Gradings on the octonions, the Albert algebra, and exceptional simple Lie algebras

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Gradings on Lie algebras have been extensively used since the beginning of Lie theory:

- the Cartan grading on a complex semisimple Lie algebra is the $\mathbb{Z}^r$-grading (r being the rank) whose homogeneous components are the root spaces relative to a Cartan subalgebra (which is the zero component),
- symmetric spaces are related to $\mathbb{Z}_2$-gradings,
- Kac–Moody Lie algebras to gradings by a finite cyclic group,
- the theory of Jordan algebras and pairs to 3-gradings on Lie algebras, etc.

In 1989, a systematic study of gradings on Lie algebras was started by Patera and Zassenhaus.

Fine gradings on the classical simple complex Lie algebras, other than D_4, by arbitrary abelian groups were considered by Havlícek, Patera, and Pelantova in 1998.

The arguments there are computational and the problem of classification of fine gradings is not completely settled. The complete classification, up to equivalence, of fine gradings on all classical simple Lie algebras (including D_4) over algebraically closed fields of characteristic zero has been obtained quite recently.

For any abelian group G, the classification of all G-gradings, up to isomorphism, on the classical simple Lie algebras other than D_4 over algebraically closed fields of characteristic different from two has been achieved in 2010 by Bahturin and Kochetov, using methods developed in the last years by a number of authors.

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The gradings on the octonions and on the Albert algebra are instrumental in obtaining the gradings on the exceptional simple Lie algebras.

Outline

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The Albert algebra

G_2 and F_4

Jordan gradings on exceptional simple Lie algebras

Composition algebras

Freudenthal's Magic Square

Gradings

Gradings on Octonions

Composition algebras

Definition

A *composition algebra* over a field $\mathbb{F}$ is a triple $(C, \cdot, n)$ where

- C is a vector space over $\mathbb{F}$,
- $x \cdot y$ is a bilinear multiplication $C \times C \to C$,
- $n : C \to \mathbb{F}$ is a multiplicative nondegenerate quadratic form:
  - its polar $n(x, y) = n(x + y) - n(x) - n(y)$ is nondegenerate,
  - $n(x \cdot y) = n(x)n(y)\ \forall x, y \in C$.

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The unital composition algebras will be called *Hurwitz algebras*.

Hurwitz algebras

Hurwitz algebras form a class of degree two algebras:

$$x^{\cdot 2} - n(x,1)x + n(x)1 = 0$$

for any x.
They are endowed with an antiautomorphism, the *standard conjugation*:

$$\bar{x} = n(x,1)1 - x,$$

satisfying

$$\bar{\bar{x}} = x, \quad x + \bar{x} = n(x,1)1, \quad x \cdot \bar{x} = \bar{x} \cdot x = n(x)1.$$

Cayley-Dickson doubling process

Let $(B, \cdot, n)$ be an associative Hurwitz algebra, and let λ be a nonzero scalar in the ground field $\mathbb{F}$. Consider the direct sum of two copies of B:

$$C = B \oplus Bu,$$

with the following multiplication and nondegenerate quadratic form that extend those on B:

$$\begin{aligned} &(a + bu) \cdot (c + du) = (a \cdot c + \lambda \bar{d} \cdot b) + (d \cdot a + b \cdot \bar{c})u, \\ &n(a + bu) = n(a) - \lambda n(b). \end{aligned}$$

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Notation: $CD(A, \mu, \lambda) := CD\big(CD(A, \mu), \lambda\big).$

Generalized Hurwitz Theorem

Theorem

Every Hurwitz algebra over a field $\mathbb{F}$ is isomorphic to one of the following:

(i) *The ground field $\mathbb{F}$ if its characteristic is $\neq 2$.*

(ii) *A quadratic commutative and associative separable algebra $K(\mu) = \mathbb{F}1 + \mathbb{F}v$, with $v^2 = v + \mu$ and $4\mu + 1 \neq 0$. The norm is given by its generic norm.*

(iii) *A quaternion algebra $Q(\mu, \beta) = CD(K(\mu), \beta)$. (These four dimensional algebras are associative but not commutative.)*

(iv) *A Cayley algebra $C(\mu, \beta, \gamma) = CD(K(\mu), \beta, \gamma)$. (These eight dimensional algebras are alternative, but not associative.)*

Symmetric composition algebras

Definition

A composition algebra $(S, *, n)$ is said to be *symmetric* if the polar form of its norm is associative:

$$n(x * y, z) = n(x, y * z),$$

for any $x, y, z \in S$.

This is equivalent to the condition:

$$(x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in S$.

Examples

- *Para-Hurwitz algebras*: Given a Hurwitz algebra $(C, \cdot, n)$, its para-Hurwitz counterpart is the composition algebra $(C, \bullet, n)$, where
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- *Okubo algebras*: Assume $\operatorname{char} \mathbb{F} \neq 3$ and $\exists\, \omega \neq 1 = \omega^3$ in $\mathbb{F}$. Consider the algebra A_0 of zero trace elements in a central simple degree 3 associative algebra with multiplication
$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3} \operatorname{tr}(xy)1,$$
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(There is a more general definition valid over arbitrary fields.)

Classification

Theorem (E.-Myung 93, E. 97)

Any symmetric composition algebra is either:

- *a para-Hurwitz algebra,*
- *a form of a two-dimensional para-Hurwitz algebra without idempotent elements (with a precise description),*
- *an Okubo algebra.*

Triality Lie algebra

Assume from now on that $\operatorname{char}\mathbb{F} \neq 2$.

Let $(S, *, n)$ be any symmetric composition algebra and consider the corresponding orthogonal Lie algebra:

$$\mathfrak{o}(S, n) = \{d \in \operatorname{End}_{\mathbb{F}}(S) : n\big(d(x), y\big) + n\big(x, d(y)\big) = 0 \ \forall x, y \in S\},$$

and the subalgebra of $\mathfrak{o}(S, n)^3$ (with componentwise multiplication):

$$\mathfrak{tri}(S, *, n) = \\ \{(d_1, d_2, d_3) \in \mathfrak{o}(S, n)^3 : d_3(x * y) = d_1(x) * y + x * d_2(y) \ \forall x, y\}$$

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This is the *triality Lie algebra*.

The map: $\theta : \mathfrak{tri}(S,*,n) \to \mathfrak{tri}(S,*,n)$, $(d_1, d_2, d_3) \mapsto (d_3, d_1, d_2)$ is an automorphism of order 3, (triality automorphism).

Principle of Local Triality

Theorem (Principle of Local Triality)

Let $(S, *, n)$ be an eight dimensional symmetric composition algebra. Then the projection

$$\begin{aligned} \pi_1 : \mathfrak{tri}(S, *, n) &\longrightarrow \mathfrak{o}(S, n) \\ (d_1, d_2, d_3) &\mapsto \quad d_1, \end{aligned}$$

is an isomorphism of Lie algebras.

Freudenthal's Magic Square

Let $(S, *, n)$ and $(S', \star, n')$ be two symmetric composition algebras. One can construct a Lie algebra as follows:

$$\mathfrak{g} = \mathfrak{g}(S, S') = \left(\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')\right) \oplus \left(\oplus_{i=1}^{3} \iota_i (S \otimes S')\right),$$

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with bracket given by:

- the Lie bracket in $\mathfrak{tri}(S) \oplus \mathfrak{tri}(S')$, which thus becomes a Lie subalgebra of $\mathfrak{g}$,
- $[(d_1, d_2, d_3), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d'_1, d'_2, d'_3), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d'_i(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((x * y) \otimes (x' \star y')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = n'(x', y')\theta^i(t_{x,y}) + n(x, y)\theta'^i(t'_{x',y'})$,

Freudenthal's Magic Square

		dim S'			
$\mathfrak{g}(S, S')$		1	2	4	8
dim S	1	A_1	A_2	C_3	F_4
	2	A_2	$A_2 \oplus A_2$	A_5	E_6
	4	C_3	A_5	D_6	E_7
	8	F_4	E_6	E_7	E_8

Definition

G abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

G-grading on $\mathcal{A}$:

$$\mathcal{A} = \oplus_{g \in G} \mathcal{A}_g,$$

$$\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

Example: Pauli matrices

$\mathcal{A} = \mathsf{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \oplus_{(\bar{\imath},\bar{\jmath}) \in \mathbb{Z}_n \times \mathbb{Z}_n} \mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F} X^i Y^j.$$

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$\mathcal{A}$ becomes a *graded division algebra*.

