Gradings on composition algebras

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Graded Algebras & Superalgebras

1. Composition algebras

2. Gradings on Hurwitz algebras

3. Symmetric composition algebras

4. Gradings on symmetric composition algebras

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1 Composition algebras

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Composition algebras

Definition

A *composition algebra* over a field k is a triple $(C, \cdot, n)$ where

- C is a vector space over k,
- $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
- $n : C \to k$ is a multiplicative nondegenerate quadratic form:
  - its polar $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate,
  - $n(x \cdot y) = n(x)n(y)\ \forall x, y \in C$.

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The unital composition algebras will be called *Hurwitz algebras*.

Hurwitz algebras

Hurwitz algebras form a class of degree two algebras:

$$x^{\cdot 2} - n(x,1)x + n(x)1 = 0$$

for any x.
They are endowed with an antiautomorphism, the *standard conjugation*:

$$\bar{x} = n(x,1)1 - x,$$

satisfying

$$\bar{\bar{x}} = x, \quad x + \bar{x} = n(x,1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1.$$

Cayley-Dickson doubling process

Let $(B, \cdot, n)$ be an associative Hurwitz algebra, and let λ be a nonzero scalar in the ground field k. Consider the direct sum of two copies of B:

$$C = B \oplus Bu,$$

with the following multiplication and nondegenerate quadratic form that extend those on B:

$$\begin{aligned}(a + bu) \cdot (c + du) &= (a \cdot c + \lambda \bar{d} \cdot b) + (d \cdot a + b \cdot \bar{c})u,\\ n(a + bu) &= n(a) - \lambda n(b).\end{aligned}$$

Then $(C, \cdot, n)$ is again a Hurwitz algebra, which is denoted by $CD(B, \lambda)$

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Notation: $CD(A, \mu, \lambda) := CD\big(CD(A, \mu), \lambda\big)$.

Generalized Hurwitz Theorem

Theorem

Every Hurwitz algebra over a field k is isomorphic to one of the following:

(i) *The ground field k if its characteristic is $\neq 2$.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = k1 + kv$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.*

(iii) *A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)*

(iv) *A Cayley algebra $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)*

The split Hurwitz algebras

There are 4 split (i.e., $\exists x$ s.t. $n(x) = 0$) Hurwitz algebras:

$$k, \qquad k \times k, \qquad \mathrm{Mat}_2(k), \qquad C(k).$$

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Canonical basis of the *split Cayley algebra* $C(k) = CD\big(\mathrm{Mat}_2(k), -1\big)$:

$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

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$$n(e_1, e_2) = n(u_i, v_i) = 1, \quad \text{(otherwise 0)}$$

$$e_1^2 = e_1, \quad e_2^2 = e_2,$$
$$e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3)$$
$$u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3)$$
$$u_i u_{i+1} = -u_{i+1} u_i = v_{i+2}, \; v_i v_{i+1} = -v_{i+1} v_i = u_{i+2}, \; \text{(indices modulo 3)}$$

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Gradings

$$A = \oplus_{g\in G} A_g,$$

$\forall g_1, g_2 \in G$, either $A_{g_1}A_{g_2} = 0$ or $\exists g_3 \in G$ such that $0 \neq A_{g_1}A_{g_2} \subseteq A_{g_3}$.

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Given another grading

$$A = \oplus_{h \in H} A_h,$$

of A, the first grading is said to be a *coarsening* of the second one, and then this latter one is called a *refinement* of the former, in case

$$\forall h \in H \text{ there is a } g \in G \text{ with } A_h \subseteq A_g.$$

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The two gradings are said to be *equivalent* if there exists $\varphi \in \mathrm{Aut}(A)$ such that for any $g \in G$ with $A_g \neq 0$, there is an $h \in H$ with $\varphi(A_g) = A_h$.

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A grading is said to be of *type* $(h_1, \dots, h_r)$ if for any i there are h_i homogeneous subspaces of dimension i.

Group gradings

The most interesting gradings are those for which the index set G is a group and for any $g_1, g_2 \in G$, $A_{g_1}A_{g_2} \subseteq A_{g_1 g_2}$. These are called *group gradings*.

