Gradings on simple Lie algebras

Alberto Elduque

December 2010

Definition

G abelian group, $\mathcal{A}$ algebra over a field $\mathbb{F}$.

G-grading on $\mathcal{A}$:

$$\mathcal{A} = \oplus_{g \in G} \mathcal{A}_g,$$

$$\mathcal{A}_g \mathcal{A}_h \subseteq \mathcal{A}_{gh} \qquad \forall g, h \in G.$$

Example: Pauli matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \oplus_{(\bar{\imath},\bar{\jmath})\in\mathbb{Z}_n\times\mathbb{Z}_n}\mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F}X^iY^j.$$

Example: Pauli matrices

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon^{n-1} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(ϵ a primitive nth root of 1)

$$X^n = 1 = Y^n, \qquad YX = \epsilon XY$$

$$\mathcal{A} = \oplus_{(\bar{\imath},\bar{\jmath})\in\mathbb{Z}_n\times\mathbb{Z}_n}\mathcal{A}_{(\bar{\imath},\bar{\jmath})}, \qquad \mathcal{A}_{(\bar{\imath},\bar{\jmath})} = \mathbb{F}X^iY^j.$$

$\mathcal{A}$ becomes a *graded division algebra*.

Example: octonions

Cayley-Dickson process:

$$\begin{aligned} \mathbb{K} &= \mathbb{F} \oplus \mathbb{F}i, & i^2 &= -1, \\ \mathbb{H} &= \mathbb{K} \oplus \mathbb{K}j, & j^2 &= -1, \\ \mathbb{O} &= \mathbb{H} \oplus \mathbb{H}l, & l^2 &= -1, \end{aligned}$$

$\mathbb{O}$ is $\mathbb{Z}_2^3$-graded with

$$\deg(i) = (\bar{1}, \bar{0}, \bar{0}), \quad \deg(j) = (\bar{0}, \bar{1}, \bar{0}), \quad \deg(l) = (\bar{0}, \bar{0}, \bar{1}).$$

Examples: Lie algebras

Examples: Lie algebras

- **Cartan grading:** $\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\right)$
  (root space decomposition of a semisimple complex Lie algebra).
  This is a grading over $\mathbb{Z}^n$, $n = \operatorname{rank}\mathfrak{g}$.

Examples: Lie algebras

- **Cartan grading:** $\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\right)$
  (root space decomposition of a semisimple complex Lie algebra).
  This is a grading over $\mathbb{Z}^n$, $n = \operatorname{rank}\mathfrak{g}$.

- **Jordan systems $\leftrightarrow$ $\mathbb{Z}$-gradings**

$$\mathfrak{g} = \mathfrak{g}_{-n} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n.$$

Examples: Lie algebras

- **Cartan grading:** $\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\right)$
(root space decomposition of a semisimple complex Lie algebra).
This is a grading over $\mathbb{Z}^n$, $n = \operatorname{rank}\mathfrak{g}$.

- **Jordan systems $\leftrightarrow$ $\mathbb{Z}$-gradings**

$$\mathfrak{g} = \mathfrak{g}_{-n} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n.$$

Examples: Lie algebras

- **Cartan grading:** $\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\right)$
  (root space decomposition of a semisimple complex Lie algebra).
  This is a grading over $\mathbb{Z}^n$, $n = \operatorname{rank}\mathfrak{g}$.

- **Jordan systems** $\leftrightarrow$ $\mathbb{Z}$**-gradings**

$$\mathfrak{g} = \mathfrak{g}_{-n} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n.$$

  I. Kantor: "There are no Jordan algebras, there are only Lie algebras."

Examples: Lie algebras

- **Cartan grading:** $\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\right)$
(root space decomposition of a semisimple complex Lie algebra).
This is a grading over $\mathbb{Z}^n$, $n = \operatorname{rank}\mathfrak{g}$.

- **Jordan systems $\leftrightarrow$ $\mathbb{Z}$-gradings**

$$\mathfrak{g} = \mathfrak{g}_{-n} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n.$$

I. Kantor: "There are no Jordan algebras, there are only Lie algebras."

K. McCrimmon: "Of course, this can be turned around: nine times out of ten, when you open up a Lie algebra you find a Jordan algebra inside which makes it tick."

Examples: Lie algebras

- **Symmetric spaces** $\leftrightarrow$ $\mathbb{Z}_2$-**gradings:** $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$.

Examples: Lie algebras

- **Symmetric spaces** $\leftrightarrow$ $\mathbb{Z}_2$**-gradings:** $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$.

- Finite order automorphisms of simple complex Lie algebras (classified by V. Kac) correspond to gradings over finite cyclic groups:

$$\varphi \in \operatorname{Aut}\mathfrak{g},\ \varphi^n = 1 \quad \leftrightarrow \quad \mathfrak{g} = \oplus_{\bar{\imath}\in\mathbb{Z}_n}\mathfrak{g}_{\bar{\imath}},$$

$$\mathfrak{g}_{\bar{\imath}} = \{x \in \mathfrak{g} : \varphi(x) = \epsilon^i x\}\ (\epsilon = e^{\frac{2\pi i}{n}}).$$

(This is important in the theory of Kac-Moody Lie algebras.)

Conventions

In what follows:

- $\mathbb{F}$ will denote an algebraically closed ground field, char $\mathbb{F} \neq 2$.
- The dimension of the algebras considered will always be finite.
- The stress will be put on the methods, and not on the results. Some of them are quite technical.

Basic definitions

Let $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \oplus_{g'\in G'}\mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

Basic definitions

Let $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \oplus_{g'\in G'}\mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
  Then Γ' is a *coarsening* of Γ.