Basic definitions (Patera-Zassenhaus)

Let $\Gamma : \mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$ be a grading on $\mathcal{A}$ $\quad(\dim_{\mathbb{F}} \mathcal{A} < \infty$, $\mathbb{F} = \bar{\mathbb{F}}$, $\operatorname{char} \mathbb{F} \neq 2)$:

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- The *support* of Γ is $\operatorname{Supp} \Gamma = \{g \in G : \mathcal{A}_g \neq 0\}$.
- The *universal grading group* of Γ is the group $U(\Gamma)$ generated by $\operatorname{Supp} \Gamma$ subject to the relations $g_1 g_2 = g_3$ if $0 \neq \mathcal{A}_{g_1} \mathcal{A}_{g_2} \subseteq \mathcal{A}_{g_3}$.

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The grading Γ is then a grading too by $U(\Gamma)$.

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- The *automorphism group*

$$\mathrm{Aut}(\Gamma) = \{\varphi \in \mathrm{Aut}\,\mathcal{A} : \\ \exists \alpha \in \mathit{Sym}(\mathrm{Supp}\ \Gamma) \text{ s.t. } \varphi(\mathcal{A}_g) \subseteq \mathcal{A}_{\alpha(g)}\ \forall g\}.$$

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- The *diagonal group*

$$\mathrm{Diag}(\Gamma) = \{\varphi \in \mathrm{Aut}(\Gamma) : \forall g \in \mathrm{Supp}\ \Gamma\ \exists \lambda_g \in \mathbb{F}^\times \text{ s.t. } \varphi|_{\mathcal{A}_g} = \lambda_g\, \mathit{id}\}.$$

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- The quotient $W(\Gamma) = \mathsf{Aut}(\Gamma)/\operatorname{Stab}(\Gamma)$ is the *Weyl group* of Γ.

$W(\Gamma)$ acts by automorphisms on $U(\Gamma)$

Each $\varphi \in \mathsf{Aut}(\Gamma)$ determines a self-bijection α of $\mathrm{Supp}\ \Gamma$ that induces an automorphism of the universal grading group $U(\Gamma)$. Then, there appears a natural group homomorphism:

$$\mathsf{Aut}(\Gamma) \to \mathsf{Aut}(U(\Gamma))$$

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Thus, the Weyl group embeds naturally in $\mathsf{Aut}(U(\Gamma))$, i.e., there is a natural action of the Weyl group on $U(\Gamma)$ by automorphisms.

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Remark

$\mathrm{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$.

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- Γ is *fine* if it admits no proper refinement.

  Any grading is a coarsening of a fine grading.

Basic definitions (Patera-Zassenhaus)

- Γ and Γ' are *equivalent* if there is an automorphism $\varphi \in \operatorname{Aut} \mathcal{A}$ such that for any $g \in G$ there is a $g' \in G'$ with $\varphi(\mathcal{A}_g) = \mathcal{A}'_{g'}$.

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- For $G = G'$, Γ and Γ' are *isomorphic* if there is an automorphism $\varphi \in \operatorname{Aut}\mathcal{A}$ such that $\varphi(\mathcal{A}_g) = \mathcal{A}'_g$ for any $g \in G$.

Composition algebras

Freudenthal's Magic Square

Gradings

Gradings on Octonions

The split octonions

Cayley-Dickson process:

$$\mathbb{K} = \mathbb{F} \oplus \mathbb{F}\,\mathbf{i}, \qquad \mathbf{i}^2 = -1,$$
$$\mathbb{H} = \mathbb{K} \oplus \mathbb{K}\,\mathbf{j}, \qquad \mathbf{j}^2 = -1,$$
$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\,\mathbf{l}, \qquad \mathbf{l}^2 = -1,$$

$\mathbb{O}$ is $\mathbb{Z}_2^3$-graded with

$$\deg(\mathbf{i}) = (\bar{1}, \bar{0}, \bar{0}), \quad \deg(\mathbf{j}) = (\bar{0}, \bar{1}, \bar{0}), \quad \deg(\mathbf{l}) = (\bar{0}, \bar{0}, \bar{1}).$$

Cartan grading on the Octonions

$\mathbb{O}$ contains canonical bases:

$$\mathcal{B} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$$

with

$n(e_1, e_2) = n(u_i, v_i) = 1, \quad$ otherwise 0.

$e_1^2 = e_1, \quad e_2^2 = e_2,$
$e_1 u_i = u_i e_2 = u_i, \quad e_2 v_i = v_i e_1 = v_i, \quad (i = 1, 2, 3)$
$u_i v_i = -e_1, \quad v_i u_i = -e_2, \quad (i = 1, 2, 3)$
$u_i u_{i+1} = -u_{i+1} u_i = v_{i+2},\ v_i v_{i+1} = -v_{i+1} v_i = u_{i+2},$ (indices modulo 3)
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otherwise 0.

The *Cartan grading* is the $\mathbb{Z}^2$-grading determined by:

$$\deg u_1 = -\deg v_1 = (1, 0), \quad \deg u_2 = -\deg v_2 = (0, 1).$$

Fine gradings on the Octonions

Theorem (E. 1998)

Up to equivalence, the fine gradings on $\mathbb{O}$ are

- *the Cartan grading, and*
- *the $\mathbb{Z}_2^3$-grading given by the Cayley-Dickson doubling process.*

Fine gradings on the Octonions

Sketch of proof:

Fine gradings on the Octonions

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- The Cayley-Hamilton equation: $x^2 - n(x,1)x + n(x)1 = 0$, implies that the norm has a well behavior relative to the grading:

$$n(\mathbb{O}_g) = 0 \text{ unless } g^2 = e, \quad n(\mathbb{O}_g, \mathbb{O}_h) = 0 \text{ unless } gh = e.$$

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- If there is a $g \in \mathrm{Supp}\,\Gamma$ with either order > 2 or $\dim \mathbb{O}_g \geq 2$, there are elements $x \in \mathbb{O}_g$, $y \in \mathbb{O}_{g^{-1}}$ with $n(x) = 0 = n(y)$, $n(x,y) = 1$. Then $e_1 = x\bar{y}$ and $e_2 = y\bar{x}$ are orthogonal primitive idempotents in $\mathbb{O}_e$, and one uses the corresponding Peirce decomposition to check that, up to equivalence, our grading is the Cartan grading.

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- Otherwise $\dim \mathbb{O}_g = 1$ and $g^2 = e$ for any $g \in \mathrm{Supp}\,\Gamma$. We get the $\mathbb{Z}_2^3$-grading.

