Okubo algebras

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Composition algebras

Definition

A **composition algebra** is a triple $(\mathcal{C}, *, n)$, where

- $(\mathcal{C}, *)$ is a (not necessarily associative) algebra,
- $n : \mathcal{C} \to \mathbb{F}$ is a nonsingular *multiplicative* quadratic form.

The unital composition algebras are called **Hurwitz algebras**.

For Hurwitz algebras, the map $x \mapsto \bar{x} = n(x,1)1 - x$ is an involution such that $x\bar{x} = \bar{x}x = n(x)1$ for any x.

Examples

The classical real division algebras

$$\mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}, \quad \mathbb{O},$$

are Hurwitz algebras whose norm is positive definite, so they are **absolute valued algebras**.

Examples

A Hurwitz algebra is said to be **split** if either its dimension is 1 or its norm is isotropic.

Up to isomorphism, the split Hurwitz algebras are the following:

- $\mathbb{F}$, with $n(\alpha) = \alpha^2$,
- $\mathbb{F} \times \mathbb{F}$, with $n\big((\alpha, \beta)\big) = \alpha\beta$,
- $\mathrm{Mat}_2(\mathbb{F})$, with $n(A) = \det(A)$,

Examples

- The algebra of **Zorn matrices** (or *split Cayley algebra*):

$$\mathcal{C}_s = \left\{ \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{F}, u, v \in \mathbb{F}^3 \right\},$$

with

$$\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \begin{pmatrix} \alpha' & u' \\ v' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + (u \mid v') & \alpha u' + \beta' u - v \times v' \\ \alpha' v + \beta v' + u \times u' & \beta\beta' + (v \mid v') \end{pmatrix},$$

and

$$n\left(\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \right) = \alpha\beta - (u \mid v).$$

In particular, over an algebraically closed field these are the only Hurwitz algebras, up to isomorphism, and over arbitrary fields the dimension of any Hurwitz algebra is restricted to 1, 2, 4 or 8.

Kaplansky's trick (1953)

Given any finite-dimensional composition algebra $(\mathcal{C}, \cdot, n)$ and an element $x \in \mathcal{C}$ with $n(x) \neq 0$, the element $u = \frac{1}{n(x)} x^{\cdot 2}$ has norm 1, so the left and right multiplications by u: L_u and R_u, are isometries.

Define a new multiplication on $\mathcal{C}$ by

$$x \circ y := R_u^{-1}(x) \cdot L_u^{-1}(y).$$

Then $(\mathcal{C}, \circ, n)$ is a Hurwitz algebra with unity $u^{\cdot 2}$.

Therefore, the multiplication on any finite-dimensional composition algebra $(\mathcal{C}, \cdot, n)$ is of the form

$$x \cdot y = f(x) \circ g(y)$$

for a Hurwitz algebra $(\mathcal{C}, \circ, n)$ and two isometries $f, g \in \mathrm{O}(\mathcal{C}, n)$.

Pseudo-octonions (Okubo 1978)

Let $\mathbb{F}$ be a field of characteristic $\neq 2, 3$ containing a primitive cubic root ω of 1. The element $\mu = \frac{1}{1-\omega}$ satisfies $3\mu(1-\mu) = 1$.

On the vector space $\mathfrak{sl}_3(\mathbb{F})$ consider the multiplication:

$$x \diamond y = \mu xy + (1-\mu)yx - \frac{1}{3}\operatorname{tr}(xy)1$$

and the quadratic form

$$q(x) = \frac{1}{6}\operatorname{tr}(x^2).$$

Then, for any x, y,

$$q(x \diamond y) = q(x)q(y), \qquad (x \diamond y) \diamond x = q(x)y = x \diamond (y \diamond x).$$

In particular, $(\mathfrak{sl}_3(\mathbb{F}), \diamond, q)$ is a *composition algebra*.