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Given a group grading $A = \oplus_{g\in G}A_g$, it will be always assumed that the group G is generated by its subset $\{g \in G : A_g \neq 0\}$.

It is known that the grading group of any Hurwitz algebra is always abelian.

Universal grading group

In order to avoid equivalent gradings, given any grading $A = \oplus_{g\in G} A_g$ of an arbitrary algebra, we will consider the *universal grading group*, which is defined as the quotient $\hat{G} = \mathbb{Z}(G)/R$ of the abelian group $\mathbb{Z}(G)$ freely generated by the set G, modulo the subgroup R generated by the set $\{a + b - c : a, b, c \in G,\ 0 \neq A_a A_b \subseteq A_c\}$. Then A is $\hat{G}$-graded with $A_\gamma = \sum\{A_g : g + R = \gamma\}$.

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If the given grading $A = \oplus_{g\in G} A_g$ is already a group grading with abelian G, then G is a quotient of the universal grading group $\hat{G}$ and the given grading is equivalent to the new grading $A = \oplus_{\gamma\in\hat{G}} A_\gamma$ (here the automorphism φ can be taken to be the identity). Therefore, in dealing with gradings over abelian groups, up to equivalence, it is enough to consider the universal grading groups.

Gradings on Cayley algebras

In order to get all the group gradings on Cayley algebras, it is enough to use the following simple facts for any such grading $C = \oplus_{g\in G} C_g$:

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- The grading group is always abelian.
- $n(C_g, C_h) = 0$ unless $g + h = 0$.
- $\forall g \in G$, $\oplus_{n\in\mathbb{Z}} C_{ng}$ is a composition subalgebra.

Gradings on Cayley algebras

Theorem (E. 98)

Let $C = \oplus_{g\in G} C_g$ *be a nontrivial group grading of a Cayley algebra over a field* k*, where* G *is the universal grading group. Then either:*

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Let $C = \oplus_{g \in G} C_g$ be a nontrivial group grading of a Cayley algebra over a field k, where G is the universal grading group. Then either:

1. $G = \mathbb{Z}_2, \quad C = CD(Q, \alpha) = Q \oplus Qu$ and
$$C_{\bar{0}} = Q, \quad C_{\bar{1}} = Q^{\perp} = Qu.$$

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2. $G = \mathbb{Z}_2^2$, $\quad C = CD(K,\alpha,\beta) = \big(K \oplus Kx\big) \oplus \big(K \oplus Kx\big)y$ and
$$C_{(\bar 0,\bar 0)} = K, \quad C_{(\bar 1,\bar 0)} = Kx, \quad C_{(\bar 0,\bar 1)} = Ky, \quad C_{(\bar 1,\bar 1)} = K(xy).$$

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$$C_{(\bar 0,\bar 0)} = K, \quad C_{(\bar 1,\bar 0)} = Kx, \quad C_{(\bar 0,\bar 1)} = Ky, \quad C_{(\bar 1,\bar 1)} = K(xy).$$

3. $G = \mathbb{Z}_2^3$ (char $k \neq 2$), $\quad C = CD(k,\alpha,\beta,\gamma)$ and
$$C_{(\bar 1,\bar 0,\bar 0)} = kx, \quad C_{(\bar 0,\bar 1,\bar 0)} = ky, \quad C_{(\bar 0,\bar 0,\bar 1)} = kz.$$

Gradings on Cayley algebras

4. $G = \mathbb{Z}_3, \quad C = C(k)$ (split) and

$$C_{\bar{0}} = \text{span}\{e_1, e_2\}, \ C_{\bar{1}} = \text{span}\{u_1, u_2, u_3\}, \ C_{\bar{2}} = \text{span}\{v_1, v_2, v_3\}.$$

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5. $G = \mathbb{Z}_4, \quad C = C(k)$ and

$$\begin{aligned} C_{\bar{0}} &= \operatorname{span}\{e_1, e_2\}, & C_{\bar{1}} &= \operatorname{span}\{u_1, u_2\}, \\ C_{\bar{2}} &= \operatorname{span}\{u_3, v_3\}, & C_{\bar{3}} &= \operatorname{span}\{v_1, v_2\}. \end{aligned}$$