$\mathbb{Z}_2^3$-grading: Octonions as a twisted group algebra

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Theorem (Albuquerque-Majid 1999)

The octonion algebra is the twisted group algebra

$$\mathbb{O} = \mathbb{F}_\sigma[\mathbb{Z}_2^3],$$

where

$$e^\alpha e^\beta = \sigma(\alpha, \beta) e^{\alpha+\beta}$$

for $\alpha, \beta \in \mathbb{Z}_2^3$, *with*

$$\sigma(\alpha, \beta) = (-1)^{\psi(\alpha,\beta)},$$

$$\psi(\alpha, \beta) = \beta_1\alpha_2\alpha_3 + \alpha_1\beta_2\alpha_3 + \alpha_1\alpha_2\beta_3 + \sum_{i \leq j} \alpha_i\beta_j.$$

$\mathbb{Z}_2^3$-grading: Octonions as a twisted group algebra

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This allows to consider the algebra of octonions as an "associative algebra in a suitable category".

Cartan grading: Weyl group

Let S be the vector subspace spanned by $(1,1,1)$ in $\mathbb{R}^3$ and consider the two-dimensional real vector space $E = \mathbb{R}^3/S$. Take the elements

$$\epsilon_1 = (1,0,0) + S,\ \epsilon_2 = (0,1,0) + S,\ \epsilon_3 = (0,0,1) + S.$$

The subgroup $G = \mathbb{Z}\epsilon_1 + \mathbb{Z}\epsilon_2 + \mathbb{Z}\epsilon_3$ is isomorphic to $\mathbb{Z}^2$, and we may think of the Cartan grading Γ on the octonions $\mathbb{O}$ as the grading in which

$$\begin{aligned} \deg(e_1) &= 0 = \deg(e_2), \\ \deg(u_i) &= \epsilon_i = -\deg(v_i),\ i = 1,2,3. \end{aligned}$$

Cartan grading: Weyl group

Then $\mathrm{Supp}\,\Gamma = \{0\} \cup \{\pm\epsilon_i \mid i = 1, 2, 3\}$ and G is the universal group.
The set

$$\Phi := \Big(\mathrm{Supp}\,\Gamma \cup \{\alpha + \beta \mid \alpha, \beta \in \mathrm{Supp}\,\Gamma, \alpha \neq \pm\beta\}\Big) \setminus \{0\}$$

is the root system of type G_2.

Cartan grading: Weyl group

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\mathrm{Aut}(G)$, and this with a subgroup of $GL(E)$, we have:

$$\begin{aligned} W(\Gamma) &\subset \{\mu \in \mathrm{Aut}(G) \mid \mu(\mathrm{Supp}\ \Gamma) = \mathrm{Supp}\ \Gamma\} \\ &\subset \{\mu \in GL(E) \mid \mu(\Phi) = \Phi\} =: \mathrm{Aut}\,\Phi. \end{aligned}$$

The latter group is the automorphism group of the root system Φ, which coincides with its Weyl group.

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Theorem

Let Γ be the Cartan grading on the octonions. Identify $\mathrm{Supp}\ \Gamma \setminus \{0\}$ with the short roots in the root system Φ of type G_2. Then $W(\Gamma) = \mathrm{Aut}\,\Phi$.

$\mathbb{Z}_2^3$-grading: Weyl group

Theorem

Let Γ be the $\mathbb{Z}_2^3$-grading on the octonions induced by the Cayley-Dickson doubling process. Then $W(\Gamma) = \mathrm{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$.

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Let Γ be the $\mathbb{Z}_2^3$-grading on the octonions induced by the Cayley-Dickson doubling process. Then $W(\Gamma) = \mathrm{Aut}(\mathbb{Z}_2^3) \cong GL_3(2)$.

Remark

As any $\varphi \in \mathrm{Stab}(\Gamma)$ multiplies each of the elements $\mathbf{i}$, $\mathbf{j}$, $\mathbf{l}$ by either 1 or -1, we see that $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^3$. Therefore, the group $\mathrm{Aut}(\Gamma)$ is a (non-split) extension of $\mathbb{Z}_2^3$ by $W(\Gamma) \cong GL_3(2)$.

Gradings on para-Hurwitz algebras

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Theorem

Gradings on para-Hurwitz algebras of dimension 4 or 8

$$\updownarrow$$

Gradings on their Hurwitz counterparts.

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Gradings on para-Hurwitz algebras of dimension 4 or 8

$$\updownarrow$$

Gradings on their Hurwitz counterparts.

Therefore, any para-Cayley algebra is endowed with a $\mathbb{Z}_2^3$-grading.

Gradings on Okubo algebras

Gradings on Okubo algebras

Assuming $\mathbb{F}$ is a field of characteristic $\neq 3$ containing a primitive third root ω of 1, then the matrix algebra $\mathrm{Mat}_3(\mathbb{F})$ is generated by the order 3 matrices:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and the assignment

$$\deg(x) = (\bar{1}, \bar{0}), \qquad \deg(y) = (\bar{0}, \bar{1}),$$

gives a $\mathbb{Z}_3^2$-grading of $\mathrm{Mat}_3(\mathbb{F})$, which is inherited by the Okubo algebra $\big(\mathfrak{sl}_3(\mathbb{F}), *, n\big)$.

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Over algebraically closed fields, any grading on an Okubo algebra is a coarsening of either the natural $\mathbb{Z}^2$-grading (Cartan grading) or this $\mathbb{Z}_3^2$-grading.

$\mathbb{Z}_3^2$-grading

Consider the order three automorphism τ of $\mathbb{O}$:

$$\tau(e_i) = e_i,\ i = 1, 2, \quad \tau(u_j) = u_{j+1},\ \tau(v_j) = v_{j+1},\ j = 1, 2, 3,$$

and define a new multiplication on $\mathbb{O}$:

$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

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and define a new multiplication on $\mathbb{O}$:

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It turns out that this is too the (split) *Okubo algebra*, defined in a characteristic free way, and the $\mathbb{Z}_3^2$-grading is now given by setting

$$\deg e_1 = (\bar{1}, \bar{0}) \quad \text{and} \quad \deg u_1 = (\bar{0}, \bar{1}).$$

$\mathbb{Z}_3^2$-grading

	e_1	e_2	u_1	v_1	u_2	v_2	u_3	v_3
e_1	e_2	0	0	$-v_3$	0	$-v_1$	0	$-v_2$
e_2	0	e_1	$-u_3$	0	$-u_1$	0	$-u_2$	0
u_1	$-u_2$	0	v_1	0	$-v_3$	0	0	$-e_1$
v_1	0	$-v_2$	0	u_1	0	$-u_3$	$-e_2$	0
u_2	$-u_3$	0	0	$-e_1$	v_2	0	$-v_1$	0
v_2	0	$-v_3$	$-e_2$	0	0	u_2	0	$-u_1$
u_3	$-u_1$	0	$-v_2$	0	0	$-e_1$	v_3	0
v_3	0	$-v_1$	0	$-u_2$	$-e_2$	0	0	u_3

Multiplication table of the (split) Okubo algebra

The Albert algebra

G_2 and F_4

Jordan gradings on exceptional simple Lie algebras

Albert algebra

$$\mathbb{A} = H_3(\mathbb{O}) = \left\{ \begin{pmatrix} \alpha_1 & \bar{a}_3 & a_2 \\ a_3 & \alpha_2 & \bar{a}_1 \\ \bar{a}_2 & a_1 & \alpha_3 \end{pmatrix} \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{F},\ a_1, a_2, a_3 \in \mathbb{O} \right\}$$

$$= \mathbb{F}E_1 \oplus \mathbb{F}E_2 \oplus \mathbb{F}E_3 \oplus \iota_1(\mathbb{O}) \oplus \iota_2(\mathbb{O}) \oplus \iota_3(\mathbb{O}),$$

where

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\iota_1(a) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{a} \\ 0 & a & 0 \end{pmatrix}, \iota_2(a) = 2\begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ \bar{a} & 0 & 0 \end{pmatrix}, \iota_3(a) = 2\begin{pmatrix} 0 & \bar{a} & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Albert algebra

The multiplication in $\mathbb{A}$ is given by $X \circ Y = \frac{1}{2}(XY + YX)$ ($\operatorname{char} \mathbb{F} \neq 2$, $\mathbb{F} = \bar{\mathbb{F}}$).

