# Graded simple algebras and modules

Alberto Elduque

Universidad de Zaragoza

1. Graded simple algebras
2. Loop algebras
3. Graded simple modules

# Gradings

$G$ abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

**$G$-grading on $\mathcal{A}$**:

$$\Gamma : \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g,$$

$$\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

# Examples

**Cartan grading:**

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\right)$$

(root space decomposition of a semisimple complex Lie algebra).

This is a grading by $\mathbb{Z}^n$, $n = \operatorname{rank} \mathfrak{g}$.

# Examples

**Pauli matrices:** $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

($\epsilon$ a primitive $n$th root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \bigoplus_{(\bar{\imath},\bar{\jmath})\in\mathbb{Z}_n\times\mathbb{Z}_n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F}X^iY^j.$$

$\mathcal{A}$ becomes a **graded division algebra**.

# Simple algebras

Let $\mathcal{B}$ be an algebra over $\mathbb{F}$:

- $\mathcal{B}$ is **simple** if it has no proper ideals.
  In other words, $\mathcal{B}$ is simple if it is simple as a module for its **multiplication algebra** $\mathrm{Mult}(\mathcal{B})$.
- The **centroid** of $\mathcal{B}$ is the centralizer of $\mathrm{Mult}(\mathcal{B})$ in $\mathsf{End}_{\mathbb{F}}(\mathcal{B})$:

  $$C(\mathcal{B}) := \{f \in \mathsf{End}_{\mathbb{F}}(\mathcal{B}) : (xy)f = (xf)y = x(yf)\ \forall x, y \in \mathcal{B}\}.$$

  $C(\mathcal{B})$ is commutative if $\mathcal{B}^2 = \mathcal{B}$, and it is a field (an extension field of $\mathbb{F}$) if $\mathcal{B}$ is simple.
- $\mathcal{B}$ is **central simple** if it is simple and *central*: $C(\mathcal{B}) = \mathbb{F}\mathrm{id}$.

# Graded-simple algebras

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a graded algebra:

- $\mathcal{B}$ is **graded-simple** if it has no proper graded ideals.

  Its centroid 'inherits' a $G$-grading:

  $$C(\mathcal{B})_g := \{f \in C(\mathcal{B}) : \mathcal{B}_h f \subseteq \mathcal{B}_{hg}\ \forall h \in G\}.$$

- $\mathcal{B}$ is **graded-central-simple** if it is graded-simple and *graded-central*: $C(\mathcal{B})_e = \mathbb{F}\mathrm{id}$.

# Graded-simple algebras

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a graded-simple algebra, then:

- $C(\mathcal{B})$ is a graded field (i.e., a commutative graded division algebra[1]).
- $\mathcal{B}$ is simple (ungraded) if and only if $C(\mathcal{B})$ is a field.
- $\mathbb{K} = C(\mathcal{B})_e$ is a field, and $\mathcal{B}$ is graded-central-simple considered as an algebra over $\mathbb{K}$.

[1] The group algebras of abelian groups are graded fields.

# Split centroid

Let $\mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ be a graded-central-simple algebra with centroid $C(\mathcal{B})$.
Let $H$ be the *support* of the induced grading on $C(\mathcal{B})$. This is a subgroup of $G$.

## Definition

$C(\mathcal{B})$ is said to **split** if it is isomorpic, as a $G$-graded algebra, to the group algebra $\mathbb{F}H$.

## Proposition

- $C(\mathcal{B})$ splits if and only if there is a homomorphism of unital algebras $C(\mathcal{B}) \to \mathbb{F}$.
- If $\mathbb{F}$ is algebraically closed[2], then $C(\mathcal{B})$ splits.

[2] $\mathbb{F}^\times$ is then a divisible group.

# Loop algebras

Let $G$ be an abelian group, $H$ a subgroup of $G$,
$\pi : G \to \overline{G} = G/H$ the canonical projection.

Let $\overline{\Gamma} : \mathcal{A} = \bigoplus_{\bar{g} \in \overline{G}} \mathcal{A}_{\bar{g}}$ be a $\overline{G}$-grading on an algebra $\mathcal{A}$.