Basic definitions

Let $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \oplus_{g'\in G'}\mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
  Then Γ' is a *coarsening* of Γ.
- Γ is *fine* if it admits no proper refinement.

Basic definitions

Let $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \oplus_{g'\in G'}\mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
  Then Γ' is a *coarsening* of Γ.
- Γ is *fine* if it admits no proper refinement.
- Γ and Γ' are *equivalent* if there is an automorphism $\varphi \in \operatorname{Aut}\mathcal{A}$ such that for any $g \in G$ there is a $g' \in G'$ with $\varphi(\mathcal{A}_g) = \mathcal{A}'_{g'}$.

Basic definitions

Let $\Gamma : \mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$ and $\Gamma' : \mathcal{A} = \oplus_{g' \in G'} \mathcal{A}'_{g'}$ be two gradings on $\mathcal{A}$:

- Γ is a *refinement* of Γ' if for any $g \in G$ there is a $g' \in G'$ such that $\mathcal{A}_g \subseteq \mathcal{A}_{g'}$.
  Then Γ' is a *coarsening* of Γ.
- Γ is *fine* if it admits no proper refinement.
- Γ and Γ' are *equivalent* if there is an automorphism $\varphi \in \operatorname{Aut} \mathcal{A}$ such that for any $g \in G$ there is a $g' \in G'$ with $\varphi(\mathcal{A}_g) = \mathcal{A}'_{g'}$.
- For $G = G'$, Γ and Γ' are *isomorphic* if there is an automorphism $\varphi \in \operatorname{Aut} \mathcal{A}$ such that $\varphi(\mathcal{A}_g) = \mathcal{A}'_g$ for any $g \in G$.

Abelian diagonalizable subgroups

Abelian diagonalizable subgroups

- Any grading $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ induces a group homomorphism

$$\begin{aligned} \eta_\Gamma : \hat{G} = \mathrm{Hom}(G, \mathbb{F}^\times) &\longrightarrow \mathrm{Aut}\,\mathcal{A} \\ \varphi \quad &\mapsto \eta_\Gamma(\varphi) : x_g \in \mathcal{A}_g \mapsto \varphi(g)x_g. \end{aligned}$$

  The homogeneous components are the common eigenspaces of the elements of the abelian diagonalizable subgroup $\eta_\Gamma(\hat{G})$.

Abelian diagonalizable subgroups

- Any grading $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ induces a group homomorphism

$$\begin{aligned}\eta_\Gamma : \hat{G} = \mathrm{Hom}(G, \mathbb{F}^\times) &\longrightarrow \operatorname{Aut}\mathcal{A}\\ \varphi \quad &\mapsto \eta_\Gamma(\varphi) : x_g \in \mathcal{A}_g \mapsto \varphi(g)x_g.\end{aligned}$$

  The homogeneous components are the common eigenspaces of the elements of the abelian diagonalizable subgroup $\eta_\Gamma(\hat{G})$.

- Conversely, for an abelian diagonalizable subgroup $Q \leq \operatorname{Aut}\mathcal{A}$, its Zariski closure $\bar{Q}$ remains abelian and diagonalizable with the same eigenspaces, and it is an algebraic subgroup of $\operatorname{Aut}\mathcal{A}$. Then

Abelian diagonalizable subgroups

- Any grading $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ induces a group homomorphism

$$\begin{aligned} \eta_\Gamma : \hat{G} = \mathrm{Hom}(G, \mathbb{F}^\times) &\longrightarrow \mathrm{Aut}\,\mathcal{A} \\ \varphi \quad &\mapsto \eta_\Gamma(\varphi) : x_g \in \mathcal{A}_g \mapsto \varphi(g)x_g. \end{aligned}$$

  The homogeneous components are the common eigenspaces of the elements of the abelian diagonalizable subgroup $\eta_\Gamma(\hat{G})$.

- Conversely, for an abelian diagonalizable subgroup $Q \leq \mathrm{Aut}\,\mathcal{A}$, its Zariski closure $\bar{Q}$ remains abelian and diagonalizable with the same eigenspaces, and it is an algebraic subgroup of $\mathrm{Aut}\,\mathcal{A}$. Then
  - $\bar{Q} = \hat{G}$, for $G = \mathrm{Hom}(\bar{Q}, \mathbb{F}^\times)$ (morphisms of algebraic groups).

Abelian diagonalizable subgroups

- Any grading $\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ induces a group homomorphism

$$\begin{aligned}\eta_\Gamma : \hat{G} = \mathrm{Hom}(G, \mathbb{F}^\times) &\longrightarrow \mathrm{Aut}\,\mathcal{A}\\ \varphi \quad &\mapsto \eta_\Gamma(\varphi) : x_g \in \mathcal{A}_g \mapsto \varphi(g)x_g.\end{aligned}$$

  The homogeneous components are the common eigenspaces of the elements of the abelian diagonalizable subgroup $\eta_\Gamma(\hat{G})$.