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6. $G = \mathbb{Z}$ (3-grading), $\quad C = C(k)$ and

$$C_0 = \text{span}\{e_1, e_2, u_3, v_3\}, \ C_1 = \text{span}\{u_1, v_2\}, \ C_{-1} = \text{span}\{u_2, v_1\}.$$

Gradings on Cayley algebras

7. $G = \mathbb{Z}$, (5-grading): $C = C(k)$ and:

$$C_0 = \text{span}\{e_1, e_2\}, \quad C_1 = \text{span}\{u_1, u_2\}, \quad C_2 = \text{span}\{v_3\},$$
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8. $G = \mathbb{Z}^2$: $C = C(k)$ and:

$$C_{(0,0)} = \text{span}\{e_1, e_2\},$$
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9. $G = \mathbb{Z} \times \mathbb{Z}_2$: $C = C(k)$ and:

$$C_{(0,\bar{0})} = \text{span}\{e_1, e_2\}, \quad C_{(0,\bar{1})} = \text{span}\{u_3, v_3\},$$
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Gradings on Cayley algebras

Corollary

If char $k \neq 2$*, all the gradings of a Cayley algebra* C *are coarsenings of either a* $\mathbb{Z}_2^3$*-grading or a* $\mathbb{Z}^2$*-grading.*

Gradings on Cayley algebras

Corollary

If char $k \neq 2$, *all the gradings of a Cayley algebra C are coarsenings of either a $\mathbb{Z}_2^3$-grading or a $\mathbb{Z}^2$-grading.*

Remark

The gradings on Hurwitz algebras of dimension ≤ 4 are obtained by restricting the ones in Cayley algebras.

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Symmetric composition algebras

Definition

A composition algebra $(S, *, n)$ is said to be *symmetric* if the polar form of its norm is associative:

$$n(x * y, z) = n(x, y * z),$$

for any $x, y, z \in S$.

This is equivalent to the condition:

$$(x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in S$.

Examples

- *Para-Hurwitz algebras*: Given a Hurwitz algebra $(C, \cdot, n)$, its para-Hurwitz counterpart is the composition algebra $(C, \bullet, n)$, where

$$x \bullet y = \bar{x} \cdot \bar{y}.$$

This algebra will be denoted by $\bar{C}$ for short.
The unity of $(C, \cdot, n)$ becomes a *para-unit* in $\bar{C}$, that is, an element e such that $e \bullet x = x \bullet e = n(e, x)e - x$. If the dimension is at least 4, the para-unit is unique, and it is the unique idempotent that spans the commutative center of the para-Hurwitz algebra.

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- *Petersson algebras*: Let τ be an automorphism of a Hurwitz algebra $(C, \cdot, n)$ with $\tau^3 = 1$, and consider the new multiplication defined on C by means of:

$$x * y = \tau(\bar{x}) \cdot \tau^2(\bar{y}).$$

  The algebra $(C, *, n)$ is a symmetric composition algebra, which will be denoted by $\bar{C}_\tau$ for short.

Okubo algebras

Let $\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ be a canonical basis of $C(k)$. Then the linear map $\tau_{st} : C(k) \to C(k)$ determined by the conditions:

$$\tau_{st}(e_i) = e_i,\ i = 1, 2; \quad \tau_{st}(u_i) = u_{i+1},\ \tau_{st}(v_i) = v_{i+1} \text{ (indices modulo 3)},$$

is clearly an order 3 automorphism of $C(k)$. ("*st*" stands for *standard*.)

Definition

The associated Petersson algebra $P_8(k) = \overline{C(k)}_{\tau_{st}}$ is called the *pseudo-octonion algebra* over the field k.

The forms of $P_8(k)$ are called *Okubo algebras*.

(This is not the original definition of the pseudo-octonion algebra due to Okubo in 1978.)