## Definition

The $G$-graded algebra

$$L_\pi(\mathcal{A}) := \bigoplus_{g \in G} \mathcal{A}_{\bar{g}} \otimes g \ \left(\leq \mathcal{A} \otimes_{\mathbb{F}} \mathbb{F}G\right),$$

where

$$(x_{\bar{g}_1} \otimes g_1)(y_{\bar{g}_2} \otimes g_2) := x_{\bar{g}_1} y_{\bar{g}_2} \otimes g_1 g_2,$$

is called the **loop algebra** of $(\mathcal{A}, \overline{\Gamma})$ relative to $\pi$.

# Loop algebras: properties

- $\mathcal{A}$ is graded-simple if and only if so is $L_\pi(\mathcal{A})$. In this case $\mathcal{A}$ is graded-central if and only if so is $L_\pi(\mathcal{A})$.

- If $\mathcal{A}$ is graded-simple, then the centroid $C\big(L_\pi(\mathcal{A})\big)$ is naturally isomorphic to the loop algebra $L_\pi\big(C(\mathcal{A})\big)$.
  In particular, if $\mathcal{A}$ is central (i.e., $C(\mathcal{A}) = \mathbb{F}\mathrm{id}$), then $C\big(L_\pi(\mathcal{A})\big)$ is isomorphic to $\mathbb{F}H$ and, hence, it splits.

# Graded-simple algebras and loop algebras

**Theorem (Allison-Berman-Faulkner-Pianzola 2008)**

*Let* $\Gamma : \mathcal{B} = \bigoplus_{g \in G} \mathcal{B}_g$ *be a grading such that* $\mathcal{B}$ *is a graded-simple algebra with split centroid:* $C(\mathcal{B}) \cong \mathbb{F}H$, *and let* $\rho : C(\mathcal{B}) \to \mathbb{F}$ *be a homomorphism of unital algebras. Then,*

- $\mathcal{A} := \mathcal{B}/\mathcal{B}(\ker \rho)$, *is naturally* $\overline{G} = G/H$*-graded. Besides,* $\mathcal{A}$ *is a central simple algebra (as an ungraded algebra!).*
- *The canonical projection* $\Pi : \mathcal{B} \to \mathcal{A}$ *restricts to linear isomorphism* $\mathcal{B}_g \to \mathcal{A}_{\bar{g}}$ *for any* $g \in G$.
- $\mathcal{B}$ *is graded isomorphic to the loop algebra* $L_\pi(\mathcal{A})$.

# Graded-simple algebras and loop algebras

## Sketch of proof

$\Gamma$ has a coarsening $\overline{\Gamma}: \mathcal{B} = \bigoplus_{\bar{g}\in\overline{G}} \mathcal{B}_{\bar{g}}$, where $\mathcal{B}_{\bar{g}} = \bigoplus_{h\in H} \mathcal{B}_{gh}$.
The ideal $\mathcal{B}(\ker\rho)$ is a graded ideal for $\overline{\Gamma}$, so that $\mathcal{A}$ inherits a grading by $\overline{G}$.
For any $g \in G$, let $\bar{g} = \pi(g)$. Then

$$\mathcal{B}_{\bar{g}} = \mathcal{B}_g C(\mathcal{B}) = \mathcal{B}_g = \mathcal{B}_g(\mathbb{F}1 \oplus \ker\rho) = \mathcal{B}_g \oplus \mathcal{B}_g \ker\rho,$$

while $\mathcal{B}_g \ker\rho = \mathcal{B}_g C(\mathcal{B}) \ker\rho = \mathcal{B}_{\bar{g}} \ker\rho$. Hence we get

$$\mathcal{B}_{\bar{g}} = \mathcal{B}_g \oplus \mathcal{B}_{\bar{g}} \ker\rho,$$

and this proves that $\Pi$ restricts to linear isomorphisms $\mathcal{B}_g \cong \mathcal{A}_{\bar{g}}$.
The map

$$\Phi : \mathcal{B} \longrightarrow L_\pi(\mathcal{A}), \qquad x_g \mapsto \Pi(x_g) \otimes g$$

gives the required isomorphism.