- Conversely, for an abelian diagonalizable subgroup $Q \leq \mathrm{Aut}\,\mathcal{A}$, its Zariski closure $\bar{Q}$ remains abelian and diagonalizable with the same eigenspaces, and it is an algebraic subgroup of $\mathrm{Aut}\,\mathcal{A}$. Then
  - $\bar{Q} = \hat{G}$, for $G = \mathrm{Hom}(\bar{Q}, \mathbb{F}^\times)$ (morphisms of algebraic groups).
  - $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$ is a grading, with

$$\mathcal{A}_g = \{x \in \mathcal{A} : \varphi(x) = g(\varphi)x \ \forall\varphi \in Q\}.$$

MAD subgroups

MAD subgroups

- The fine gradings on $\mathcal{A}$ up to equivalence are in bijection with the conjugacy classes of *maximal abelian diagonalizable* (MAD) subgroups of $\operatorname{Aut}\mathcal{A}$. **(Patera-Zassenhaus 1989)**

MAD subgroups

- The fine gradings on $\mathcal{A}$ up to equivalence are in bijection with the conjugacy classes of *maximal abelian diagonalizable* (MAD) subgroups of $\operatorname{Aut}\mathcal{A}$. **(Patera-Zassenhaus 1989)**

- Algebras with isomorphic groups of automorphisms present "the same" classification of fine gradings up to equivalence.

Classical simple Lie algebras (char $\mathbb{F} = 0$)

Classical simple Lie algebras (char $\mathbb{F} = 0$)

- Havlícek, Patera and Pelantova (1998) gave a list of MAD subgroups of $\operatorname{Aut} \mathfrak{g}$ for $\mathfrak{g}$ classical (but D_4 is excluded).

Classical simple Lie algebras (char $\mathbb{F} = 0$)

- Havlícek, Patera and Pelantova (1998) gave a list of MAD subgroups of $\operatorname{Aut}\mathfrak{g}$ for $\mathfrak{g}$ classical (but D_4 is excluded).

  Any MAD subgroup is conjugated to (at least) one in the list, but the problem of equivalence is not completely settled.

Classical simple Lie algebras (char $\mathbb{F} = 0$)

- Havlícek, Patera and Pelantova (1998) gave a list of MAD subgroups of $\operatorname{Aut}\mathfrak{g}$ for $\mathfrak{g}$ classical (but D_4 is excluded).

  Any MAD subgroup is conjugated to (at least) one in the list, but the problem of equivalence is not completely settled.

  The methods are quite computational.

Classical simple Lie algebras (char $\mathbb{F} = 0$)

- Havlícek, Patera and Pelantova (1998) gave a list of MAD subgroups of $\operatorname{Aut}\mathfrak{g}$ for $\mathfrak{g}$ classical (but D_4 is excluded).

  Any MAD subgroup is conjugated to (at least) one in the list, but the problem of equivalence is not completely settled.

  The methods are quite computational.

  An explicit (and irredundant) description of the corresponding fine gradings has been given for some classical simple Lie algebras of small rank (Patera, Pelantova, Svobodova).

Classical simple Lie algebras (char $\mathbb{F} = 0$)

Classical simple Lie algebras (char $\mathbb{F} = 0$)

- Draper, Martín-González and Viruel (2008-2010) gave the classification of the conjugacy classes of the MAD subgroups of the simple Lie algebra of type D_4 which contain outer automorphisms.

Classical simple Lie algebras (char $\mathbb{F} = 0$)

- Draper, Martín-González and Viruel (2008-2010) gave the classification of the conjugacy classes of the MAD subgroups of the simple Lie algebra of type D_4 which contain outer automorphisms.

- The classification of the fine gradings, up to equivalence, on the classical simple Lie algebras (including D_4!) has been recently completed (E. 2010).

Classical simple Lie algebras (char $\mathbb{F} = 0$)

- Draper, Martín-González and Viruel (2008-2010) gave the classification of the conjugacy classes of the MAD subgroups of the simple Lie algebra of type D_4 which contain outer automorphisms.

- The classification of the fine gradings, up to equivalence, on the classical simple Lie algebras (including D_4!) has been recently completed (E. 2010).
  It uses a different approach, which assigns certain invariants to each fine grading. It is based on the study of gradings on associative algebras, following an approach introduced by Bahturin et al.

Classical simple Lie algebras (char $\mathbb{F} = 0$)

- Draper, Martín-González and Viruel (2008-2010) gave the classification of the conjugacy classes of the MAD subgroups of the simple Lie algebra of type D_4 which contain outer automorphisms.

- The classification of the fine gradings, up to equivalence, on the classical simple Lie algebras (including D_4!) has been recently completed (E. 2010).
  It uses a different approach, which assigns certain invariants to each fine grading. It is based on the study of gradings on associative algebras, following an approach introduced by Bahturin et al.

  (The results are involved. For instance, there appear 17 non equivalent fine gradings for D_4.)

Exceptional simple Lie algebras (char $\mathbb{F} = 0$)

Exceptional simple Lie algebras ($\text{char}\, \mathbb{F} = 0$)

G_2: The fine gradings have been classified by Draper and Martín-González (and independently by Bahturin and Tvalavadze), using the results on gradings on the octonions (E. 1998).

G_2: The fine gradings have been classified by Draper and Martín-González (and independently by Bahturin and Tvalavadze), using the results on gradings on the octonions (E. 1998).

Up to equivalence, there appear only the Cartan grading and the $\mathbb{Z}_2^3$-grading induced by the corresponding grading of the octonions.

Exceptional simple Lie algebras (char $\mathbb{F} = 0$)

Exceptional simple Lie algebras (char $\mathbb{F} = 0$)

F_4: Again the fine gradings have been classified by Draper and Martín-González (2009).

Exceptional simple Lie algebras (char $\mathbb{F} = 0$)

F_4: Again the fine gradings have been classified by Draper and Martín-González (2009).

The proof relies on MAD subgroups and on the description of the simple Lie algebra of type F_4 as the Lie algebra of derivations of the exceptional simple Jordan algebra (or Albert algebra) $\mathbb{A} = H_3(\mathbb{O})$.