Classification

Theorem (E.-Myung 93, E. 97)

Any symmetric composition algebra is either:

- *a para-Hurwitz algebra,*
- *a form of a two-dimensional para-Hurwitz algebras without idempotent elements (with a precise description),*
- *an Okubo algebra.*

Classification

Moreover:

- If char $k \neq 3$ and $\exists \omega \neq 1 = \omega^3$ in k, then any Okubo algebra is, up to isomorphism, the algebra A_0 of zero trace elements in a central simple degree 3 associative algebra with multiplication

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy),$$

and norm $n(x) = -\frac{1}{2} \operatorname{tr}(x^2)$.

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- If char $k \neq 3$ and $\nexists \omega \neq 1 = \omega^3$ in k, then any Okubo algebra is, up to isomorphism, the algebra $S(A, j)_0 = \{x \in A_0 : j(x) = -x\}$, where (A, j) is a central simple degree three associative algebra over $k[\omega]$ and j is a $k[\omega]/k$-involution of second kind, with multiplication and norm as above.

Classification

Finally, if char $k = 3$, for any Okubo algebra there are nonzero scalars $\alpha, \beta \in k$ and a basis such that the multiplication table is:

$*$	$x_{1,0}$	$x_{-1,0}$	$x_{0,1}$	$x_{0,-1}$	$x_{1,1}$	$x_{-1,-1}$	$x_{-1,1}$	$x_{1,-1}$
$x_{1,0}$	$-\alpha x_{-1,0}$	0	0	$x_{1,-1}$	0	$x_{0,-1}$	0	$\alpha x_{-1,-1}$
$x_{-1,0}$	0	$-\alpha^{-1}x_{1,0}$	$x_{-1,1}$	0	$x_{0,1}$	0	$\alpha^{-1}x_{1,1}$	0
$x_{0,1}$	$x_{1,1}$	0	$-\beta x_{0,-1}$	0	$\beta x_{1,-1}$	0	0	$x_{1,0}$
$x_{0,-1}$	0	$x_{-1,-1}$	0	$-\beta^{-1}x_{0,1}$	0	$\beta^{-1}x_{-1,1}$	$x_{-1,0}$	0
$x_{1,1}$	$\alpha x_{-1,1}$	0	0	$x_{1,0}$	$-(\alpha\beta)x_{-1,-1}$	0	$\beta x_{0,-1}$	0
$x_{-1,-1}$	0	$\alpha^{-1}x_{1,-1}$	$x_{-1,0}$	0	0	$-(\alpha\beta)^{-1}x_{1,1}$	0	$\beta^{-1}x_{0,1}$
$x_{-1,1}$	$x_{0,1}$	0	$\beta x_{-1,-1}$	0	0	$\alpha^{-1}x_{1,0}$	$-\alpha^{-1}\beta x_{1,-1}$	0
$x_{1,-1}$	0	$x_{0,-1}$	0	$\beta^{-1}x_{1,1}$	$\alpha x_{-1,0}$	0	0	$-\alpha\beta^{-1}x_{-1,1}$

Okubo algebras with isotropic norm

Remark

Okubo algebras with isotropic norm present this same multiplication table, no matter what the characteristic of the ground field is.

Notation: $\mathcal{O}_{\alpha,\beta}$.

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Remark

This multiplication table gives a $\mathbb{Z}_3^2$-grading of the corresponding Okubo algebra of type (8), which will be referred to as the *standard* $\mathbb{Z}_3^2$*-grading*.

Different presentations of the pseudo-octonion algebra

Let $\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ be a canonical basis of $C(k)$. Consider the following order 3 automorphisms:

τ_{st}: $\tau_{st}(e_i) = e_i$ (i=1,2), $\tau_{st}(u_i) = u_{i+1}$, $\tau_{st}(v_i) = v_{i+1}$ (indices modulo 3).

τ_{nst}: $\tau_{nst}(e_1) = e_1$, $\tau_{nst}(e_2) = e_2$,
$\tau_{nst}(u_1) = u_2$, $\tau_{nst}(u_2) = -u_1 - u_2$, $\tau_{nst}(u_3) = u_3$,
$\tau_{nst}(v_1) = -v_1 + v_2$, $\tau_{nst}(v_2) = -v_1$, $\tau_{nst}(v_3) = v_3$.