Pseudo-octonions

Alternatively, scale $\diamond$ by $\omega - \omega^2$ and q by $(\omega - \omega^2)^2 = -3$, to get:

$$x * y = \omega xy - \omega^2 yx - \frac{\omega - \omega^2}{3}\,\mathrm{tr}(xy)1,$$

$$n(x) = -\frac{1}{2}\,\mathrm{tr}(x^2),$$

so

$$n(x * y) = n(x)n(y), \qquad (x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in \mathfrak{sl}_3(\mathbb{F})$.

Denote by $P_8(\mathbb{F})$ the algebra thus defined (**algebra of pseudo-octonions**).

- $P_8(\mathbb{F})$ makes sense in characteristic 2, because $\mathrm{tr}(x^2)$ 'is a multiple of 2' if $\mathrm{tr}(x) = 0$.
- Okubo and Osborn (1981) gave an 'ad hoc' definition of $P_8(\mathbb{F})$ over fields of characteristic 3 by means of its multiplication table.

Petersson algebras

Theorem (Petersson 1969)

Let $\mathbb{F}$ be an algebraically closed field of characteristic $\neq 2, 3$. Then any simple finite-dimensional algebra satisfying

$$(xy)x = x(yx), \quad ((xz)y)(xz) = \big(x((zy)z)\big)x$$

for any x, y, z is, up to isomorphism, one of the following:

- *The algebra $(\mathcal{B}, \bullet)$, where $\mathcal{B}$ is a Hurwitz algebra and $x \bullet y = \bar{x}\bar{y}$.*
- *The algebra $(\mathcal{C}_s, *)$, where $\mathcal{C}_s$ is the split Cayley algebra, and $x * y = \varphi(\bar{x})\varphi^2(\bar{y})$, where φ is a precise order 3 automorphism of $\mathcal{C}_s$.*

Petersson algebras

$P_8(\mathbb{F})$ satisfies the hypotheses of Petersson's Theorem, so the algebra of pseudo-octonions over an algebraically closed field of characteristic $\neq 2, 3$ must be isomorphic to the last algebra in the Theorem.

Definition

Let $(\mathcal{C}, \cdot, n)$ be a Hurwitz algebra, and let $\varphi \in \mathrm{Aut}(\mathcal{C}, \cdot, n)$ be an automorphism with $\varphi^3 = \mathrm{id}$.

- The composition algebra $(\mathcal{C}, *, n)$, with

$$x * y = \varphi(\bar{x}) \cdot \varphi^2(\bar{y})$$

  is called a **Petersson algebra**, and denoted by $\mathcal{C}_\varphi$.
- $\mathcal{C}_{\mathrm{id}}$ is called a **para-Hurwitz** algebra.

Modern definition

Inspired by Okubo's definition (E. 1999)

In order to define Okubo algebras over arbitrary fields consider the Pauli matrices:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

in $\mathrm{Mat}_3(\mathbb{C})$, which satisfy

$$x^3 = y^3 = 1, \quad xy = \omega yx.$$

For $i, j \in \mathbb{Z}/3\mathbb{Z}$, $(i, j) \neq (0, 0)$, define

$$x_{i,j} := \frac{\omega^{ij}}{\omega - \omega^2} x^i y^j.$$

$\{x_{i,j} : (i, j) \neq (0, 0)\}$ is a basis of $\mathfrak{sl}_3(\mathbb{C})$.

Modern definition

Inspired by Okubo's definition

$$x_{i,j} * x_{k,l} = \omega x_{i,j} x_{k,l} - \omega^2 x_{k,l} x_{i,j} - \frac{\omega - \omega^2}{3} \operatorname{tr}(x_{i,j} x_{k,l}) 1$$

$$= \begin{cases} x_{i+k,j+l} \\ 0 \\ -x_{i+k,j+l} \end{cases} \qquad \big(x_{0,0} := 0\big)$$

depending on $\begin{vmatrix} i & j \\ k & l \end{vmatrix}$ being equal to 0, 1 or 2 (modulo 3).

Miraculously, the ω's disappear!