Then E_i are orthogonal idempotents with $E_1 + E_2 + E_3 = 1$. The rest of the products are as follows:

$$E_i \circ \iota_i(a) = 0, \quad E_{i+1} \circ \iota_i(a) = \frac{1}{2}\iota_i(a) = E_{i+2} \circ \iota_i(a),$$

$$\iota_i(a) \circ \iota_{i+1}(b) = \iota_{i+2}(a \bullet b), \quad \iota_i(a) \circ \iota_i(b) = 2n(a,b)(E_{i+1} + E_{i+2}),$$

for any $a, b \in \mathbb{O}$, with $i = 1, 2, 3$ taken modulo 3, where $a \bullet b = \bar{a}\bar{b}$ is the para-Hurwitz multiplication.

Cartan grading

Consider the following elements in $\mathbb{Z}^4 = \mathbb{Z}^2 \times \mathbb{Z}^2$:

$$a_1 = (1, 0, 0, 0), \qquad a_2 = (0, 1, 0, 0), \qquad a_3 = (-1, -1, 0, 0),$$
$$g_1 = (0, 0, 1, 0), \qquad g_2 = (0, 0, 0, 1), \qquad g_3 = (0, 0, -1, -1).$$

Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$.

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Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$.

Take a canonical basis of the octonions. The assignment

$$\deg e_1 = \deg e_2 = 0, \quad \deg u_i = g_i = -\deg v_i$$

gives the Cartan grading on $\mathbb{O}$.

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$$g_1 = (0,0,1,0), \qquad g_2 = (0,0,0,1), \qquad g_3 = (0,0,-1,-1).$$

Then $a_1 + a_2 + a_3 = 0 = g_1 + g_2 + g_3$.

Take a canonical basis of the octonions. The assignment

$$\deg e_1 = \deg e_2 = 0, \quad \deg u_i = g_i = -\deg v_i$$

gives the Cartan grading on $\mathbb{O}$.

Now, the *Cartan grading* on $\mathbb{A}$ is given by:

$$\begin{aligned}
&\deg E_i = 0,\ \deg \iota_i(e_1) = a_i = -\deg \iota_i(e_2),\\
&\deg \iota_i(u_i) = g_i = -\deg \iota_i(v_i),\\
&\deg \iota_i(u_{i+1}) = a_{i+2} + g_{i+1} = -\deg \iota_i(v_{i+1}),\\
&\deg \iota_i(u_{i+2}) = -a_{i+1} + g_{i+2} = -\deg \iota_i(v_{i+2}).
\end{aligned}$$

Cartan grading: Weyl group

The universal group of the Cartan grading is $\mathbb{Z}^4$, which is contained in $E = \mathbb{R}^4$. Consider the following elements of $\mathbb{Z}^4$:

$$\begin{aligned}
\epsilon_0 &= \deg \iota_1(e_1) = a_1 = (1, 0, 0, 0), \\
\epsilon_1 &= \deg \iota_1(u_1) = g_1 = (0, 0, 1, 0), \\
\epsilon_2 &= \deg \iota_1(u_2) = a_3 + g_2 = (-1, -1, 0, 1), \\
\epsilon_3 &= \deg \iota_1(u_3) = -a_2 + g_3 = (0, -1, -1, -1).
\end{aligned}$$

Note that the ϵ_i's, $0 \leq i \leq 3$, are linearly independent, but do not form a basis of $\mathbb{Z}^4$. For instance,

$$\begin{aligned}
\deg \iota_2(e_1) &= a_2 = \frac{1}{2}(-\epsilon_0 - \epsilon_1 - \epsilon_2 - \epsilon_3), \\
\deg \iota_3(e_1) &= a_3 = \frac{1}{2}(-\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3).
\end{aligned}$$

Cartan grading: Weyl group

The supports of the Cartan grading Γ on each of the subspaces $\iota_i(\mathbb{O})$ are:

$$\mathrm{Supp}\ \iota_1(\mathbb{O}) = \{\pm\epsilon_i \mid 0 \leq i \leq 3\},$$

$$\begin{aligned}\mathrm{Supp}\ \iota_2(\mathbb{O}) &= \mathrm{Supp}\ \iota_1(\mathbb{O})(\iota_3(e_1) + \iota_3(e_2))\\ &= \{\frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{even number of } + \text{ signs}\},\end{aligned}$$

$$\begin{aligned}\mathrm{Supp}\ \iota_3(\mathbb{O}) &= \mathrm{Supp}\ \iota_1(\mathbb{O})(\iota_2(e_1) + \iota_2(e_2))\\ &= \{\frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3) \mid \text{odd number of } + \text{ signs}\}.\end{aligned}$$

Cartan grading: Weyl group

$$\begin{aligned}\Phi &:= \Big(\mathrm{Supp}\ \Gamma \cup \{\alpha+\beta \mid \alpha,\beta \in \mathrm{Supp}\ \iota_1(\mathbb{O}),\ \alpha \neq \pm\beta\}\Big) \setminus \{0\}\\ &= \mathrm{Supp}\ \iota_1(\mathbb{O}) \cup \mathrm{Supp}\ \iota_2(\mathbb{O}) \cup \mathrm{Supp}\ \iota_3(\mathbb{O})\\ &\qquad \cup \{\pm\epsilon_i \pm \epsilon_j \mid 0 \leq i \neq j \leq 3\},\end{aligned}$$

is the root system of type F_4. (Note that the ϵ_i's, $i = 0, 1, 2, 3$, form an orthogonal basis of E relative to the unique (up to scalar) inner product that is invariant under the Weyl group of Φ.)

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is the root system of type F_4. (Note that the ϵ_i's, $i = 0, 1, 2, 3$, form an orthogonal basis of E relative to the unique (up to scalar) inner product that is invariant under the Weyl group of Φ.)

Identifying the Weyl group $W(\Gamma)$ with a subgroup of $\mathrm{Aut}(\mathbb{Z}^4)$, and this with a subgroup of $GL(E)$, we have:

Theorem

Let Γ be the Cartan grading on the Albert algebra. Identify $\mathrm{Supp}\,\Gamma \setminus \{0\}$ with the short roots in the root system Φ of type F_4. Then $W(\Gamma) = \mathrm{Aut}\,\Phi$.

$\mathbb{Z}_2^5$-grading

$\mathbb{A}$ is naturally $\mathbb{Z}_2^2$-graded with

$$\mathbb{A}_{(\bar{0},\bar{0})} = \mathbb{F}E_1 + \mathbb{F}E_2 + \mathbb{F}E_3,$$
$$\mathbb{A}_{(\bar{1},\bar{0})} = \iota_1(\mathbb{O}), \qquad \mathbb{A}_{(\bar{0},\bar{1})} = \iota_2(\mathbb{O}), \qquad \mathbb{A}_{(\bar{1},\bar{1})} = \iota_3(\mathbb{O}).$$

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This $\mathbb{Z}_2^2$-grading may be combined with the fine $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ to obtain a fine $\mathbb{Z}_2^5$-grading:

$$\begin{aligned}
\deg E_i &= (\bar{0},\bar{0},\bar{0},\bar{0},\bar{0}), \ \ i = 1,2,3,\\
\deg \iota_1(x) &= (\bar{1},\bar{0},\deg x),\\
\deg \iota_2(x) &= (\bar{0},\bar{1},\deg x),\\
\deg \iota_3(x) &= (\bar{1},\bar{1},\deg x).
\end{aligned}$$

