Graded modules over simple Lie algebras

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(joint work with Mikhail Kochetov)

Dedicated to Yuri Bahturin

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- $\mathcal{L}$: finite-dimensional semisimple G-graded Lie algebra/ $\mathbb{F}$ (algebraically closed ground field of characteristic 0):

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By complete reducibility, W is a direct sum of **graded simple modules**.

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(Q2) Which $\mathcal{L}$-modules admit a compatible G-grading?

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If $\mathcal{R}$ is graded simple, then

$$\mathcal{R} \cong \mathrm{End}_{\mathcal{D}}(W),$$

for a **graded division algebra** $\mathcal{D}$ and a G-graded right $\mathcal{D}$-module W.

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$[M_r(\mathbb{F})] = 1$ if and only if the grading on $M_r(\mathbb{F})$ is elementary.

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- $[\mathcal{D}]$ is determined by the pair (T, β).

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$\mathcal{D} = M_n(\mathbb{F})$, ϵ a primitive nth root of 1:

$$x = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

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For any G containing a subgroup $T \simeq \mathbb{Z}_n^2$, $\mathcal{D}$ is a G-graded division algebra with support T with $\mathcal{D}_{(\bar r,\bar s)} = \mathbb{F}X_{(\bar r,\bar s)}$ ($X_{(\bar r,\bar s)} := x^r y^s$), and

$$X_{(\bar r,\bar s)}X_{(\bar r',\bar s')} = \beta\Big((\bar r,\bar s),(\bar r',\bar s')\Big)X_{(\bar r',\bar s')}X_{(\bar r,\bar s)}$$

with

$$\beta\Big((\bar r,\bar s),(\bar r',\bar s')\Big) = \epsilon^{sr'-rs'}.$$

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If $\mathcal{R}_1$ and $\mathcal{R}_2$ are finite-dimensional simple G-graded associative algebras, then so is $\mathcal{R}_1 \otimes \mathcal{R}_2$, so we may define a product:

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We thus obtain an abelian group: the **G-graded Brauer group of $\mathbb{F}$**, whose elements are the isomorphism classes of the finite-dimensional simple G-graded associative algebras over $\mathbb{F}$.

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- If $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, then the formula

$$\alpha_\chi(a) = \chi(g)a, \qquad \text{for any } g \in G \text{ and } a \in \mathcal{A}_g,$$

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Any $\chi \in \widehat{G}$ determines an automorphism α_χ of $\mathcal{R}$, which is the conjugation by an element of the form

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Then

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$\hat{\beta} : \widehat{G} \times \widehat{G} \to \mathbb{F}^\times$ is an alternating bicharacter: the **commutation factor** for the action of $\widehat{G}$.

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T and β are recovered from $\hat{\beta}$ as

- $T = \left(\operatorname{rad}\hat{\beta}\right)^{\perp} \left(= \{g \in G : \chi(g) = 1\ \forall \chi \in \operatorname{rad}\hat{\beta}\}\right)$,
- $\beta(t_1, t_2) = \hat{\beta}(\chi_1, \chi_2)$, where χ_i is any character such that $\hat{\beta}(\psi, \chi_i) = \psi(t_i)$ for any $\psi \in \widehat{G}$, $i = 1, 2$.

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If $[\mathcal{R}_i] \simeq \hat{\beta}_i$, $i = 1, 2$, then

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Let W be a module for $\mathcal{L}$ endowed with a compatible G-grading, and let $\varphi : \widehat{G} \to GL(W) : \chi \mapsto \varphi_\chi$ the associated action.
The compatibility condition is equivalent to:

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That is, φ_χ is an isomorphism $W \to W^{\alpha_\chi}$, so

any module with a compatible G-grading must satisfy $W \cong W^{\alpha_\chi}$ for any $\chi \in \widehat{G}$.

Induced action on isomorphism classes of modules

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$\mathrm{Aut}(\mathcal{L})$ acts (on the right) on the set of isomorphism classes of $\mathcal{L}$-modules: for any $\alpha \in \mathrm{Aut}(\mathcal{L})$ and $\mathcal{L}$-module V, V^{α} denotes the $\mathcal{L}$-module defined on the same vector space V, but with the 'twisted action':

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If $\alpha \in \mathsf{Int}(\mathcal{L})$, then $V^\alpha \cong V$, so the action of $\mathsf{Aut}(\mathcal{L})$ factors through $\mathsf{Out}(\mathcal{L}) = \mathsf{Aut}(\mathcal{L})/\mathsf{Int}(\mathcal{L}) \simeq \mathsf{Aut}(\mathrm{Dyn})$.