# When is a graded algebra a loop algebra?

## Proposition

$\Gamma : \mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$, $H \leq G$, $\pi : G \to \overline{G} = G/H$ canonical projection.
Then $\mathcal{B}$ is graded isomorphic to a loop algebra $L_\pi(\mathcal{A})$ if and only if $C(\mathcal{B})$ contains a graded subalgebra isomorphic, as a graded algebra, to the group algebra $\mathbb{F}H$.

## Sketch of proof

It is enough to substitute $C(\mathcal{B})$ by its subalgebra isomorphic to $\mathbb{F}H$, and $\rho$ by the 'augmentation map' in the proof above.

# Graded simple algebras and 'graded and simple' algebras

Consider the following groupoids:

- $\mathfrak{A}(\pi)$: the groupoid of central simple algebras with a $\overline{G}$-grading.
- $\mathfrak{B}(\pi)$: the groupoid of $G$-graded-central-simple algebras $\mathcal{B}$ such that $C(\mathcal{B})$ splits and it is graded isomorphic to the group algebra $\mathbb{F}H$.

## Proposition

The following are equivalent:

- $\mathcal{A} \in \mathfrak{A}(\pi)$,
- $L_\pi(\mathcal{A}) \in \mathfrak{B}(\pi)$,
- $L_\pi(\mathcal{A})$ is graded-central-simple with $H = \mathrm{Supp}\Big(C\big(L_\pi(\mathcal{A})\big)\Big)$.

# Graded simple algebras and 'graded and simple' algebras

**Theorem**

- *If* $\mathcal{A} \in \mathfrak{A}(\pi)$*, then* $L_\pi(\mathcal{A}) \in \mathfrak{B}(\pi)$*.*
- *If* $\mathcal{B} \in \mathfrak{B}(\pi)$*, then there is an* $\mathcal{A} \in \mathfrak{A}(\pi)$ *such that* $\mathcal{B}$ *is graded isomorphic to* $L_\pi(\mathcal{A})$*.*

Moreover, under some mild restrictions, if $\mathcal{A}, \mathcal{A}'$ are in $\mathfrak{A}(\pi)$ and their loop algebras are graded isomorphic, then $\mathcal{A}$ is graded isomorphic to $\mathcal{A}'$.

# Application

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, and let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ a grading on a finite-dimensional semisimple Lie algebra. Then:

- $\mathcal{L}$ is uniquely the direct sum of graded-simple ideals, and
- each such ideal is, up to isomorphism, a loop algebra of a graded and simple Lie algebra.

**Consequence**

In order to classify gradings on semisimple Lie algebras, it is enough to classify gradings on simple Lie algebras.

# Graded modules

$\mathcal{R}$ $G$-graded unital associative $\mathbb{F}$-algebra[3], $\mathcal{W}$ a left $\mathcal{R}$-module,

- $\mathcal{W}$ is a $G$-graded left $\mathcal{R}$-module if it is endowed with a grading as a vector space $\mathcal{W} = \bigoplus_{g \in G} \mathcal{W}_g$ and $\mathcal{R}_g \mathcal{W}_{g'} \subseteq \mathcal{W}_{gg'}$ for any $g, g' \in G$.
- In this case, $\mathcal{W}$ is **graded-simple** if it has no proper graded submodules.
- The **centralizer** $C(\mathcal{W})$ is the graded algebra $C(\mathcal{W}) = \bigoplus_{g \in G} C(\mathcal{W})_g$, where,

  $$C(\mathcal{W})_g := \{f \in \mathrm{End}_{\mathcal{R}}(\mathcal{W}) : \mathcal{W}_{g'} f \subseteq \mathcal{W}_{g'g} \ \forall g' \in G\}.$$

  If $\mathcal{W}$ is graded-simple, then $C(\mathcal{W})$ is a graded division algebra.
- $\mathcal{W}$ is graded-central if $C(\mathcal{W})_e = \mathbb{F}\mathrm{id}$.