τ_ω: (Assuming char $k \neq 3$ and $\exists \omega \neq 1 = \omega^3$ in k)
$\tau_\omega(e_1) = e_1$, $\tau_\omega(e_2) = e_2$,
$\tau_\omega(u_i) = \omega^i u_i$, $\tau_\omega(v_i) = \omega^{-i} v_i$ $(i = 1, 2, 3)$.

Different presentations of the pseudo-octonion algebra

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Lemma

The Petersson algebras $\overline{C(k)}_{\tau_{st}}$, $\overline{C(k)}_{\tau_{nst}}$ and $\overline{C(k)}_{\tau_\omega}$ are all isomorphic to the pseudo-octonion algebra $P_8(k)$.

1 Composition algebras

2 Gradings on Hurwitz algebras

3 Symmetric composition algebras

4 Gradings on symmetric composition algebras

5 Gradings on exceptional simple Lie algebras

Gradings on para-Hurwitz algebras

Theorem

Gradings on para-Hurwitz algebras of dimension 4 *or* 8

$$\updownarrow$$

Gradings on their Hurwitz counterparts.

Theorem (E. 08)

Let $S = \oplus_{g \in G} S_g$ *be a nontrivial group grading of an Okubo algebra over a field* k*, where* G *is the universal grading group. Then either:*

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$$S_{\bar{0}} = K \oplus Kx, \qquad S_{\bar{1}} = (K \oplus Kx)y.$$

Gradings on Okubo algebras

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2. $G = \mathbb{Z}_2$ (char $k = 3$), $S = P_8(k) = \overline{C(k)}_{\tau_{nst}}$ and
$$S_{\bar{0}} = \text{span}\{e_1, e_2, u_3, v_3\}, \quad S_{\bar{1}} = \text{span}\{u_1, u_2, v_1, v_2\}.$$

3. $G = \mathbb{Z}_2^2$ (char $k \neq 3$), S is the Petersson algebra $\bar{C}_\tau$, where $C = CD(K, \mu, \nu) = (K \oplus Kx) \oplus (K \oplus Kx)y$, $K = k1 + kw$, $w^2 + w + 1 = 0$, and τ is the identity on $K \oplus Kx$ and $\tau(y) = wy$, and

$$S_{(\bar{0},\bar{0})} = K,\ S_{(\bar{1},\bar{0})} = Kx,\ S_{(\bar{0},\bar{1})} = Ky,\ S_{(\bar{1},\bar{1})} = K(xy).$$

Gradings on Okubo algebras

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Gradings on Okubo algebras

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5. $G = \mathbb{Z}_3$ (nonstandard, char $k = 3$), $S = P_8(k) = \overline{C(k)}_{\tau_{nst}}$ and

$$S_{\bar{0}} = \text{span}\{e_1, e_2\},\ S_{\bar{1}} = \text{span}\{u_1, u_2, u_3\},\ S_{\bar{2}} = \text{span}\{v_1, v_2, v_3\}.$$

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7. $G = \mathbb{Z}_4$, $\quad S = P_8(k) = \overline{C(k)}_{\tau_{nst}}$ and

$$\begin{aligned} S_{\bar 0} &= \operatorname{span}\{e_1, e_2\}, & S_{\bar 1} &= \operatorname{span}\{u_1, u_2\}, \\ S_{\bar 2} &= \operatorname{span}\{u_3, v_3\}, & S_{\bar 3} &= \operatorname{span}\{v_1, v_2\}. \end{aligned}$$

Gradings on Okubo algebras

6. $G = \mathbb{Z}_3^2$, $\quad S = \mathcal{O}_{\alpha,\beta}$ with its standard $\mathbb{Z}_3^2$-grading.

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8. $G = \mathbb{Z}$ (3-grading, char $k \neq 3$), $\quad S = P_8(k) = \overline{C(k)}_{\tau_\omega}$ and

$$S_0 = \text{span}\{e_1, e_2, u_3, v_3\}, \; S_1 = \text{span}\{u_1, v_2\}, \; S_{-1} = \text{span}\{u_2, v_1\}.$$