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Fix a Cartan subalgebra and a system $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots, and let Λ^+ be the set of dominant integral weights.

Then we get a 'bijection':

{Action of $\mathrm{Aut}(\mathcal{L})$ on isomorphism classes of irreducible $\mathcal{L}$-modules}

$$\updownarrow$$

{Action of $\mathrm{Out}(\mathcal{L})$ on Λ^+ obtained by permutation of the vertices of the Dynkin diagram}

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Then $\widehat{G}$ acts on the isomorphism classes of irreducible $\mathcal{L}$-modules and, for any $\chi \in \widehat{G}$, the automorphism $\alpha_\chi \in \mathrm{Aut}(\mathcal{L})$ projects onto some $\tau_\chi \in \mathrm{Out}(\mathcal{L})$.

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For any dominant integral weight $\lambda \in \Lambda^+$ consider the **inertia group**

$$\begin{aligned} K_\lambda &:= \{\chi \in \widehat{G} : \tau_\chi(\lambda) = \lambda\} \\ &= \{\chi \in \widehat{G} : V_\lambda \cong (V_\lambda)^{\alpha_\chi}\}. \end{aligned}$$

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Therefore, $H_\lambda := \big(K_\lambda\big)^\perp$ is a finite subgroup of G, of size $|H_\lambda| = |\widehat{G}\lambda|$ (the size of the orbit of λ), and K_λ is isomorphic to the group of characters of G/H_λ.

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The homomorphism

$$K_\lambda \longrightarrow \mathrm{Aut}\big(\mathrm{End}(V_\lambda)\big), \quad \chi \mapsto \mathrm{Ad}_{u_\chi},$$

corresponds to a compatible $\bar{G} := G/H_\lambda$-grading on $\mathrm{End}(V_\lambda)$.

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The associated commutation factor $\hat{\beta}_\lambda : K_\lambda \times K_\lambda \to \mathbb{F}^\times$ is determined by the commutation of the u_χ's:

$$u_{\chi_1} u_{\chi_2} = \hat{\beta}_\lambda(\chi_1, \chi_2) u_{\chi_2} u_{\chi_1}.$$

Brauer invariant and Schur index

The class $[\mathrm{End}(V_\lambda)]$ in the $\Big(G/H_\lambda\Big)$-graded Brauer group is called the **Brauer invariant** of λ. (Notation: $\mathrm{Br}(\lambda)$)

The associated commutation factor $\hat{\beta}_\lambda : K_\lambda \times K_\lambda \to \mathbb{F}^\times$ is determined by the commutation of the u_χ's:

$$u_{\chi_1} u_{\chi_2} = \hat{\beta}_\lambda(\chi_1, \chi_2) u_{\chi_2} u_{\chi_1}.$$

The degree of the graded division algebra $\mathcal{D}$ representing $\mathrm{Br}(\lambda)$ is called the **Schur index** of λ.

Brauer invariant and Schur index

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Proposition

The $\mathcal{L}$-module $(V_\lambda)^k$ admits a $\bar{G} = G/H_\lambda$-grading that makes it a graded simple $\mathcal{L}$-module (where $\mathcal{L}$ is endowed with the natural induced $\bar{G}$-grading) if and only if k equals the Schur index of λ.

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Sketch of proof:

$\mathrm{End}((V_\lambda)^k) \cong M_k(\mathbb{F}) \otimes \mathrm{End}(V_\lambda)$. If k is the Schur index of λ and $\mathcal{D}$ represents $\mathrm{Br}(\lambda)$, then $\mathcal{D}^{\mathrm{op}} \cong M_k(\mathbb{F})$.
Thus $\mathrm{End}((V_\lambda)^k)$ admits a $\bar{G}$-grading with

$$\mathrm{End}((V_\lambda)^k) \cong \mathcal{D}^{\mathrm{op}} \otimes \mathrm{End}(V_\lambda) \cong \mathcal{D}^{\mathrm{op}} \otimes M_r(\mathcal{D}).$$