$\mathbb{Z}_2^5$-grading: Weyl group

Write $\mathbb{Z}_2^5 = \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 c_1 \oplus \mathbb{Z}_2 c_2 \oplus \mathbb{Z}_2 c_3$. Then the $\mathbb{Z}_2^5$-grading Γ is defined by setting

$$\deg \iota_1(1) = a, \quad \deg \iota_2(1) = b,$$

$$\deg \iota_3(\mathbf{i}) = a + b + c_1,\ \deg \iota_3(\mathbf{j}) = a + b + c_2,\ \deg \iota_3(\mathbf{l}) = a + b + c_3.$$

Theorem

Let Γ be the $\mathbb{Z}_2^5$-grading on the Albert algebra. Let $T = \oplus_{i=1}^3 \mathbb{Z}_2 c_i$. Then

$$W(\Gamma) = \{\mu \in \mathrm{Aut}(\mathbb{Z}_2^5) : \mu(T) = T\}.$$

$\mathbb{Z}_2^5$-grading: Weyl group

Write $\mathbb{Z}_2^5 = \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 c_1 \oplus \mathbb{Z}_2 c_2 \oplus \mathbb{Z}_2 c_3$. Then the $\mathbb{Z}_2^5$-grading Γ is defined by setting

$$\deg \iota_1(1) = a, \quad \deg \iota_2(1) = b,$$

$$\deg \iota_3(\mathbf{i}) = a + b + c_1, \ \deg \iota_3(\mathbf{j}) = a + b + c_2, \ \deg \iota_3(\mathbf{l}) = a + b + c_3.$$

Theorem

Let Γ be the $\mathbb{Z}_2^5$-grading on the Albert algebra. Let $T = \oplus_{i=1}^3 \mathbb{Z}_2 c_i$. Then

$$W(\Gamma) = \{\mu \in \mathrm{Aut}(\mathbb{Z}_2^5) : \mu(T) = T\}.$$

Remark

Any $\psi \in \mathrm{Stab}(\Gamma)$ fixes E_i and multiplies $\iota_1(1), \iota_2(1), \iota_3(\mathbf{i}), \iota_3(\mathbf{j}), \iota_3(\mathbf{l})$, by either 1 or -1. Hence $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$ is isomorphic to $\mathbb{Z}_2^5$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading

Take an element $\mathbf{i} \in \mathbb{F}$ with $\mathbf{i}^2 = -1$ and consider the following elements in $\mathbb{A}$:

$$\begin{aligned}
&E = E_1, \ \widetilde{E} = 1 - E = E_2 + E_3, \\
&\nu(a) = \mathbf{i}\iota_1(a) \quad \text{for all} \quad a \in \mathbb{O}_0, \\
&\nu_\pm(x) = \iota_2(x) \pm \mathbf{i}\iota_3(\bar{x}) \quad \text{for all} \quad x \in \mathbb{O}, \\
&S^\pm = E_3 - E_2 \pm \frac{\mathbf{i}}{2}\iota_1(1).
\end{aligned}$$

$\mathbb{A}$ is then 5-graded:

$$\mathbb{A} = \mathbb{A}_{-2} \oplus \mathbb{A}_{-1} \oplus \mathbb{A}_0 \oplus \mathbb{A}_1 \oplus \mathbb{A}_2,$$

with $\mathbb{A}_{\pm 2} = \mathbb{F}S^\pm$, $\mathbb{A}_{\pm 1} = \nu_\pm(\mathbb{O})$, and $\mathbb{A}_0 = \mathbb{F}E \oplus \left(\mathbb{F}\widetilde{E} \oplus \nu(\mathbb{O}_0)\right)$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading

The $\mathbb{Z}_2^3$-grading on $\mathbb{O}$ combines with this $\mathbb{Z}$-grading

$$\mathbb{A} = \mathbb{F}S^- \oplus \nu^-(\mathbb{O}) \oplus \mathbb{A}_0 \oplus \nu^+(\mathbb{O}) \oplus \mathbb{F}S^+$$

to give a fine $\mathbb{Z} \times \mathbb{Z}_2^3$-grading as follows:

$$\begin{aligned}
&\deg S^\pm = (\pm 2, \bar{0}, \bar{0}, \bar{0}),\\
&\deg \nu_\pm(x) = (\pm 1, \deg x),\\
&\deg E = 0 = \deg \widetilde{E},\\
&\deg \nu(a) = (0, \deg a),
\end{aligned}$$

for homogeneous elements $x \in \mathbb{O}$ and $a \in \mathbb{O}_0$.

$\mathbb{Z} \times \mathbb{Z}_2^3$-grading: Weyl group

Theorem

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$$W(\Gamma) = \mathrm{Aut}(\mathbb{Z} \times \mathbb{Z}_2^3).$$

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Remark

One can show that $\mathrm{Stab}(\Gamma) = \mathrm{Diag}(\Gamma)$, which is isomorphic to $\mathbb{F}^\times \times \mathbb{Z}_2^3$.

$\mathbb{Z}_3^3$-grading

Recall that the Okubo algebra can be defined on the octonions, with new multiplication:

$$x * y = \tau(\bar{x})\tau^2(\bar{y}).$$

where τ is the order three automorphism of $\mathbb{O}$ given by:

$$\tau(e_i) = e_i,\ i = 1, 2, \quad \tau(u_j) = u_{j+1},\ \tau(v_j) = v_{j+1},\ j = 1, 2, 3.$$

$\mathbb{Z}_3^3$-grading

Define $\tilde{\iota}_i(x) = \iota_i(\tau^i(x))$ for all $i = 1, 2, 3$ and $x \in \mathbb{O}$. Then the multiplication in the Albert algebra

$$\mathbb{A} = \oplus_{i=1}^3 \big(\mathbb{F}E_i \oplus \tilde{\iota}_i(\mathbb{O})\big)$$

becomes:

$$E_i^{\circ 2} = E_i, \quad E_i \circ E_{i+1} = 0,$$

$$E_i \circ \tilde{\iota}_i(x) = 0, \quad E_{i+1} \circ \tilde{\iota}_i(x) = \frac{1}{2}\tilde{\iota}_i(x) = E_{i+2} \circ \tilde{\iota}_i(x),$$

$$\tilde{\iota}_i(x) \circ \tilde{\iota}_{i+1}(y) = \tilde{\iota}_{i+2}(x * y), \quad \tilde{\iota}_i(x) \circ \tilde{\iota}_i(y) = 2n(x, y)(E_{i+1} + E_{i+2}),$$

for $i = 1, 2, 3$ and $x, y \in \mathbb{O}$.

$\mathbb{Z}_3^3$-grading

Assume now $\operatorname{char}\mathbb{F} \neq 3$. Then the $\mathbb{Z}_3^2$-grading on the Okubo algebra is determined by two commuting order 3 automorphisms $\varphi_1, \varphi_2 \in \mathrm{Aut}(\mathbb{O}, *)$:

$$\begin{aligned} \varphi_1(e_1) &= \omega e_1, & \varphi_1(u_1) &= u_1, \\ \varphi_2(e_1) &= e_1, & \varphi_2(u_1) &= \omega u_1, \end{aligned}$$

where ω is a primitive cubic root of unity in $\mathbb{F}$.

$\mathbb{Z}_3^3$-grading

The commuting order 3 automorphisms φ_1, φ_2 of $(\mathbb{O}, *)$ extend to commuting order 3 automorphisms of $\mathbb{A}$:

$$\varphi_j(E_i) = E_i, \quad \varphi_j\big(\tilde{\iota}_i(x)\big) = \tilde{\iota}_i(\varphi_j(x)).$$

On the other hand, the linear map $\varphi_3 \in \mathrm{End}(\mathcal{A})$ defined by

$$\varphi_3(E_i) = E_{i+1}, \quad \varphi_3\big(\tilde{\iota}_i(x)\big) = \tilde{\iota}_{i+1}(x),$$

is another order 3 automorphism, which commutes with φ_1 and φ_2.

