simple graded modules.

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(Q2) Which $\mathcal{L}$-modules admit a compatible G-grading?

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Brauer invariants for the representations of the classical simple Lie algebras

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If $\mathcal{R}$ is graded-simple, then

$$\mathcal{R} \cong \mathrm{End}_{\mathcal{D}}(W),$$

for a graded division algebra $\mathcal{D}$ and a G-graded right $\mathcal{D}$-module W.

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- $\mathcal{R} \cong M_k(\mathcal{D}) \cong M_k(\mathbb{F}) \otimes \mathcal{D}$, where $M_k(\mathbb{F})$ is endowed with an elementary grading: there are $g_1, \dots, g_k \in G$ with

$$\deg(E_{ij}) = g_i g_j^{-1}.$$

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$[M_r(\mathbb{F})] = 1$ if and only if the grading on $M_r(\mathbb{F})$ is elementary.

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- $[\mathcal{D}]$ is determined by the pair (T, β).

Graded Brauer group

If $\mathcal{R}_1$ and $\mathcal{R}_2$ are simple G-graded associative algebras, then so is $\mathcal{R}_1 \otimes \mathcal{R}_2$, so we may define a product:

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We thus obtain an abelian group: the graded Brauer group of $\mathbb{F}$.

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$\widehat{G}$-actions and bicharacters

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Any $\chi \in \widehat{G}$ determines an automorphism α_χ of $\mathcal{R}$, which is the conjugation by an element of the form

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Then

$$u_{\chi_1} u_{\chi_2} = \hat{\beta}(\chi_1, \chi_2) u_{\chi_2} u_{\chi_1}, \qquad \text{with } \hat{\beta}(\chi_1, \chi_2) = \beta(t_1, t_2).$$

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$\hat{\beta} : \widehat{G} \times \widehat{G} \to \mathbb{F}^\times$ is an alternating bicharacter: the commutation factor for the action of $\widehat{G}$.

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T and β are recovered from $\hat{\beta}$ as

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If $[\mathcal{R}_i] \simeq \hat{\beta}_i$, $i = 1, 2$, then

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Let W be a module for $\mathcal{L}$ endowed with a compatible G-grading, and let $\varphi : \widehat{G} \to GL(W) : \chi \mapsto \varphi_\chi$ the associated action.
The compatibility condition is equivalent to:

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That is, φ_χ is an isomorphism $W \to W^{\alpha_\chi}$, so any module with a compatible G-grading must satisfy $W \cong W^{\alpha_\chi}$ for any $\chi \in \widehat{G}$.

Induced action on isomorphism classes of modules

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$\mathrm{Aut}(\mathcal{L})$ acts (on the right) on the set of isomorphism classes of $\mathcal{L}$-modules: for any $\alpha \in \mathrm{Aut}(\mathcal{L})$ and $\mathcal{L}$-module V, V^α denotes the $\mathcal{L}$-module defined on the same vector space V, but with the 'twisted action':

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If $\alpha \in \mathrm{Int}(\mathcal{L})$, then $V^\alpha \cong V$, so the action of $\mathrm{Aut}(\mathcal{L})$ factors through $\mathrm{Out}(\mathcal{L}) = \mathrm{Aut}(\mathcal{L})/\,\mathrm{Int}(\mathcal{L})$.

Induced action on dominant integral weights

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Fix a Cartan subalgebra and a system $\{\alpha_1, \dots, \alpha_r\}$ of simple roots, and let Λ^+ be the set of dominant integral weights.

Then we get a 'bijection':

$$\{\text{Action of } \mathrm{Aut}(\mathcal{L}) \text{ on isomorphism classes of irreducible } \mathcal{L}\text{-modules}\}$$

$$\updownarrow$$

$$\{\text{Action of } \mathrm{Out}(\mathcal{L}) \text{ on } \Lambda^+ \text{ obtained by permutation of the vertices of the Dynkin diagram}\}$$

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Then $\widehat{G}$ acts on the isomorphism classes of irreducible $\mathcal{L}$-modules and, for any $\chi \in \widehat{G}$, the automorphism $\alpha_\chi \in \mathrm{Aut}(\mathcal{L})$ projects onto some $\tau_\chi \in \mathrm{Out}(\mathcal{L})$.