Hence $[\mathrm{End}((V_\lambda)^k)] = 1$, so the $\bar{G}$-grading on $(V_\lambda)^k$ is elementary, i.e., it is induced by a $\bar{G}$-grading on $(V_\lambda)^k$. □

Induced graded vector space

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If U is a $\bar{G}$-graded $\mathcal{L}$-module, then W is a G-graded $\mathcal{L}$-module:

$$x.(\chi \otimes u) := \chi \otimes \alpha_{\chi^{-1}}(x)u.$$

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Theorem

Up to isomorphisms and shifts, the $W(\mathcal{O})$'s are the graded-simple finite dimensional $\mathcal{L}$-modules.

Modules admitting compatible gradings: (Q2)

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Theorem

An $\mathcal{L}$-module V admits a compatible G-grading if and only if for any $\lambda \in \Lambda^+$ the multiplicities of V_μ in V, for all the elements μ in the orbit $\widehat{G}\lambda$, are equal and divisible by the Schur index of λ.

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In particular, for $\lambda \in \Lambda^+$,

V_λ admits a compatible G-grading
if and only if
H_λ and $\mathrm{Br}(\lambda)$ are trivial.

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$\mathcal{F}$ is isomorphic to a group algebra $\mathbb{F}H$ for a subgroup H of G.
Let $\pi : G \to G/H$ be the natural projection and let $\rho : \mathcal{F} \to \mathbb{F}^{\times}$ be a homomorphism of unital algebras.

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Theorem

- *$V := W/W\ker(\rho)$ is a simple G/H-graded module.*
- *W is isomorphic, as a G-graded module, to the loop module*

$$L_\pi(V) := \bigoplus_{g\in G} V_{\bar{g}} \otimes g \ \left(\subseteq V \otimes \mathbb{F}G\right).$$

Background on algebraic groups

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- Let $\mathcal{G}$ be a semisimple algebraic group with $\mathrm{Lie}(\mathcal{G}) = \mathcal{L}$. Consider the central isogenies

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- $Z(\mathcal{G}^{\mathrm{sc}}) = \ker\big(\mathcal{G}^{\mathrm{sc}} \to \mathcal{G}^{\mathrm{ad}}\big)$ is isomorphic to the group of characters of $\Lambda/\Lambda^{\mathrm{r}}$.
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- $\mathrm{Aut}(\mathcal{L}) = \mathcal{G}^{\mathrm{ad}} \rtimes \mathrm{Aut}(\mathrm{Dyn})$.

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- $\eta : \widehat{G} \to \mathrm{Aut}(\mathcal{L})$, $\chi \mapsto \alpha_\chi$, a 'G-grading' on $\mathcal{L}$.

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- $\lambda \in \Lambda^+$, $\rho : \mathcal{L} \to \mathfrak{gl}(V_\lambda)$ the associated representation.
  If S^λ is the stabilizer of λ in $\mathsf{Aut}(\mathrm{Dyn})$, ρ integrates to a representation
  $$\tilde{\rho} : \mathcal{G}^{\mathrm{sc}} \rtimes S^\lambda \to GL(V_\lambda).$$
  The elements of $Z(\mathcal{G}^{\mathrm{sc}})$ act by scalar multiplication on V_λ, so $\tilde{\rho}$ induces a homomorphism
  $$\Psi_\lambda : Z(\mathcal{G}^{\mathrm{sc}}) \to \mathbb{F}^\times.$$

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- Let $\pi : \mathcal{G}^{\mathrm{sc}} \rtimes S^{\lambda} \to \mathcal{G}^{\mathrm{ad}} \rtimes S^{\lambda}$ be the natural quotient map.
  $\big(K_{\lambda} \to \mathcal{G}^{\mathrm{ad}} \rtimes S^{\lambda} \hookrightarrow \mathrm{Aut}(\mathcal{L})\big)$
  For $\chi \in K_{\lambda}$, let $\tilde{\alpha}_{\chi} \in \mathcal{G}^{\mathrm{sc}} \rtimes S^{\lambda}$ be a preimage of α_{χ}. Then

$$\rho\big(\alpha_{\chi}(x)\big) = \tilde{\rho}(\tilde{\alpha}_{\chi})\rho(x)\tilde{\rho}(\tilde{\alpha}_{\chi})^{-1}.$$