Gradings on Okubo algebras

9. $G = \mathbb{Z}$ (5-grading), $S = P_8(k) = \overline{C(k)}_{\tau_{nst}}$ and

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10. $G = \mathbb{Z}^2$ (char $k \neq 3$), $S = P_8(k) = \overline{C(k)}_{\tau_\omega}$ and

$$S_{(0,0)} = \text{span}\{e_1, e_2\},$$
$$S_{(1,0)} = \text{span}\{u_1\}, \quad S_{(0,1)} = \text{span}\{u_2\}, \quad S_{(1,1)} = \text{span}\{v_3\},$$
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11. $G = \mathbb{Z} \times \mathbb{Z}_2$ (char $k \neq 3$), $S = P_8(k) = \overline{C(k)}_{\tau_\omega}$ and

$$S_{(0,\bar{0})} = \text{span}\{e_1, e_2\}, \quad S_{(0,\bar{1})} = \text{span}\{u_3, v_3\},$$
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Sketch of the proof:

- First one has to prove that either S_0 is a para-Hurwitz algebra (possibly after a field extension), or $S_0 = 0$ and then necessarily the grading is a standard $\mathbb{Z}_3^2$-grading.

  (This is the most difficult part.)

- Then, if $S_0 \neq 0$, one has to show that the Okubo algebra can be presented as a Petersson algebra $\bar{C}_\tau$, for a graded Cayley algebra, and an order 3 automorphism τ which stabilizes all the homogeneous components of C.

  Now, it is enough to check all the possibilities for the different gradings on Cayley algebras.

Gradings on Okubo algebras

Remark

Over an algebraically closed field of characteristic $\neq 3$, all the gradings of the pseudo-octonion algebra are, up to isomorphism, coarsenings of either the $\mathbb{Z}^2$-grading or the standard $\mathbb{Z}_3^2$-grading.

Gradings on Okubo algebras

Remark

The pseudo-octonion algebra $P_8(k)$ was introduced by Okubo (1978), assuming the characteristic is $\neq 3$ and containing the cubic roots of 1, as the set of zero trace 3×3-matrices with multiplication

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy),$$

and norm $n(x) = -\frac{1}{2}\,\mathrm{tr}(x^2)$.

It follows that any grading by an abelian group of the algebra of matrices $\mathrm{Mat}_3(k)$ is inherited by $P_8(k)$, and conversely.

Over an algebraically closed field of characteristic 0, there are just two fine gradings by abelian groups of $\mathrm{Mat}_3(k)$ (Bahturin-Sehgal-Zaicev 2001): an "elementary" $\mathbb{Z}^2$-grading and a $\mathbb{Z}_3^2$-grading. These two gradings induce the two fine gradings of $P_8(k)$.

1 Composition algebras

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Triality Lie algebra

Assume from now on that char $k \neq 2, 3$ and $\omega \in k$.

Let $(S, *, n)$ be any symmetric composition algebra and consider the corresponding orthogonal Lie algebra:

$$\mathfrak{o}(S, n) = \{d \in \mathrm{End}_k(S) : n\big(d(x), y\big) + n\big(x, d(y)\big) = 0 \ \forall x, y \in S\},$$

and the subalgebra of $\mathfrak{o}(S, n)^3$ (with componentwise multiplication):

$$\mathfrak{tri}(S, *, n) = \{(d_0, d_1, d_2) \in \mathfrak{o}(S, n)^3 : d_0(x*y) = d_1(x)*y + x*d_2(y) \ \forall x, y\}$$

This is the *triality Lie algebra*.
The map:

$$\begin{aligned} \theta : \mathfrak{tri}(S, *, n) &\to \mathfrak{tri}(S, *, n) \\ (d_0, d_1, d_2) &\mapsto (d_2, d_0, d_1) \end{aligned}$$

is an automorphism of order 3.