Exceptional simple Lie algebras (char $\mathbb{F} = 0$)

F_4: Again the fine gradings have been classified by Draper and Martín-González (2009).

The proof relies on MAD subgroups and on the description of the simple Lie algebra of type F_4 as the Lie algebra of derivations of the exceptional simple Jordan algebra (or Albert algebra) $\mathbb{A} = H_3(\mathbb{O})$.

Apart from the Cartan grading, there appear three other non-equivalent fine gradings.

Exceptional simple Lie algebras (char $\mathbb{F} = 0$)

Exceptional simple Lie algebras (char $\mathbb{F} = 0$)

E_6: A classification has been announced by Draper and Viruel.

Exceptional simple Lie algebras (char $\mathbb{F} = 0$)

E_6: A classification has been announced by Draper and Viruel.

E_7, E_8: ??

Gradings and comodule algebras

In the modular case, gradings are no longer given by eigenspaces of abelian diagonalizable subgroups. A different approach is needed.

Gradings and comodule algebras

In the modular case, gradings are no longer given by eigenspaces of abelian diagonalizable subgroups. A different approach is needed.

G-grading $\longleftrightarrow$ comodule algebra over the group algebra $\mathbb{F}G$

Gradings and comodule algebras

In the modular case, gradings are no longer given by eigenspaces of abelian diagonalizable subgroups. A different approach is needed.

G-grading $\longleftrightarrow$ comodule algebra over the group algebra $\mathbb{F}G$

$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Rightarrow \qquad \begin{aligned} \rho_\Gamma : \mathcal{A} &\longrightarrow \mathcal{A}\otimes \mathbb{F}G \\ x_g &\mapsto x_g \otimes g \end{aligned}$$

(algebra morphism and comodule structure)

Gradings and comodule algebras

In the modular case, gradings are no longer given by eigenspaces of abelian diagonalizable subgroups. A different approach is needed.

G-grading $\longleftrightarrow$ comodule algebra over the group algebra $\mathbb{F}G$

$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Rightarrow \qquad \begin{aligned} \rho_\Gamma : \mathcal{A} &\longrightarrow \mathcal{A}\otimes \mathbb{F}G \\ x_g &\mapsto x_g \otimes g \end{aligned}$$

(algebra morphism and comodule structure)

$$\Gamma_\rho : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \qquad \Leftarrow \qquad \rho : \mathcal{A} \to \mathcal{A}\otimes \mathbb{F}G$$

$(\mathcal{A}_g = \{x \in \mathcal{A} : \rho(x) = x \otimes g\})$

Affine group schemes

Affine group schemes

$$\rho : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \quad \Longleftrightarrow \quad \eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}$$

(comodule algebra) (morphism of affine group schemes)

Affine group schemes

$$\rho : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \quad \Longleftrightarrow \quad \eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}$$

(comodule algebra) (morphism of affine group schemes)

where

- $G^D = \mathsf{Hom}_{\mathsf{alg}}(\mathbb{F}G, .) : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$,
- $\mathbf{Aut}\,\mathcal{A} : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}, \quad R \mapsto \mathsf{Aut}_{R\text{-alg}}(\mathcal{A} \otimes R)$.

Affine group schemes

$$\rho : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \quad \Longleftrightarrow \quad \eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}$$

(comodule algebra) (morphism of affine group schemes)

where

- $G^D = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{F}G, .) : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$,
- $\mathbf{Aut}\,\mathcal{A} : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}, \quad R \mapsto \mathrm{Aut}_{R\text{-alg}}(\mathcal{A} \otimes R)$.

For any $\varphi \in G^D(R)$, $\eta_\Gamma(\varphi) \in \mathrm{Aut}_{R\text{-alg}}(\mathcal{A} \otimes R)$ is given by:

$$\eta_\Gamma(\varphi)(x_g \otimes r) = x_g \otimes \varphi(g)r.$$

Affine group schemes

$$\rho : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \quad \Longleftrightarrow \quad \eta_\Gamma : G^D \to \mathbf{Aut}\,\mathcal{A}$$

(comodule algebra) (morphism of affine group schemes)

where

- $G^D = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{F}G, .) : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}$,
- $\mathbf{Aut}\,\mathcal{A} : \mathrm{Alg}_{\mathbb{F}} \to \mathrm{Grp}, \quad R \mapsto \mathrm{Aut}_{R\text{-alg}}(\mathcal{A} \otimes R)$.

For any $\varphi \in G^D(R)$, $\eta_\Gamma(\varphi) \in \mathrm{Aut}_{R\text{-alg}}(\mathcal{A} \otimes R)$ is given by:

$$\eta_\Gamma(\varphi)(x_g \otimes r) = x_g \otimes \varphi(g) r.$$

and ρ is recovered as

$$\rho(x) = \eta_\Gamma(id_{\mathbb{F}G})(x \otimes 1) \qquad \Big(\eta_\Gamma(id_{\mathbb{F}G}) \in \mathrm{Aut}_{\mathbb{F}G\text{-alg}}(\mathcal{A} \otimes \mathbb{F}G)\Big)$$

The rational points of the affine group scheme G^D are precisely the characters of G:

$$G^D(\mathbb{F}) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{F}G, \mathbb{F}) = \mathrm{Hom}_{\mathrm{groups}}(G, \mathbb{F}^\times) = \hat{G}$$

Gradings and affine group schemes

The rational points of the affine group scheme G^D are precisely the characters of G:

$$G^D(\mathbb{F}) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{F}G, \mathbb{F}) = \mathrm{Hom}_{\mathrm{groups}}(G, \mathbb{F}^\times) = \hat{G}$$

Message:

It is not enough to deal with $\hat{G}$ and $\mathrm{Aut}\,\mathcal{A}$, but also with their extensions to unital commutative and associative $\mathbb{F}$-algebras.