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The subgroup of $\mathrm{Aut}(\mathbb{A})$ generated by $\varphi_1, \varphi_2, \varphi_3$ is isomorphic to $\mathbb{Z}_3^3$ and induces a $\mathbb{Z}_3^3$-grading on $\mathbb{A}$.

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All the homogeneous components have dimension 1.

$\mathbb{Z}_3^3$-grading: Weyl group

The $\mathbb{Z}_3^3$-grading is determined by

$$\begin{aligned}
\deg\left(\sum_{i=1}^3 \tilde{\iota}_i(e_1)\right) &= (\bar{1}, \bar{0}, \bar{0}), \\
\deg\left(\sum_{i=1}^3 \tilde{\iota}_i(u_1)\right) &= (\bar{0}, \bar{1}, \bar{0}), \\
\deg\left(\sum_{i=1}^3 \omega^{-i} E_i\right) &= (\bar{0}, \bar{0}, \bar{1}),
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\end{aligned}$$

Theorem

Let Γ be the $\mathbb{Z}_3^3$-grading on the Albert algebra. Then $W(\Gamma)$ is the commutator subgroup of $\mathrm{Aut}(\mathbb{Z}_3^3)$, i.e.,

$$W(\Gamma) \cong SL_3(3).$$

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Why $SL_3(3)$ and not $GL_3(3)$?

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Then, for $X_1 \in \mathbb{A}_{(\bar{1},\bar{0},\bar{0})}$, $X_2 \in \mathbb{A}_{(\bar{0},\bar{1},\bar{0})}$, $X_3 \in \mathbb{A}_{(\bar{0},\bar{0},\bar{1})}$, we have:

$$(X_1 \circ X_2) \circ X_3 = \begin{cases} \omega X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma,\\ \omega^{-1} X_1 \circ (X_2 \circ X_3), & \text{for } \Gamma^-. \end{cases}$$

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Hence Γ and Γ^- are equivalent, but NOT isomorphic, gradings.

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Hence Γ and Γ^- are equivalent, but NOT isomorphic, gradings.

Besides, any fine $\mathbb{Z}_3^3$-grading on $\mathbb{A}$ is isomorphic to either Γ or Γ^-, so $W(\Gamma)$ has index two in $\mathrm{Aut}(U(\Gamma)) \cong GL_3(3)$.

$\mathbb{Z}_3^3$-grading and the Tits construction

Let $\mathcal{R} = \mathsf{Mat}_3(\mathbb{F})$. Then

$$\mathbb{A} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2,$$

with $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$ copies of $\mathcal{R}$.

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with $\mathcal{R}_0$, $\mathcal{R}_1$, $\mathcal{R}_2$ copies of $\mathcal{R}$.
The product in $\mathbb{A}$ satisfies $\mathcal{R}_i \circ \mathcal{R}_j \subseteq \mathcal{R}_{i+j}$ (mod 3) and:

$\circ$	a'_0	b'_1	c'_2
a_0	$(a \circ a')_0$	$(\bar{a}b')_1$	$(c'\bar{a})_2$
b_1	$(\bar{a}'b)_1$	$(b \times b')_2$	$(\overline{bc'})_2$
c_2	$(c\bar{a}')_2$	$(\overline{b'c})_0$	$(c \times c')_1$

where

- $a \circ a' = \frac{1}{2}(aa' + a'a)$,
- $a \times b = a \circ b - \frac{1}{2}(\mathrm{tr}(a)b + \mathrm{tr}(b)a) + \frac{1}{2}\big(\mathrm{tr}(a)\,\mathrm{tr}(b) - \mathrm{tr}(ab)\big)1$,
- $\bar{a} = a \times 1 = \frac{1}{2}\big(\mathrm{tr}(a)1 - a\big)$.

$\mathbb{Z}_3^3$-grading and the Tits construction

Assume $\operatorname{char}\mathbb{F} \neq 3$. Take Pauli matrices in $\mathcal{R}$:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

where ω, ω^2 are the primitive cubic roots of 1, which satisfy

$$x^3 = 1 = y^3, \quad yx = \omega xy.$$

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These Pauli matrices give a grading by $\mathbb{Z}_3^2$ on $\mathcal{R}$, with

$$\mathcal{R}_{(\alpha_1,\alpha_2)} = \mathbb{F}x^{\alpha_1}y^{\alpha_2}.$$

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This grading combines with the $\mathbb{Z}_3$-grading on $\mathbb{A}$ induced by Tits construction, to give the unique, up to equivalence, fine grading by $\mathbb{Z}_3^3$ of the Albert algebra.

$\mathbb{Z}_3^3$-grading and the Tits construction

For $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_3^3$ consider the element

$$Z^\alpha := (x^{\alpha_1} y^{\alpha_2})_{\alpha_3} \in \mathcal{R}_{\alpha_3} \subseteq \mathbb{A}.$$

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Then, for any $\alpha, \beta \in \mathbb{Z}_3^3$:

$$Z^\alpha \circ Z^\beta = \begin{cases} \omega^{\tilde{\psi}(\alpha,\beta)} Z^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\tilde{\psi}(\alpha,\beta)} Z^{\alpha+\beta} & \text{otherwise,} \end{cases}$$

where

$$\tilde{\psi}(\alpha, \beta) = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_3 - \beta_3) - (\alpha_1\beta_2 + \alpha_2\beta_1).$$

$\mathbb{Z}_3^3$-grading and the Tits construction

Consider now the elements (Racine 1990, unpublished)

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$$W^\alpha \circ W^\beta = \omega^{-\alpha_1\alpha_2-\beta_1\beta_2} Z^\alpha \circ Z^\beta$$

$$= \begin{cases} \omega^{\tilde\psi(\alpha,\beta)-(\alpha_1\alpha_2+\beta_1\beta_2)} Z^{\alpha+\beta} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha+\mathbb{Z}_3\beta)\leq 1, \\ -\frac{1}{2}\omega^{\tilde\psi(\alpha,\beta)-(\alpha_1\alpha_2+\beta_1\beta_2)} Z^{\alpha+\beta} & \text{otherwise,} \end{cases}$$

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The Albert algebra as a twisted group algebra

Theorem (Griess 1990)

The Albert algebra is, up to isomorphism, the twisted group algebra

$$\mathbb{A} = \mathbb{F}_\sigma[\mathbb{Z}_3^3],$$

with

$$\sigma(\alpha, \beta) = \begin{cases} \omega^{\psi(\alpha,\beta)} & \text{if } \dim_{\mathbb{Z}_3}(\mathbb{Z}_3\alpha + \mathbb{Z}_3\beta) \leq 1, \\ -\frac{1}{2}\omega^{\psi(\alpha,\beta)} & \text{otherwise,} \end{cases}$$

where

$$\psi(\alpha, \beta) = (\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_3 - \beta_3).$$

Fine gradings on the Albert algebra

Theorem (Draper–Martín-González 2009 (char $= 0$), E.–Kochetov 2012)

Up to equivalence, the fine gradings of the Albert algebra are:

1. *The Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$).*
2. *The $\mathbb{Z}_2^5$-grading obtained by combining the natural $\mathbb{Z}_2^2$-grading on 3×3 hermitian matrices with the fine grading by $\mathbb{Z}_2^3$ of $\mathbb{O}$.*
3. *The $\mathbb{Z} \times \mathbb{Z}_2^3$-grading obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
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All the gradings up to isomorphism on $\mathbb{A}$ have been classified too (E.–Kochetov).