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For any dominant integral weight $\lambda \in \Lambda^+$ consider the inertia group

$$\begin{aligned} K_\lambda &:= \{\chi \in \widehat{G} : \tau_\chi(\lambda) = \lambda\} \\ &= \{\chi \in \widehat{G} : V_\lambda \cong (V_\lambda)^{\alpha_\chi}\}. \end{aligned}$$

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Therefore, $H_\lambda := \left(K_\lambda\right)^\perp$ is a finite subgroup of G, of size $|H_\lambda| = |\widehat{G}\lambda|$ (the size of the orbit of λ), and K_λ is isomorphic to the group of characters of G/H_λ.

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- There is a representation (as algebraic groups):

$$\begin{aligned} K_\lambda &\longrightarrow \mathrm{Aut}\big(\mathrm{End}(V_\lambda)\big) \\ \chi &\mapsto \quad \tilde{\alpha}_\chi, \end{aligned}$$

  where $\tilde{\alpha}_\chi\big(\rho(x)\big) = \rho\big(\alpha_\chi(x)\big)$ for any $x \in U(\mathcal{L})$, which corresponds to a compatible $\bar{G} := G/H_\lambda$-grading on $\mathrm{End}(V_\lambda)$.

(Recall that K_λ is isomorphic to the character group of G/H_λ.)

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The degree of the graded division algebra $\mathcal{D}$ representing $\mathrm{Br}(\lambda)$ is called the Schur index of λ.

Brauer invariant and Schur index

Proposition

The $\mathcal{L}$-module $(V_\lambda)^k$ admits a $\bar{G} = G/H_\lambda$-grading that makes it a graded simple $\mathcal{L}$-module (where $\mathcal{L}$ is endowed with the natural induced $\bar{G}$-grading) if and only if k equals the Schur index of λ.

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This grading is unique up to isomorphism and shift.

Sketch of proof: $\operatorname{End}((V_\lambda)^k) \cong M_k(\mathbb{F}) \otimes \operatorname{End}(V_\lambda)$. If k is the Schur index of λ and $\mathcal{D}$ represents $\operatorname{Br}(\lambda)$, then $\mathcal{D}^{\mathrm{op}} \cong M_k(\mathbb{F})$. Thus $\operatorname{End}((V_\lambda)^k)$ admits a $\bar{G}$-grading with

$$\operatorname{End}((V_\lambda)^k) \cong \mathcal{D}^{\mathrm{op}} \otimes \operatorname{End}(V_\lambda) \cong \mathcal{D}^{\mathrm{op}} \otimes M_r(\mathcal{D}).$$

Hence $[\operatorname{End}((V_\lambda)^k)] = 1$, so the $\bar{G}$-grading on $(V_\lambda)^k$ is elementary, i.e., it is induced by a $\bar{G}$-grading on $(V_\lambda)^k$.

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Then $K = H^{\perp}$ is a finite index subgroup of $\widehat{G}$ and

$$W = \mathrm{Ind}_K^{\widehat{G}} U := \mathbb{F}\widehat{G} \otimes_{\mathbb{F}K} U$$

is a $\widehat{G}$-module; i.e., a G-graded vector space.

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If U is a $\bar{G}$-graded $\mathcal{L}$-module, then W is a G-graded $\mathcal{L}$-module:

$$x.(\chi \otimes u) := \chi \otimes \alpha_{\chi^{-1}}(x)u.$$

Graded simple modules: (Q1)

For each $\widehat{G}$-orbit $\mathcal{O}$ in Λ^+, select a representative λ.

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Theorem

Up to isomorphisms and shifts, the $W(\mathcal{O})$'s are the graded-simple finite dimensional $\mathcal{L}$-modules.