Gradings and affine group schemes

The rational points of the affine group scheme G^D are precisely the characters of G:

$$G^D(\mathbb{F}) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{F}G, \mathbb{F}) = \mathrm{Hom}_{\mathrm{groups}}(G, \mathbb{F}^\times) = \hat{G}$$

Message:

It is not enough to deal with $\hat{G}$ and $\mathrm{Aut}\,\mathcal{A}$, but also with their extensions to unital commutative and associative $\mathbb{F}$-algebras.

If $\mathbf{Aut}\,\mathcal{A} \cong \mathbf{Aut}\,\mathcal{B}$, then the problem of the classification of (fine) gradings on $\mathcal{A}$ and $\mathcal{B}$ are equivalent.

Orthogonal and symplectic Lie algebras

Orthogonal and symplectic Lie algebras

$$\mathbf{Aut}\,\mathfrak{so}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \tau_o), \qquad n \geq 5,\ n \neq 6, 8,$$

$$\mathbf{Aut}\,\mathfrak{sp}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \tau_s), \qquad n \text{ even},\ n \geq 4.$$

(τ_o is an orthogonal involution and τ_s a symplectic involution)

Orthogonal and symplectic Lie algebras

$$\mathbf{Aut}\,\mathfrak{so}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \tau_o), \qquad n \geq 5,\ n \neq 6, 8,$$

$$\mathbf{Aut}\,\mathfrak{sp}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \tau_s), \qquad n \text{ even},\ n \geq 4.$$

(τ_o is an orthogonal involution and τ_s a symplectic involution)

With $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$ and τ an involution of $\mathcal{A}$:

$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \quad (\text{with } \tau(\mathcal{A}_g) = \mathcal{A}_g\ \forall g \in G)$$

$$\Leftrightarrow\ \rho_\Gamma : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \text{ ('commuting with } \tau\text{')}$$

$$\Leftrightarrow \eta_\Gamma : G^D \to \mathbf{Aut}(\mathcal{A}, \tau).$$

Orthogonal and symplectic Lie algebras

$$\mathbf{Aut}\,\mathfrak{so}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \tau_o), \qquad n \geq 5,\ n \neq 6, 8,$$

$$\mathbf{Aut}\,\mathfrak{sp}_n(\mathbb{F}) \cong \mathbf{Aut}(\mathrm{Mat}_n(\mathbb{F}), \tau_s), \qquad n \text{ even},\ n \geq 4.$$

(τ_o is an orthogonal involution and τ_s a symplectic involution)

With $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$ and τ an involution of $\mathcal{A}$:

$$\Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g \quad (\text{with } \tau(\mathcal{A}_g) = \mathcal{A}_g\ \forall g \in G)$$

$$\Leftrightarrow\ \rho_\Gamma : \mathcal{A} \to \mathcal{A} \otimes \mathbb{F}G \text{ (‘commuting with } \tau\text{’)}$$

$$\Leftrightarrow \eta_\Gamma : G^D \to \mathbf{Aut}(\mathcal{A}, \tau).$$

It is enough to study gradings on matrix algebras which are compatible with an involution.

Special linear Lie algebras

Special linear Lie algebras

$$\mathbf{Aut}\,\mathfrak{sl}_2(\mathbb{F}) \cong \mathbf{Aut}\,\mathrm{Mat}_2(\mathbb{F}),$$

$$\mathbf{Aut}\,\mathfrak{psl}_n(\mathbb{F}) \cong \overline{\mathbf{Aut}}\,\mathrm{Mat}_n(\mathbb{F}), \qquad n \geq 3, \text{ unless } n = 3 = \operatorname{char}\mathbb{F}$$

Special linear Lie algebras

$$\mathbf{Aut}\,\mathfrak{sl}_2(\mathbb{F}) \cong \mathbf{Aut}\,\mathrm{Mat}_2(\mathbb{F}),$$

$$\mathbf{Aut}\,\mathfrak{psl}_n(\mathbb{F}) \cong \overline{\mathbf{Aut}}\,\mathrm{Mat}_n(\mathbb{F}), \qquad n \geq 3, \text{ unless } n = 3 = \operatorname{char}\mathbb{F}$$

Here $\overline{\mathbf{Aut}}$ denotes the affine group scheme of automorphisms and antiautomorphisms.

Special linear Lie algebras

$$\mathbf{Aut}\,\mathfrak{sl}_2(\mathbb{F}) \cong \mathbf{Aut}\,\mathrm{Mat}_2(\mathbb{F}),$$

$$\mathbf{Aut}\,\mathfrak{psl}_n(\mathbb{F}) \cong \overline{\mathbf{Aut}}\,\mathrm{Mat}_n(\mathbb{F}), \qquad n \geq 3, \text{ unless } n = 3 = \operatorname{char}\mathbb{F}$$

Here $\overline{\mathbf{Aut}}$ denotes the affine group scheme of automorphisms and antiautomorphisms.

Therefore, the problem of classification of gradings on the classical Lie algebras (other than D_4 and, for $\operatorname{char}\mathbb{F} = 3$, A_2), reduces to a problem about certain gradings on algebras of matrices.

Gradings on matrix algebras

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F}), \qquad \Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$:

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F}), \qquad \Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$:

- There is a graded division algebra $\mathcal{D}$ and a right graded module V for $\mathcal{D}$ such that
$$\mathcal{A} \cong \mathrm{End}_{\mathcal{D}}(V),$$
and the grading on $\mathcal{A}$ is the one induced by the grading on V.