[3] For instance, the universal enveloping algebra of a graded Lie algebra

# Loop modules

Let $G$ be an abelian group, $H$ a subgroup of $G$,
$\pi : G \to \overline{G} = G/H$ the canonical projection.

Let $\mathcal{R}$ be a $G$-graded unital associative $\mathbb{F}$-algebra, then $\mathcal{R}$ is naturally $\overline{G}$-graded (a coarsening): $\mathcal{R}_{\bar{g}} := \bigoplus_{g\in\pi^{-1}(\bar{g})} \mathcal{R}_g$.
Let $\mathcal{V} = \bigoplus_{\bar{g}\in\overline{G}} \mathcal{V}_{\bar{g}}$ be a $\overline{G}$-graded $\mathcal{R}$-module.

## Definition

The $G$-graded module

$$L_\pi(\mathcal{V}) := \bigoplus_{g\in G} \mathcal{V}_{\bar{g}} \otimes g \ \left(\leq \mathcal{V} \otimes_{\mathbb{F}} \mathbb{F}G\right)$$

where

$$r_{g_1}(v_{\bar{g}_2} \otimes g_2) := r_{g_1} v_{\bar{g}_2} \otimes g_1 g_2,$$

is called the **loop module** of $\mathcal{V}$ relative to $\pi$.

# Loop modules

**Proposition**

A $G$-graded left $\mathcal{R}$-module $\mathcal{W}$ is graded isomorphic to a loop module $L_\pi(\mathcal{V})$ for a $\overline{G}$-graded left $\mathcal{R}$-module $\mathcal{V}$ if and only if its centralizer $C(\mathcal{W})$ contains a graded subfield isomorphic to the group algebra $\mathbb{F}H$.

# Graded simple modules

**Proposition**

Let $\mathcal{W}$ be a $G$-graded-simple left $\mathcal{R}$-module.

1. $C(\mathcal{W})$ contains maximal graded subfields.
2. If $\mathcal{W}$ is graded-central and $\mathbb{F}$ is algebraically closed, any graded subfield of $C(\mathcal{W})$ is isomorphic to the group algebra of its support.
3. If $\mathbb{F}$ is algebraically closed and $\dim \mathcal{W} < |\mathbb{F}|$ (these may be infinite cardinals), then $\mathcal{W}$ is graded-central.

# Graded simple modules

**Proposition**

Let $\mathcal{V}$ be a $\overline{G}$-graded left $\mathcal{R}$-module.

1. If $L_\pi(\mathcal{V})$ is $G$-graded-simple, then $\mathcal{V}$ is $\overline{G}$-graded-simple.
2. The converse is not true, unless an extra restriction is fulfilled: the $G$-pregrading on $\mathcal{V}$ associated to its $\overline{G}$-grading is thin.

# Graded-simple modules and ‘graded and simple’ modules

Consider the following groupoids:

- $\mathfrak{M}(\pi)$: the groupoid whose objects are the simple, central, and $\overline{G}$-graded left $\mathcal{R}$-modules $\mathcal{V} = \bigoplus_{\bar{g}\in\overline{G}} \mathcal{V}_{\bar{g}}$ such that the $G$-pregrading associated to the $\overline{G}$-grading is thin.
- $\mathfrak{N}(\pi)$ is the groupoid whose objects are the pairs $(\mathcal{W}, \mathcal{F})$, where $\mathcal{W}$ is a $G$-graded-simple left $\mathcal{R}$-module and $\mathcal{F}$ is a maximal graded subfield of $C(\mathcal{W})$ isomorphic to the group algebra $\mathbb{F}H$ as a $G$-graded algebra.

**Proposition**

If $\mathcal{V}$ is and object in $\mathfrak{M}(\pi)$, then $\big(L_\pi(\mathcal{V}), L_\pi(\mathbb{F}\mathrm{id})\big)$ is in $\mathfrak{N}(\pi)$.

In this way, we obtain a **loop functor** $L_\pi : \mathfrak{M}(\pi) \to \mathfrak{N}(\pi)$.