Principle of Local Triality

Theorem

Let $(S, *, n)$ be an eight dimensional symmetric composition algebra . Then:

(i) **(Principle of Local Triality)** *The projection $\pi_0 : \mathfrak{tri}(S, *, n) \to \mathfrak{o}(S, n) : (d_0, d_1, d_2) \mapsto d_0$, is an isomorphism of Lie algebras.*

(ii) *For any $x, y \in S$, consider the triple:*

$$t_{x,y} = \Big(\sigma_{x,y}, \frac{1}{2}n(x, y)id - r_x l_y, \frac{1}{2}q(x, y)id - l_x r_y\Big),$$

where $\sigma_{x,y} : z \mapsto n(x, z)y - n(y, z)x$. Then

$$\mathfrak{tri}(S, *, n) = t_{S,S} \,(= span\,\{t_{x,y} : x, y \in S\}),$$

$$[t_{a,b}, t_{x,y}] = t_{\sigma_{a,b}(x),y} + t_{x,\sigma_{a,b}(y)}.$$

Gradings on D_4

By taking together gradings on a symmetric composition algebra and the order 3 automorphism given by triality, one obtains the following gradings on D_4:

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Theorem

- *A $\mathbb{Z}_2^3$-grading of a para-Cayley algebra $(\bar{C}, \bullet, n)$ induces a $\mathbb{Z}_2^3 \times \mathbb{Z}_3$-grading of the orthogonal Lie algebra $\mathfrak{o}(C, n)$ of type $(14, 7)$.*
- *The standard $\mathbb{Z}_3^2$-grading on an Okubo algebra $(\mathcal{O}, *, n)$ induces a $\mathbb{Z}_3^3$-grading on the orthogonal Lie algebra $\mathfrak{o}(\mathcal{O}, n)$ of type $(24, 2)$.*

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Over an algebraically closed field of characteristic 0, these are two of the 14 different fine gradings of D_4, obtained by Draper-Martín-Viruel.

Freudenthal Magic Square

Let $(S, *, n)$ and $(S', \star, n')$ be two symmetric composition algebras. One can construct a Lie algebra as follows:

$$\mathfrak{g} = \mathfrak{g}(S, S') = \big(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\big) \oplus \Big(\oplus_{i=0}^{2} \iota_i(S \otimes S')\Big),$$

with bracket given by:

- the Lie bracket in $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d'_0, d'_1, d'_2), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d'_i(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' \star y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = n'(x', y')\theta^i(t_{x,y}) + n(x, y)\theta'^i(t'_{x',y'})$,

Freudenthal Magic Square

$\mathfrak{g}(S, S')$		dim S': 1	2	4	8
dim S	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

Freudenthal Magic Square

The Lie algebra $\mathfrak{g}(S, S')$ is naturally $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded with

$$\mathfrak{g}_{(\bar{0},\bar{0})} = \mathfrak{tri}(S) \oplus \mathfrak{tri}(S'),$$

$$\mathfrak{g}_{(\bar{1},\bar{0})} = \iota_0(S \otimes S'), \qquad \mathfrak{g}_{(\bar{0},\bar{1})} = \iota_1(S \otimes S'), \qquad \mathfrak{g}_{(\bar{1},\bar{1})} = \iota_2(S \otimes S').$$

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Also, the order 3 automorphisms θ and θ' extend to an order 3 automorphism Θ of $\mathfrak{g}(S, S')$. The eigenspaces of Θ constitute a $\mathbb{Z}_3$-grading of $\mathfrak{g}(S, S')$.

Albert algebras

Theorem

Let $(S, *, n)$ be any symmetric composition algebra. On the vector space $\mathbb{A} = \mathbb{A}(S) = k^3 \oplus \big(\oplus_{i=0}^{2} \iota_i(S)\big)$ define a commutative multiplication by:

$$\begin{cases} (\alpha_0, \alpha_1, \alpha_2) \circ (\beta_1, \beta_2, \beta_3) = (\alpha_0\beta_0, \alpha_1\beta_1, \alpha_2\beta_2), \\ (\alpha_0, \alpha_1, \alpha_2) \circ \iota_i(a) = \frac{1}{2}(\alpha_{i+1} + \alpha_{i+2})\iota_i(a), \\ \iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(a * b), \\ \iota_i(a) \circ \iota_i(b) = 2n(a,b)\big(e_{i+1} + e_{i+2}\big). \end{cases}$$

Then

- $\mathbb{A}$ *is a central simple Jordan algebra.*

  If $\dim S = 8$*, this is an Albert algebra.*

- *Its Lie algebra of derivations is, up to isomorphism,* $\mathfrak{g}(k, S)$*.*