Gradings on matrix algebras

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F}), \qquad \Gamma : \mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$:

- There is a graded division algebra $\mathcal{D}$ and a right graded module V for $\mathcal{D}$ such that

$$\mathcal{A} \cong \mathrm{End}_{\mathcal{D}}(V),$$

and the grading on $\mathcal{A}$ is the one induced by the grading on V.

($\mathcal{D}$ is a tensor product of matrix algebras graded by Pauli matrices.)

Gradings on matrix algebras

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F}), \qquad \Gamma : \mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$:

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$, $\quad \Gamma : \mathcal{A} = \oplus_{g \in G} \mathcal{A}_g$:

- If φ is an antiautomorphism of $\mathcal{A}$ preserving the grading, and such that φ^2 acts as a scalar on each homogeneous space, then there is a graded division algebra $\mathcal{D}$ with a graded involution τ, and a right graded module V for D endowed with a *balanced hermitian form*

  $$B : V \times V \to \mathcal{D},$$

  such that

  $$\mathcal{A} \cong \mathrm{End}_{\mathcal{D}}(V)$$

  and φ corresponds to the adjoint map relative to the hermitian form B.

$\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$, $\qquad \Gamma : \mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$:

- If φ is an antiautomorphism of $\mathcal{A}$ preserving the grading, and such that φ^2 acts as a scalar on each homogeneous space, then there is a graded division algebra $\mathcal{D}$ with a graded involution τ, and a right graded module V for D endowed with a *balanced hermitian form*

$$B : V \times V \to \mathcal{D},$$

such that

$$\mathcal{A} \cong \mathrm{End}_{\mathcal{D}}(V)$$

and φ corresponds to the adjoint map relative to the hermitian form B.

The involution τ imposes severe restriction on $\mathcal{D}$, which must be a tensor product of quaternion algebras each one graded over $\mathbb{Z}_2^2$.

Type A_r

Type A_r

Assume $n \neq 3$ if $\operatorname{char} \mathbb{F} = 3$. Let $\mathfrak{g} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{A}, \mathcal{A}]/Z(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}]$ with $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

Type A_r

Assume $n \neq 3$ if $\operatorname{char} \mathbb{F} = 3$. Let $\mathfrak{g} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{A},\mathcal{A}]/Z(\mathcal{A}) \cap [\mathcal{A},\mathcal{A}]$ with $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

- For $n = 2$, any grading on $\mathfrak{g}$ is induced by a grading on $\mathcal{A}$: $\mathfrak{g}_g = \mathfrak{g} \cap \mathcal{A}_g \ \forall g \in G$.

Type A_r

Assume $n \neq 3$ if $\operatorname{char} \mathbb{F} = 3$. Let $\mathfrak{g} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{A}, \mathcal{A}]/Z(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}]$ with $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

- For $n = 2$, any grading on $\mathfrak{g}$ is induced by a grading on $\mathcal{A}$: $\mathfrak{g}_g = \mathfrak{g} \cap \mathcal{A}_g \ \forall g \in G$.
- For $n > 2$, any grading on $\mathfrak{g}$ is either

Type A_r

Assume $n \neq 3$ if $\operatorname{char} \mathbb{F} = 3$. Let $\mathfrak{g} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{A}, \mathcal{A}]/Z(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}]$ with $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

- For $n = 2$, any grading on $\mathfrak{g}$ is induced by a grading on $\mathcal{A}$: $\mathfrak{g}_g = \mathfrak{g} \cap \mathcal{A}_g \ \forall g \in G$.

- For $n > 2$, any grading on $\mathfrak{g}$ is either
  - induced by a grading on $\mathcal{A}$, or

Type A_r

Assume $n \neq 3$ if $\operatorname{char}\mathbb{F} = 3$. Let $\mathfrak{g} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{A},\mathcal{A}]/Z(\mathcal{A}) \cap [\mathcal{A},\mathcal{A}]$ with $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

- For $n = 2$, any grading on $\mathfrak{g}$ is induced by a grading on $\mathcal{A}$: $\mathfrak{g}_g = \mathfrak{g} \cap \mathcal{A}_g \ \forall g \in G$.

- For $n > 2$, any grading on $\mathfrak{g}$ is either
  - induced by a grading on $\mathcal{A}$, or
  - there are $h \in G$, $h^2 = 1$, a $\bar{G} = G/\langle h\rangle$-grading on $\mathcal{A}$, an antiautomorphism φ such that φ preserves the $\bar{G}$-grading and its square acts as a scalar on each homogeneous component, and a character $\chi \in \hat{G}$ with $\chi(h) = -1$, such that

$$\mathfrak{g}_g = \{x \in \mathcal{A}_{\bar{g}} : -\varphi(x) = \chi(g)x\} \cap [\mathcal{A},\mathcal{A}] \pmod{Z(\mathcal{A})}.$$

Type A_r

Assume $n \neq 3$ if $\operatorname{char} \mathbb{F} = 3$. Let $\mathfrak{g} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{A}, \mathcal{A}]/Z(\mathcal{A}) \cap [\mathcal{A}, \mathcal{A}]$ with $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

- For $n = 2$, any grading on $\mathfrak{g}$ is induced by a grading on $\mathcal{A}$: $\mathfrak{g}_g = \mathfrak{g} \cap \mathcal{A}_g \ \forall g \in G$.