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$$\rho\big(\alpha_\chi(x)\big) = \tilde{\rho}(\tilde{\alpha}_\chi)\rho(x)\tilde{\rho}(\tilde{\alpha}_\chi)^{-1}.$$

- $\widehat{G}$ is abelian, so the commutators $[\tilde{\alpha}_{\chi_1}, \tilde{\alpha}_{\chi_2}]$ lie in $Z(\mathcal{G}^{\mathrm{sc}})$, and the commutation factor is given by:

$$\hat{\beta}_\lambda(\chi_1, \chi_2) = \Psi_\lambda\big([\tilde{\alpha}_{\chi_1}, \tilde{\alpha}_{\chi_2}]\big).$$

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Corollary

If $\mathcal{L}$ *is simple of type* G_2, F_4, *or* E_8, *then any* $\mathcal{L}$*-module admits a compatible grading.*

Brauer invariants for the classical simple Lie algebras

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- If $\hat{\beta}_\lambda$ is not trivial, it can be described in terms of the commutation factor of the natural module, or of the spin modules.

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- For E_6, the Brauer invariant is either trivial or isomorphic to $[(M_3(\mathbb{F}), \text{Pauli grading})]$.

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- For E_7, the Brauer invariant is either trivial or isomorphic to $[(M_2(\mathbb{F}), \text{Pauli grading})]$.

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That's all. Thanks

Type A: inner

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$\mathcal{L} = \mathfrak{sl}_{r+1}(\mathbb{F})$ and assume that the image of $\widehat{G} \to \mathrm{Aut}(\mathcal{L})$ is contained in $\mathrm{Int}(\mathcal{L})$.

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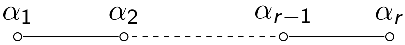

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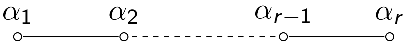

In this case the G-grading in $\mathcal{L}$ is induced by a G-grading on $\mathcal{R} = M_{r+1}(\mathbb{F})$.

For any $\lambda = \sum_{i=1}^r m_i\varpi_i \in \Lambda^+$, $H_\lambda = 1$ and

$$\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^r im_i},$$

where $\hat{\beta} : \widehat{G} \times \widehat{G} \to \mathbb{F}$ is the commutation factor for the action of $\widehat{G}$ on $\mathcal{R}$.

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- If $m_i \neq m_{r+1-i}$ for some i, then $H_\lambda = \langle h\rangle$ and

$$\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{r} im_i},$$

  where $\hat{\beta}$ is the commutation factor for the action of $(G/\langle h\rangle)\widehat{}$ on $\mathcal{R}$.
- If r is even and $m_i = m_{r+1-i}$ for all i, then $H_\lambda = 1$ and $\mathrm{Br}(\lambda) = 1$.
- If r is odd and $m_i = m_{r+1-i}$ for all i, then $H_\lambda = 1$, but $\mathrm{Br}(\lambda)$ may be nontrivial (the description is quite technical).

Type B

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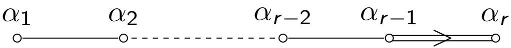

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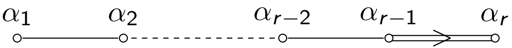

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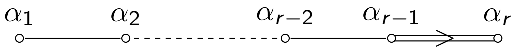

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The G-grading on $\mathcal{L}$ is always induced by a compatible G-grading on V_{ϖ_1}.

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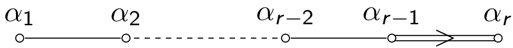

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The G-grading on $\mathcal{L}$ is always induced by a compatible G-grading on V_{ϖ_1}.

For any $\lambda = \sum_{i=1}^{r} m_i \varpi_i \in \Lambda^+$, $H_\lambda = 1$ and

$$\mathrm{Br}(\lambda) = \hat{\gamma}^{m_r} \qquad \text{(it depends only on } m_r\text{!)}$$

where $\hat{\gamma}$ is the commutation factor of the induced action of $\widehat{G}$ on $\mathfrak{Cl}_{\bar{0}}(V_{\varpi_1})$.