Modules admitting compatible gradings: (Q2)

Theorem

An $\mathcal{L}$-module V admits a compatible G-grading if and only if for any $\lambda \in \Lambda^+$ the multiplicities of V_μ in V, for all the elements μ in the orbit $\widehat{G}\lambda$, are equal and divisible by the Schur index of λ.

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In particular, for $\lambda \in \Lambda^+$, V_λ admits a compatible G-grading if and only if $H_\lambda = 1$ and $\mathrm{Br}(\lambda) = 1$.

Graded modules. Main questions

Graded Brauer group

Brauer invariant

Solution to the main questions

Brauer invariants for the representations of the classical simple Lie algebras

Given $\lambda \in \Lambda^+$, the computation of $\mathsf{Br}(\lambda)$ can be reduced to the computation of $\mathsf{Br}(\omega_1), \ldots, \mathsf{Br}(\omega_r)$, where $\omega_1, \ldots, \omega_r$ are the fundamental weights.

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For the simple Lie algebras of types G_2, F_4 and E_8, these Brauer invariants are always trivial.
Therefore, any module admits a compatible grading.

Type A: inner

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$\mathcal{L} = \mathfrak{sl}_{r+1}(\mathbb{F})$ and assume that the image of $\widehat{G} \to \mathrm{Aut}(\mathcal{L})$ is contained in $\mathrm{Int}(\mathcal{L})$.

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For any $\lambda = \sum_{i=1}^{r} m_i\omega_i \in \Lambda^+$, $H_\lambda = 1$ and

$$\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{r} i m_i},$$

where $\hat{\beta} : \widehat{G} \times \widehat{G} \to \mathbb{F}$ is the commutation factor for the action of $\widehat{G}$ on $\mathcal{R}$.

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For any $\lambda = \sum_{i=1}^{r} m_i \omega_i \in \Lambda^+$,

- If $m_i \neq m_{r+1-i}$ for some i, then $H_\lambda = \langle h \rangle$ and

$$\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{r} i m_i},$$

  where $\hat{\beta}$ is the commutation factor for the action of $\widehat{(G/\langle h \rangle)}$ on $\mathcal{R}$.
- If r is even and $m_i = m_{r+1-i}$ for all i, then $H_\lambda = 1$ and $\mathrm{Br}(\lambda) = 1$.
- If r is odd and $m_i = m_{r+1-i}$ for all i, then $H_\lambda = 1$, but $\mathrm{Br}(\lambda)$ may be nontrivial (the description is quite technical).

Type B

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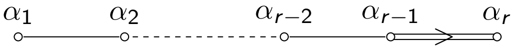
$\mathcal{L} = \mathfrak{so}_{2r+1}(\mathbb{F})$, $r \geq 2$.



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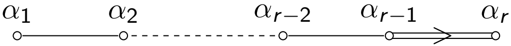

Then the module V_{ω_1} is the natural $(2r+1)$-dimensional module, for $i = 2, \ldots, r-1$, $V_{\omega_i} = \wedge^i V_{\omega_1}$, and V_{ω_r} is the spin module (i.e., the irreducible module for the even Clifford algebra $\mathfrak{Cl}_{\bar{0}}(V_{\omega_1})$).

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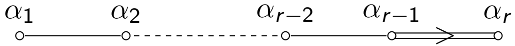

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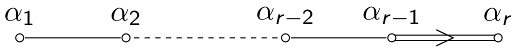

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The G-grading on $\mathcal{L}$ is always induced by a compatible G-grading on V_{ω_1}.

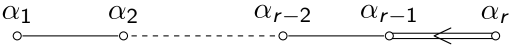
For any $\lambda = \sum_{i=1}^{r} m_i \omega_i \in \Lambda^+$, $H_\lambda = 1$ and

$$\mathrm{Br}(\lambda) = \hat{\gamma}^{m_r} \qquad \text{(it depends only on } m_r\text{!)}$$

where $\hat{\gamma}$ is the commutation factor of the induced action of $\widehat{G}$ on $\mathfrak{Cl}_{\bar{0}}(V_{\omega_1})$.