- For $n > 2$, any grading on $\mathfrak{g}$ is either
    - induced by a grading on $\mathcal{A}$, or
    - there are $h \in G$, $h^2 = 1$, a $\bar{G} = G/\langle h \rangle$-grading on $\mathcal{A}$, an antiautomorphism φ such that φ preserves the $\bar{G}$-grading and its square acts as a scalar on each homogeneous component, and a character $\chi \in \hat{G}$ with $\chi(h) = -1$, such that

$$\mathfrak{g}_g = \{x \in \mathcal{A}_{\bar{g}} : -\varphi(x) = \chi(g)x\} \cap [\mathcal{A}, \mathcal{A}] \pmod{Z(\mathcal{A})}.$$

Type A_r

Assume $n \neq 3$ if $\operatorname{char}\mathbb{F} = 3$. Let $\mathfrak{g} = \mathfrak{psl}_n(\mathbb{F}) = [\mathcal{A},\mathcal{A}]/Z(\mathcal{A}) \cap [\mathcal{A},\mathcal{A}]$ with $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$.

- For $n = 2$, any grading on $\mathfrak{g}$ is induced by a grading on $\mathcal{A}$: $\mathfrak{g}_g = \mathfrak{g} \cap \mathcal{A}_g \ \forall g \in G$.

- For $n > 2$, any grading on $\mathfrak{g}$ is either
  - induced by a grading on $\mathcal{A}$, or
  - there are $h \in G$, $h^2 = 1$, a $\bar{G} = G/\langle h\rangle$-grading on $\mathcal{A}$, an antiautomorphism φ such that φ preserves the $\bar{G}$-grading and its square acts as a scalar on each homogeneous component, and a character $\chi \in \hat{G}$ with $\chi(h) = -1$, such that

$$\mathfrak{g}_g = \{x \in \mathcal{A}_{\bar{g}} : -\varphi(x) = \chi(g)x\} \cap [\mathcal{A},\mathcal{A}] \pmod{Z(\mathcal{A})}.$$

The exceptional case $\mathfrak{psl}_3(\mathbb{F})$ in characteristic 3 is dealt with in an unexpected way, as $\mathfrak{psl}_3(\mathbb{F}) \cong [\mathbb{O},\mathbb{O}]$, and its gradings are in bijection with the gradings on the octonions. (E. 1998)

B_r $(r \geq 2)$, C_r $(r \geq 3)$, D_r $(r \geq 5)$

B_r $(r \geq 2)$, C_r $(r \geq 3)$, D_r $(r \geq 5)$

In all these cases, $\mathfrak{g} \cong \mathcal{K}(\mathcal{A}, \tau) = \{x \in \mathcal{A} : \tau(x) = -x\}$, for $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$ and τ an involution, and

B_r $(r \geq 2)$, C_r $(r \geq 3)$, D_r $(r \geq 5)$

In all these cases, $\mathfrak{g} \cong \mathcal{K}(\mathcal{A}, \tau) = \{x \in \mathcal{A} : \tau(x) = -x\}$, for $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$ and τ an involution, and

Any grading on $\mathfrak{g}$ is the restriction of a grading on $\mathcal{A}$ preserved by τ.

B_r $(r \geq 2)$, C_r $(r \geq 3)$, D_r $(r \geq 5)$

In all these cases, $\mathfrak{g} \cong \mathcal{K}(\mathcal{A}, \tau) = \{x \in \mathcal{A} : \tau(x) = -x\}$, for $\mathcal{A} = \mathrm{Mat}_n(\mathbb{F})$ and τ an involution, and

Any grading on $\mathfrak{g}$ is the restriction of a grading on $\mathcal{A}$ preserved by τ.

Most of the previous results on the classical simple Lie algebras are due to Bahturin and Kochetov (2010), and use earlier results of a number of authors.

$$\mathbf{Aut}\,\mathfrak{g}_2 \cong \mathbf{Aut}\,\mathbb{O}$$

$$\textbf{Aut}\,\mathfrak{g}_2 \cong \textbf{Aut}\,\mathbb{O}$$

The gradings on $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$ are induced by gradings on $\mathbb{O}$.

$$\textbf{Aut}\,\mathfrak{g}_2 \cong \textbf{Aut}\,\mathbb{O}$$

The gradings on $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$ are induced by gradings on $\mathbb{O}$.

This is valid for arbitrary fields of characteristic $\neq 2, 3$.

G_2

$$\mathbf{Aut}\,\mathfrak{g}_2 \cong \mathbf{Aut}\,\mathbb{O}$$

The gradings on $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$ are induced by gradings on $\mathbb{O}$.

This is valid for arbitrary fields of characteristic $\neq 2, 3$.
In characteristic 3 there are no simple Lie algebras of type G_2.
(Note that $\mathfrak{Der}(\mathbb{O})$ is no longer simple in characteristic 3, as it contains the simple ideal $\mathrm{ad}_{[\mathbb{O},\mathbb{O}]} \cong \mathfrak{psl}_3(\mathbb{F})$.)

$$\mathbf{Aut}\,\mathfrak{g}_2 \cong \mathbf{Aut}\,\mathbb{O}$$

The gradings on $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$ are induced by gradings on $\mathbb{O}$.

This is valid for arbitrary fields of characteristic $\neq 2, 3$.
In characteristic 3 there are no simple Lie algebras of type G_2.
(Note that $\mathfrak{Der}(\mathbb{O})$ is no longer simple in characteristic 3, as it contains the simple ideal $\mathrm{ad}_{[\mathbb{O},\mathbb{O}]} \cong \mathfrak{psl}_3(\mathbb{F})$.)