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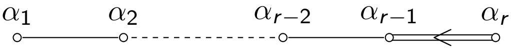

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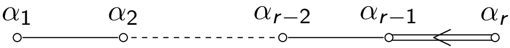

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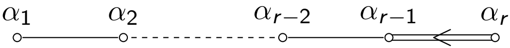

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$$\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{\lfloor (r+1)/2 \rfloor} m_{2i-1}}$$

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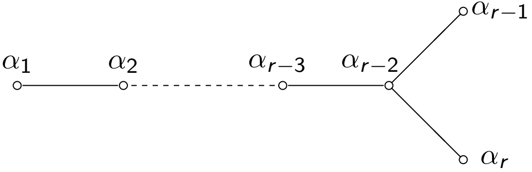

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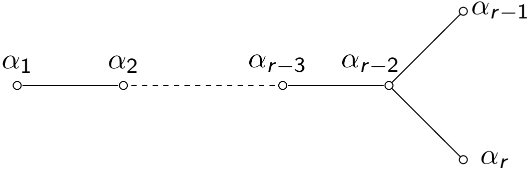

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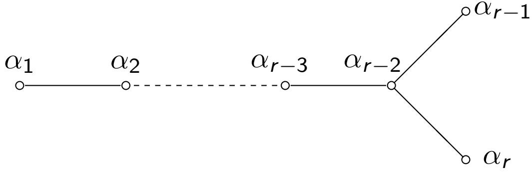

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The G-grading on $\mathcal{L}$ is induced by a grading on $\mathcal{R} = M_{2r}(\mathbb{F}) \simeq \mathrm{End}(V_{\varpi_1})$.
It is said to be *inner* if the image of $\widehat{G} \to \mathrm{Aut}(\mathcal{L})$ is contained in $\mathrm{Int}(\mathcal{L})$; otherwise it is called *outer*. (For $r = 4$ there are two possibilities here.)

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For any $\lambda = \sum_{i=1}^{r} m_i \varpi_i \in \Lambda^+$, $H_\lambda = 1$ and:

- If $m_{r-1} \equiv m_r \pmod 2$, then $\mathrm{Br}(\lambda)$ depends only on the commutation factor of the action of $\widehat{G}$ on $\mathcal{R}$.
- Otherwise it also depends on the commutation factors of the induced action of $\widehat{G}$ on the two simple ideals of $\mathfrak{Cl}_{\bar{0}}(V_{\varpi_1})$.

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Here there exists a distinguished order 2 element $h \in G$ such that the induced $\bar{G} = G/\langle h \rangle$-grading on $\mathcal{L}$ is inner.

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For any $\lambda = \sum_{i=1}^{r} m_i \varpi_i \in \Lambda^+$:

- If $m_{r-1} \neq m_r$ but $m_{r-1} \equiv m_r \pmod 2$, then $H_\lambda = \langle h \rangle$ and $\mathrm{Br}(\lambda) = 1$ (in the $G/\langle h \rangle$-graded Brauer group!).
- If $m_{r-1} \not\equiv m_r \pmod 2$, then $H_\lambda = \langle h \rangle$ and $\mathrm{Br}(\lambda)$ is given in terms of the commutation factor of $\widehat{(G/\langle h \rangle)}$ on $\mathfrak{Cl}_{\bar{0}}(V_{\varpi_1})$.
- If $m_{r-1} = m_r$, then $H_\lambda = 1$ and

$$\mathrm{Br}(\lambda) = \hat{\beta}^{\sum_{i=1}^{\lfloor r/2 \rfloor} m_{2i-1}},$$

where $\hat{\beta}$ is the commutation factor of the action of $\widehat{G}$ on $\mathcal{R}$.

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Here there exists a distinguished order 3 element $h \in G$ such that the induced $\bar{G} = G/\langle h \rangle$-grading on $\mathcal{L}$ is inner.

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For any $\lambda = \sum_{i=1}^{r} m_i \varpi_i \in \Lambda^+$:

- If $m_1 = m_3 = m_4$, then $H_\lambda = 1$ and $\mathrm{Br}(\lambda) = 1$.
- Otherwise $H_\lambda = \langle h \rangle$ and $\mathrm{Br}(\lambda) = 1$ (in the $G/\langle h \rangle$-graded Brauer group!).