The Albert algebra

G_2 and F_4

Jordan gradings on exceptional simple Lie algebras

Gradings and comodule algebras

G-grading $\longleftrightarrow$ comodule algebra over the group algebra $\mathbb{F}G$

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$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Rightarrow \qquad \begin{aligned} \rho_\Gamma : \mathcal{A} &\longrightarrow \mathcal{A}\otimes \mathbb{F}G \\ x_g &\mapsto x_g \otimes g \end{aligned}$$

(algebra morphism and comodule str.)

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$$\Gamma_\rho : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Leftarrow \qquad \rho : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G$$

$$(\mathcal{A}_g = \{x \in \mathcal{A} : \rho(x) = x\otimes g\})$$

Gradings and comodule algebras

A comodule algebra map

$$\rho : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G$$

induces a *generic automorphism* of $\mathbb{F}G$-algebras

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All the information on the grading Γ attached to ρ is contained in this single automorphism!

Gradings and affine group schemes

$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \quad \Leftrightarrow \quad \rho_\Gamma : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G$$

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Now,

$$\rho_\Gamma : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G \quad \Longleftrightarrow \quad \eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}$$

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(comodule algebra) (morphism of affine group schemes)

For any $\varphi \in G^D(\mathcal{R})$, $\eta_\Gamma(\varphi) \in \mathrm{Aut}_\mathcal{R}(\mathcal{A} \otimes \mathcal{R})$ is given by:

$$\eta_\Gamma(\varphi)(x_g \otimes r) = x_g \otimes \varphi(g) r.$$

Gradings and affine group schemes

$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \quad \Leftrightarrow \quad \rho_\Gamma : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G$$

(comodule algebra structure)

Now,

$$\rho_\Gamma : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G \quad \Longleftrightarrow \quad \eta_\Gamma : G^D \to \textbf{Aut}\,\mathcal{A}$$

(comodule algebra) (morphism of affine group schemes)

For any $\varphi \in G^D(\mathcal{R})$, $\eta_\Gamma(\varphi) \in \mathrm{Aut}_{\mathcal{R}}(\mathcal{A}\otimes\mathcal{R})$ is given by:

$$\eta_\Gamma(\varphi)(x_g\otimes r) = x_g \otimes \varphi(g)r.$$

and ρ_Γ is recovered as

$$\rho_\Gamma(x) = \eta_\Gamma(id_{\mathbb{F}G})(x\otimes 1) \qquad \Big(\eta_\Gamma(id_{\mathbb{F}G}) \in \mathrm{Aut}_{\mathbb{F}G}(\mathcal{A}\otimes\mathbb{F}G)\Big)$$

Gradings and affine group schemes

Consider a homomorphism $\Phi : \mathbf{Aut}\,\mathcal{A} \longrightarrow \mathbf{Aut}\,\mathcal{A}'$ of affine group schemes.

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If $\Gamma_1 : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma_2 : \mathcal{A} = \oplus_{h\in H}\mathcal{A}'_h$ are weakly isomorphic through the automorphisms $\psi \in \mathrm{Aut}\,\mathcal{A}$ and $\varphi : G \to H$, then the induced gradings Γ'_1 and Γ'_2 on $\mathcal{A}'$ are weakly isomorphic too through the automorphisms $\Phi_{\mathbb{F}}(\psi) \in \mathrm{Aut}\,\mathcal{A}'$ and $\varphi : G \to H$.

Gradings and affine group schemes

For $\mathbf{G} = \mathbf{Aut}\,\mathcal{A}$, $\mathrm{Lie}(\mathbf{G}) = \mathfrak{Der}(\mathcal{A})$, so

$$\mathrm{Ad} : \mathbf{Aut}\,\mathcal{A} \to \mathbf{Aut}\big(\mathfrak{Der}(\mathcal{A})\big)$$

is a homomorphism, and any grading $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ induces a grading

$$\Gamma' : \mathfrak{Der}(\mathcal{A}) = \oplus_{g\in G}\,\mathfrak{Der}(\mathcal{A})_g,$$

$$\mathfrak{Der}(\mathcal{A})_g = \{d \in \mathfrak{Der}(\mathcal{A}) : d(\mathcal{A}_h) \subseteq \mathcal{A}_{gh}\ \forall h \in G\}.$$

Gradings on G_2 and F_4

If $\mathbf{Aut}\,\mathcal{A} \cong \mathbf{Aut}\,\mathcal{B}$, then the problem of the classification of fine gradings up to equivalence, and of gradings up to isomorphism, on $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

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If the characteristic of the ground field $\mathbb{F}$ is $\neq 2, 3$, then

$$\mathrm{Ad} : \mathbf{Aut}\,\mathbb{O} \to \mathbf{Aut}\,\mathfrak{g}_2$$

is an isomorphism, and (assuming just $\mathrm{char}\,\mathbb{F} \neq 2$),

$$\mathrm{Ad} : \mathbf{Aut}\,\mathbb{A} \to \mathbf{Aut}\,\mathfrak{f}_4$$

is an isomorphism too.

Gradings on G_2

Theorem

Up to equivalence, the fine gradings on $\mathfrak{g}_2$ are

- *the Cartan grading, and*
- *a $\mathbb{Z}_2^3$-grading with $(\mathfrak{g}_2)_0 = 0$ and where $(\mathfrak{g}_2)_g$ is a Cartan subalgebra of $\mathfrak{g}_2$ for any $0 \neq g \in \mathbb{Z}_2^3$.*

Gradings on F_4

Theorem

Up to equivalence, the fine gradings on $\mathfrak{f}_4$ are

- *the Cartan grading,*
- *a grading by $\mathbb{Z}_2^5$, obtained by combining the $\mathbb{Z}_2^2$-grading given by the decomposition $\mathfrak{f}_4 = \mathfrak{d}_4 \oplus natural \oplus spin \oplus \overline{spin}$, with the $\mathbb{Z}_2^3$-grading on the octonions (which is the vector space behind the natural and spin representations of $\mathfrak{d}_4$).*
- *a grading by $\mathbb{Z} \times \mathbb{Z}_2^3$, obtained by looking at $\mathfrak{f}_4$ as the Kantor Lie algebra of a structurable algebra: $\mathfrak{f}_4 = \mathcal{K}(\mathbb{O}, -)$, and combining the natural 5-grading on $\mathcal{K}(\mathbb{O}, -)$ and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
- *a $\mathbb{Z}_3^3$-grading (only if $\operatorname{char} \mathbb{F} \neq 3$), with $(\mathfrak{f}_4)_0 = 0$ and where $(\mathfrak{f}_4)_g \oplus (\mathfrak{f}_4)_{-g}$ is a Cartan subalgebra of $\mathfrak{f}_4$ for any $0 \neq g \in \mathbb{Z}_3^3$.*

The Albert algebra

G_2 and F_4

Jordan gradings on exceptional simple Lie algebras

Jordan subgroups

Definition (Alekseevskiĭ 1974)

Given a simple Lie algebra $\mathfrak{g}$ and a complex Lie group G with $\mathrm{Int}(\mathfrak{g}) \leq G \leq \mathrm{Aut}(\mathfrak{g})$, an abelian subgroup A of G is a *Jordan subgroup* if:

(i) its normalizer $N_G(A)$ is finite,

(ii) A is a minimal normal subgroup of its normalizer, and

(iii) its normalizer is maximal among the normalizers of those abelian subgroups satisfying (i) and (ii).

Jordan gradings

The Jordan subgroups are elementary ($\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ for some prime number p), and they induce gradings, called *Jordan gradings*, in the Lie algebra $\mathfrak{g}$.

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The classification of Jordan subgroups by Alekseevskiĭ splits in two types: classical and exceptional.