The gradings on $\mathbb{O}$ are well-known (E. 1998).

$$\textbf{Aut}\,\mathfrak{g}_2 \cong \textbf{Aut}\,\mathbb{O}$$

The gradings on $\mathfrak{g}_2 = \mathfrak{Der}(\mathbb{O})$ are induced by gradings on $\mathbb{O}$.

This is valid for arbitrary fields of characteristic $\neq 2, 3$.
In characteristic 3 there are no simple Lie algebras of type G_2.
(Note that $\mathfrak{Der}(\mathbb{O})$ is no longer simple in characteristic 3, as it contains the simple ideal $\mathrm{ad}_{[\mathbb{O},\mathbb{O}]} \cong \mathfrak{psl}_3(\mathbb{F})$.)

The gradings on $\mathbb{O}$ are well-known (E. 1998).
Up to equivalence, the fine gradings on $\mathfrak{g}_2$ are

- the Cartan grading, and
- a $\mathbb{Z}_2^3$-grading with $(\mathfrak{g}_2)_0 = 0$ and where $(\mathfrak{g}_2)_g$ is a Cartan subalgebra of $\mathfrak{g}_2$ for any $0 \neq g \in \mathbb{Z}_2^3$.

F_4

$\mathbf{Aut}\,\mathfrak{f}_4 \cong \mathbf{Aut}\,\mathbb{A}$ $\qquad$ ($\mathbb{A} = H_3(\mathbb{O})$ is the Albert algebra)

F_4

$$\textbf{Aut}\,\mathfrak{f}_4 \cong \textbf{Aut}\,\mathbb{A} \qquad (\mathbb{A} = H_3(\mathbb{O}) \text{ is the Albert algebra})$$

Theorem (Draper–Martín-González (char $= 0$), E.-Kochetov)

Up to equivalence, the fine gradings of the Albert algebra are:

1. *The Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$).*
2. *A $\mathbb{Z}_2^5$-grading obtained by combining a natural $\mathbb{Z}_2^2$-grading on 3×3 hermitian matrices with the fine grading over $\mathbb{Z}_2^3$ of $\mathbb{O}$.*
3. *A $\mathbb{Z} \times \mathbb{Z}_2^3$-grading obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
4. *A $\mathbb{Z}_3^3$-grading with $\dim \mathbb{A}_g = 1\ \forall g$.*

The fine gradings on $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$ are the ones induced by these.

F_4

Aut $\mathfrak{f}_4 \cong$ **Aut** $\mathbb{A}$ $\qquad$ ($\mathbb{A} = H_3(\mathbb{O})$ is the Albert algebra)

Theorem (Draper–Martín-González (char $= 0$), E.-Kochetov)

Up to equivalence, the fine gradings of the Albert algebra are:

1. *The Cartan grading (weight space decomposition relative to a Cartan subalgebra of $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$).*
2. *A $\mathbb{Z}_2^5$-grading obtained by combining a natural $\mathbb{Z}_2^2$-grading on 3×3 hermitian matrices with the fine grading over $\mathbb{Z}_2^3$ of $\mathbb{O}$.*
3. *A $\mathbb{Z} \times \mathbb{Z}_2^3$-grading obtained by combining a 5-grading and the $\mathbb{Z}_2^3$-grading on $\mathbb{O}$.*
4. *A $\mathbb{Z}_3^3$-grading with $\dim \mathbb{A}_g = 1 \ \forall g$.*

The fine gradings on $\mathfrak{f}_4 = \mathfrak{Der}(\mathbb{A})$ are the ones induced by these.

All the gradings up to isomorphism on $\mathbb{A}$ and $\mathfrak{f}_4$ have been classified too (E.-Kochetov).

And now?

And now?

- Bahturin, Kochetov and their students have classified gradings on some Cartan type simple modular Lie algebras. This requires different methods.

And now?

- Bahturin, Kochetov and their students have classified gradings on some Cartan type simple modular Lie algebras. This requires different methods.
- A classification of gradings up to isomorphism of D_4 is missing even in characteristic 0, as it is the classification of fine gradings up to equivalence in prime characteristic.

And now?

- Bahturin, Kochetov and their students have classified gradings on some Cartan type simple modular Lie algebras. This requires different methods.
- A classification of gradings up to isomorphism of D_4 is missing even in characteristic 0, as it is the classification of fine gradings up to equivalence in prime characteristic.
- Many interesting gradings on E_6, E_7, E_8 are known, some of them are related either to gradings on octonions, the so called Okubo algebras, or the Albert algebra, but a classification is missing. The classification of fine gradings on E_6 in characteristic 0 has been announced by Draper and Viruel.

And now?

- Bahturin, Kochetov and their students have classified gradings on some Cartan type simple modular Lie algebras. This requires different methods.
- A classification of gradings up to isomorphism of D_4 is missing even in characteristic 0, as it is the classification of fine gradings up to equivalence in prime characteristic.
- Many interesting gradings on E_6, E_7, E_8 are known, some of them are related either to gradings on octonions, the so called Okubo algebras, or the Albert algebra, but a classification is missing. The classification of fine gradings on E_6 in characteristic 0 has been announced by Draper and Viruel.

And now?

- Bahturin, Kochetov and their students have classified gradings on some Cartan type simple modular Lie algebras. This requires different methods.
- A classification of gradings up to isomorphism of D_4 is missing even in characteristic 0, as it is the classification of fine gradings up to equivalence in prime characteristic.
- Many interesting gradings on E_6, E_7, E_8 are known, some of them are related either to gradings on octonions, the so called Okubo algebras, or the Albert algebra, but a classification is missing. The classification of fine gradings on E_6 in characteristic 0 has been announced by Draper and Viruel.

That's all. Thanks