# Graded-simple modules and 'graded and simple' modules

**Theorem (E.–Kochetov 2016)**

*The loop functor* $L_\pi : \mathfrak{M}(\pi) \to \mathfrak{N}(\pi)$ *has the following properties:*

1. $L_\pi$ *is faithful, that is, injective on the set of morphisms* $\mathcal{V} \to \mathcal{V}'$, *for any objects* $\mathcal{V}$ *and* $\mathcal{V}'$ *in* $\mathfrak{M}(\pi)$.
2. $L_\pi$ *is essentially surjective, that is, any object* $(\mathcal{W}, \mathcal{F})$ *in* $\mathfrak{N}(\pi)$ *is isomorphic to* $\big(L_\pi(\mathcal{V}), L_\pi(\mathbb{F}1)\big)$ *for some object* $\mathcal{V}$ *in* $\mathfrak{M}(\pi)$.
3. *If* $\mathcal{V}$ *and* $\mathcal{V}'$ *are objects in* $\mathfrak{M}(\pi)$ *such that their images under* $L_\pi$ *are isomorphic in* $\mathfrak{N}(\pi)$, *then* $\mathcal{V}'$ *is a 'twist' of* $\mathcal{V}$ *obtained by means of a character of* $H$.

**Corollary**

*Over an algebraically closed field, any finite-dimensional graded-simple module is, up to graded isomorphism, a loop module of a graded and simple module.*

# Application

Graded simple modules appear naturally in the study of gradings on Lie superalgebras: in any $G$-graded Lie superalgebra $\mathcal{L}$, $\mathcal{L}_{\bar{1}}$ is a $G$-graded module for $U(\mathcal{L}_{\bar{0}})$.

For example, let $\mathcal{L} = \mathcal{L}_{\bar{0}} \oplus \mathcal{L}_{\bar{1}}$ be the simple Lie superalgebra of type $F(4)$ over an algebraically closed field of characteristic 0:

$$\mathcal{L}_{\bar{0}} = \mathfrak{so}_7 \oplus \mathfrak{sl}_2, \qquad \mathcal{L}_{\bar{1}} = \textit{spin} \otimes \textit{natural}.$$

As a module for $\mathfrak{so}_7$, $\mathcal{L}_{\bar{1}}$ is the direct sum of two copies of the spin module.
Hence its centralizer $C(\mathcal{L}_{\bar{1}})$ is isomorphic to $M_2(\mathbb{F})$, and $\mathfrak{sl}_2$ embeds in $C(\mathcal{L}_{\bar{1}})$ as a graded (Lie) subalgebra.

# Application

$$\mathcal{L}_{\bar{0}} = \mathfrak{so}_7 \oplus \mathfrak{sl}_2, \qquad \mathcal{L}_{\bar{1}} = \textit{spin} \otimes \textit{natural}.$$

$\mathcal{L}_{\bar{1}}$ is graded-simple as a module for $\mathfrak{so}_7$ if and only if $C(\mathcal{L}_{\bar{1}})$ is a graded division algebra. This module is then a loop module.
In this situation, the restriction of the grading to $\mathfrak{sl}_2$ is a 'Pauli grading'.

Otherwise the grading on $C(\mathcal{L}_{\bar{1}})$, and hence on $\mathfrak{sl}_2$, is toral.

This explain the existence of two very different types of gradings on $\mathcal{L}$ (Draper–E.–Martín-González 2011).

# References

B. Allison, S. Berman, J. Faulkner, and A. Pianzola.
*Realization of graded-simple algebras as loop algebras*,
Forum Math. **20** (2008), no. 3, 395-432.

A. Elduque and M. Kochetov.
*Graded simple modules and loop modules*,
to appear in Contemporary Mathematics: Groups, Rings, Group Rings and Hopf Algebras, dedicated to Don Passman.
Preprint arXiv:1601.03008.

V. Mazorchuk and K. Zhao.
*Graded simple algebras and graded simple representations*.
Preprint arXiv:1504.05114.

Thank you!