Jordan subgroups: classical cases

$\mathfrak{g}$	A
A_{p^n-1}	$\mathbb{Z}_p^{2n}$
B_n $(n \geq 3)$	$\mathbb{Z}_2^{2n}$
$C_{2^{n-1}}$ $(n \geq 2)$	$\mathbb{Z}_2^{2n}$
D_{n+1} $(n \geq 3)$	$\mathbb{Z}_2^{2n}$
$D_{2^{n-1}}$ $(n \geq 3)$	$\mathbb{Z}_2^{2n}$

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The dimension of all nonzero homogeneous spaces is always 1 in these classical cases, which are well-known.

Jordan subgroups: exceptional cases

$\mathfrak{g}$	A	$\dim \mathfrak{g}_\alpha$ $(\alpha \neq 0)$
G_2	$\mathbb{Z}_2^3$	2
F_4	$\mathbb{Z}_3^3$	2
E_8	$\mathbb{Z}_5^3$	2
D_4	$\mathbb{Z}_2^3$	4
E_8	$\mathbb{Z}_2^5$	8
E_6	$\mathbb{Z}_3^3$	3

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E_6	$\mathbb{Z}_3^3$	3

Models of these gradings?

Gradings on Freudenthal's Magic Square

Given two symmetric composition algebras, the Lie algebra $\mathfrak{g}(S, S')$ is naturally $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded with

$$\mathfrak{g}_{(\bar{0},\bar{0})} = \mathfrak{tri}(S) \oplus \mathfrak{tri}(S'),$$

$$\mathfrak{g}_{(\bar{1},\bar{0})} = \iota_1(S \otimes S'), \qquad \mathfrak{g}_{(\bar{0},\bar{1})} = \iota_2(S \otimes S'), \qquad \mathfrak{g}_{(\bar{1},\bar{1})} = \iota_3(S \otimes S').$$

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Also, the triality automorphisms θ and θ' extend to an order 3 automorphism Θ of $\mathfrak{g}(S, S')$. The eigenspaces of Θ constitute a $\mathbb{Z}_3$-grading of $\mathfrak{g}(S, S')$.

Induced gradings

(From now on, assume that our ground field $\mathbb{F}$ is algebraically closed of characteristic 0.)

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- The $\mathbb{Z}_3^2$-grading on the Okubo algebra $\mathcal{O}$ induces a $\mathbb{Z}_3^3$-grading on both the simple Lie algebra $\mathfrak{g}(\mathbb{F}, \mathcal{O})$ of type F_4 (our fine $\mathbb{Z}_3^3$-grading!!) and the simple Lie algebra $\mathfrak{g}(S, \mathcal{O})$ (for the two-dimensional para-Hurwitz algebra S) of type E_6.
  In both cases $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.

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  In both cases $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_3^3$.
- The $\mathbb{Z}_2^3$-grading on a para-Cayley algebra $\bar{C}$ induces a $\mathbb{Z}_2^5$-grading on the simple Lie algebra $\mathfrak{g}(\bar{C}, \bar{C})$ of type E_8.
  Moreover, $\mathfrak{g}_0 = 0$ and $\mathfrak{g}_\alpha$ is a Cartan subalgebra of $\mathfrak{g}$ for any $0 \neq \alpha \in \mathbb{Z}_2^5$.

Exceptional Jordan gradings

Theorem

The gradings:

1. *a $\mathbb{Z}_2^3$-grading on the simple Lie algebra of type G_2 induced by the $\mathbb{Z}_2^3$-grading of the Cayley algebra,*
2. *a $\mathbb{Z}_2^3$-grading on the simple Lie algebra of type D_4 induced by the $\mathbb{Z}_2^3$-grading of the Cayley algebra,*
3. *a $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type F_4 induced by the $\mathbb{Z}_3^2$-grading of the Okubo algebra,*
4. *a $\mathbb{Z}_3^3$-grading on the simple Lie algebra of type E_6 induced by the $\mathbb{Z}_3^2$-grading of the Okubo algebra,*
5. *a $\mathbb{Z}_2^5$-grading on the simple Lie algebra of type E_8 induced by the $\mathbb{Z}_2^3$-grading of the Cayley algebra,*

are exceptional Jordan gradings.

The missing exceptional Jordan grading

Only one exceptional Jordan grading does not fit in the Theorem above: the $\mathbb{Z}_5^3$-grading on E_8.

The missing exceptional Jordan grading

Only one exceptional Jordan grading does not fit in the Theorem above: the $\mathbb{Z}_5^3$-grading on E_8.

Let V_1 and V_2 be two vector spaces over $\mathbb{F}$ of dimension 5, and consider the $\mathbb{Z}_5$-graded vector space

$$\mathfrak{g} = \oplus_{i=0}^{4} \mathfrak{g}_{\bar{\imath}},$$

where

$$\begin{aligned}
\mathfrak{g}_{\bar{0}} &= \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2),\\
\mathfrak{g}_{\bar{1}} &= V_1 \otimes \textstyle\bigwedge^2 V_2,\\
\mathfrak{g}_{\bar{2}} &= \textstyle\bigwedge^2 V_1 \otimes \bigwedge^4 V_2,\\
\mathfrak{g}_{\bar{3}} &= \textstyle\bigwedge^3 V_1 \otimes V_2,\\
\mathfrak{g}_{\bar{4}} &= \textstyle\bigwedge^4 V_1 \otimes \bigwedge^3 V_2.
\end{aligned}$$

This is a $\mathbb{Z}_5$-graded Lie algebra in a unique way: the exceptional simple Lie algebra of type E_8.

The missing exceptional Jordan grading

Up to conjugation in $\operatorname{Aut}\mathfrak{g}$, there is a unique order 5 automorphism of the simple Lie algebra $\mathfrak{g}$ of type E_8 such that the dimension of the subalgebra of fixed elements is 48.

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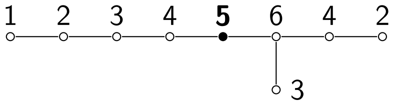

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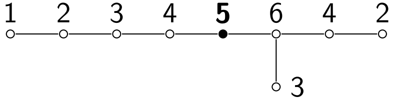

The uniqueness shows us that, up to conjugation, this is the automorphism of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_{\bar{i}}$ is ξ^i times the identity, where ξ is a fixed primitive fifth root of unity.

The missing exceptional Jordan grading

The missing exceptional Jordan grading

Consider the following automorphisms $\sigma_1, \sigma_2, \sigma_3$ of $\mathfrak{g}$:

$$\begin{aligned}
&\sigma_1(x) = \xi^i x \quad \text{for any } x \in \mathfrak{g}_{\bar{\imath}} \text{ and } 0 \le i \le 4,\\
&\sigma_2|_{\mathfrak{g}_{\bar{1}}} = b_1 \otimes \wedge^2 b_2,\\
&\sigma_3|_{\mathfrak{g}_{\bar{1}}} = c_1 \otimes \wedge^2 c_2,
\end{aligned}$$

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where on fixed bases of V_1 and V_2, the coordinate matrices of b_1, c_1, b_2, c_2 are:

$$b_1 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 & 0 \\ 0 & 0 & 0 & \xi^3 & 0 \\ 0 & 0 & 0 & 0 & \xi^4 \end{pmatrix}, \quad c_1 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$b_2 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 & 0 \\ 0 & 0 & \xi^4 & 0 & 0 \\ 0 & 0 & 0 & \xi & 0 \\ 0 & 0 & 0 & 0 & \xi^3 \end{pmatrix}, \quad c_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The missing exceptional Jordan grading

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The grading of E_8 induced by the order 5 commuting automorphisms $\sigma_1, \sigma_2, \sigma_3$ is the Jordan grading by $\mathbb{Z}_5^3$.

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There are models of the Jordan gradings of F_4 and E_6 by $\mathbb{Z}_3^3$ constructed along the same lines.

That's all.
Thanks