Type C

$\mathcal{L} = \mathfrak{sp}_{2r}(\mathbb{F})$, $r \geq 2$.



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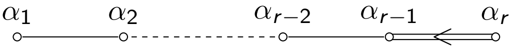

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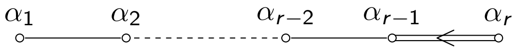

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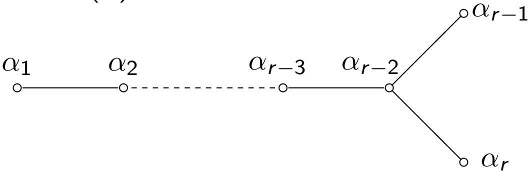
For any $\lambda = \sum_{i=1}^{r} m_i\omega_i \in \Lambda^+$, $H_\lambda = 1$ and

$$\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{\lfloor (r+1)/2 \rfloor} m_{2i-1}}$$

where $\hat{\beta}$ is the commutation factor of the action of $\widehat{G}$ on $\mathcal{R}$.

Type D

$\mathcal{L} = \mathfrak{so}_{2r}(\mathbb{F})$, $r \geq 5$ (!!).



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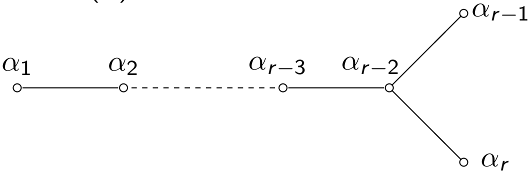

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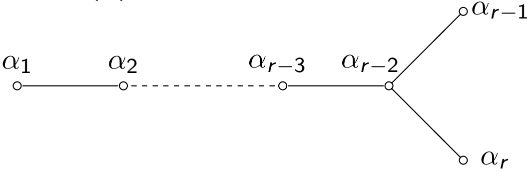

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The G-grading on $\mathcal{L}$ is induced by a grading on $\mathcal{R} = M_{2r}(\mathbb{F}) \simeq \text{End}(V_{\omega_1})$.
It is said to be *inner* if the image of $\widehat{G} \to \text{Aut}(\mathcal{L})$ is contained in $\text{Int}(\mathcal{L})$; otherwise it is called *outer*.

Type D: inner

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For any $\lambda = \sum_{i=1}^{r} m_i \omega_i \in \Lambda^+$, $H_\lambda = 1$ and:

- If $m_{r-1} \equiv m_r \pmod 2$, then $\mathrm{Br}(\lambda)$ depends only on the commutation factor of the action of $\widehat{G}$ on $\mathcal{R}$.
- Otherwise it also depends on the commutation factors of the induced action of $\widehat{G}$ on the two simple ideals of $\mathfrak{Cl}_{\bar{0}}(V_{\omega_1})$.

Type D: outer

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Here there exists a distinguished order 2 element $h \in G$ such that the induced $\bar{G} = G/\langle h \rangle$-grading on $\mathcal{L}$ is inner.

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For any $\lambda = \sum_{i=1}^{r} m_i \omega_i \in \Lambda^+$:

- If $m_{r-1} \neq m_r$ but $m_{r-1} \equiv m_r \pmod 2$, then $H_\lambda = \langle h \rangle$ and $\mathrm{Br}(\lambda) = 1$ (in the $G/\langle h \rangle$-graded Brauer group!).
- If $m_{r-1} \not\equiv m_r \pmod 2$, then $H_\lambda = \langle h \rangle$ and $\mathrm{Br}(\lambda)$ is given in terms of the commutation factor of $\widehat{(G/\langle h \rangle)}$ on $\mathfrak{Cl}_{\bar{0}}(V_{\omega_1})$.
- If $m_{r-1} = m_r$, then $H_\lambda = 1$ and

$$\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{\lfloor r/2 \rfloor} m_{2i-1}},$$

where $\hat{\beta}$ is the commutation factor of the action of $\widehat{G}$ on $\mathcal{R}$.

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That's all.
Thanks