  If $\dim S = 8$*, this is a central simple Lie algebra of type* F_4*.*

Gradings on F_4

Theorem

1. *A $\mathbb{Z}_2^3$-grading in a para-Cayley algebra $\bar{C}$ induces $\mathbb{Z}_2^5$-gradings on the Albert algebra $\mathbb{A}(\bar{C})$ and on its Lie algebra of derivations of respective types $(24, 0, 1)$ and $(24, 0, 0, 7)$.*
2. *A standard $\mathbb{Z}_3^2$-grading on an Okubo algebra $\mathcal{O}$ induces $\mathbb{Z}_3^3$-gradings on the Albert algebra $\mathbb{A}(\mathcal{O})$ and on its Lie algebra of derivations of respective types (27) and $(0, 26)$.*

Gradings on F_4

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2. *A standard $\mathbb{Z}_3^2$-grading on an Okubo algebra $\mathcal{O}$ induces $\mathbb{Z}_3^3$-gradings on the Albert algebra $\mathbb{A}(\mathcal{O})$ and on its Lie algebra of derivations of respective types* (27) *and* $(0, 26)$.

For an algebraically closed field of characteristic 0, these gradings are among the four fine gradings on either the Albert algebra or the exceptional simple Lie algebra of type F_4 obtained by Draper and Martín. The other two fine gradings are the "Cartan grading" over $\mathbb{Z}^4$, and a $\mathbb{Z}_2^3 \times \mathbb{Z}$-grading, which is related too to the $\mathbb{Z}_2^3$-gradings on para-Cayley algebras.

Remark

1. The $\mathbb{Z}_3^3$-grading of type $(0, 26)$ on the simple Lie algebra $\mathfrak{g} = \mathfrak{g}(k, \mathcal{O})$ of type F_4 satisfies that

$$\mathfrak{g}_\mu \oplus \mathfrak{g}_{-\mu}$$

is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \mu \in \mathbb{Z}_3^3$.

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2. This $\mathbb{Z}_3^3$-grading can be extended to a $\mathbb{Z}_3^3$-grading of the simple Lie algebra $\mathfrak{g}(\overline{k \times k}, \mathcal{O})$ of type E_6, of type $(0, 0, 26)$, satisfying the same property.

Gradings on E_8

Theorem

1. *If* $\bar{C}$ *and* $\bar{C}'$ *are two* $\mathbb{Z}_2^3$*-graded para-Cayley algebras, these gradings induce a* $\mathbb{Z}_2^8 = \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \times \mathbb{Z}_2^2$*-grading on the simple Lie algebra* $\mathfrak{g}(\bar{C}, \bar{C}')$ *of type* $(192, 0, 0, 14)$.

2. *If* $\mathcal{O}$ *and* $\mathcal{O}'$ *are two* $\mathbb{Z}_3^2$*-graded Okubo algebra, these gradings induce a* $\mathbb{Z}_3^5$*-grading on the simple Lie algebra* $\mathfrak{g}(\mathcal{O}, \mathcal{O}')$ *of type* $(240, 0, 0, 2)$.

Gradings on E_8

A coarsening of the previous $\mathbb{Z}_2^8$-grading can be obtained by means of the projection

$$\begin{aligned} \mathbb{Z}_2^8 = \mathbb{Z}_2^3 \times \mathbb{Z}_2^3 \times \mathbb{Z}_2^2 &\longrightarrow \mathbb{Z}_2^3 \times \mathbb{Z}_2^2 = \mathbb{Z}_2^5 \\ (\alpha, \beta, \gamma) &\mapsto (\alpha + \beta, \gamma), \end{aligned}$$

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Theorem

The corresponding $\mathbb{Z}_2^5$-grading on the simple Lie algebra of type E_8 satisfies that all its homogeneous subspaces are Cartan subalgebras (a Dempwolff decomposition).

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Similar gradings can be obtained for E_6 and E_7.

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Theorem

The corresponding $\mathbb{Z}_2^5$-grading on the simple Lie algebra of type E_8 satisfies that all its homogeneous subspaces are Cartan subalgebras (a Dempwolff decomposition).

Similar gradings can be obtained for E_6 and E_7.

That's all. Thanks