Besides, $n(x_{i,j}) = 0$ for any i, j, and

$$n(x_{i,j}, x_{k,l}) = \begin{cases} 1 & \text{for } (i,j) = -(k,l), \\ 0 & \text{otherwise.} \end{cases}$$

Modern definition

Inspired by Okubo's definition

Thus, the $\mathbb{Z}$-span

$$\mathcal{O}_{\mathbb{Z}} = \mathbb{Z}\text{-span}\,\{x_{i,j} : -1 \leq i,j \leq 1,\ (i,j) \neq (0,0)\}$$

is closed under $*$, and n restricts to a nonsingular multiplicative quadratic form on $\mathcal{O}_{\mathbb{Z}}$.

Definition

Let $\mathbb{F}$ be an arbitrary field. Then

$$\mathcal{O}_{\mathbb{F}} := \mathcal{O}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F},$$

with the induced multiplication and nonsingular quadratic form, is called the **split Okubo algebra** over $\mathbb{F}$.
The twisted forms of $\mathcal{O}_{\mathbb{F}}$ are called the **Okubo algebras** over $\mathbb{F}$.

Modern definition

Inspired by Okubo's definition

$*$	$x_{1,0}$	$x_{-1,0}$	$x_{0,1}$	$x_{0,-1}$	$x_{1,1}$	$x_{-1,-1}$	$x_{-1,1}$	$x_{1,-1}$
$x_{1,0}$	$x_{-1,0}$	0	0	$-x_{1,-1}$	0	$-x_{0,-1}$	0	$-x_{-1,-1}$
$x_{-1,0}$	0	$x_{1,0}$	$-x_{-1,1}$	0	$-x_{0,1}$	0	$-x_{1,1}$	0
$x_{0,1}$	$-x_{1,1}$	0	$x_{0,-1}$	0	$-x_{1,-1}$	0	0	$-x_{1,0}$
$x_{0,-1}$	0	$-x_{-1,-1}$	0	$x_{0,1}$	0	$-x_{-1,1}$	$-x_{-1,0}$	0
$x_{1,1}$	$-x_{-1,1}$	0	0	$-x_{1,0}$	$x_{-1,-1}$	0	$-x_{0,-1}$	0
$x_{-1,-1}$	0	$-x_{1,-1}$	$-x_{-1,0}$	0	0	$x_{1,1}$	0	$-x_{0,1}$
$x_{-1,1}$	$-x_{0,1}$	0	$-x_{-1,-1}$	0	0	$-x_{1,0}$	$x_{1,-1}$	0
$x_{1,-1}$	0	$-x_{0,-1}$	0	$-x_{1,1}$	$-x_{-1,0}$	0	0	$x_{-1,1}$

Modern definition

Inspired by Petersson's Theorem (E.-Pérez-Izquierdo 1996)

The split Cayley algebra is endowed with a natural order 3 automorphism:

$$\tau : \begin{pmatrix} \alpha & (u_1, u_2, u_3) \\ (v_1, v_2, v_3) & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & (u_3, u_1, u_2) \\ (v_3, v_1, v_2) & \beta \end{pmatrix}.$$

Definition

The **split Okubo algebra** is the Petersson algebra $(\mathcal{C}_s)_\tau$.
The twisted forms of $(\mathcal{C}_s)_\tau$ are called the **Okubo algebras** over $\mathbb{F}$.

Remark

There is no conflict between the two definitions: $\mathcal{O}_\mathbb{F}$ and $(\mathcal{C}_s)_\tau$ are isomorphic.

Symmetric composition algebras

Definition

A composition algebra $(\mathcal{S}, *, n)$ is said to be **symmetric** if the polar form of its norm is associative:

$$n(x * y, z) = n(x, y * z),$$

for any $x, y, z \in \mathcal{S}$.

This is equivalent to the condition:

$$(x * y) * x = n(x)y = x * (y * x),$$

for any $x, y \in \mathcal{S}$.

Para-Hurwitz and Okubo algebras are examples of symmetric composition algebras.

Symmetric composition algebras are (almost always) either para-Hurwitz or Okubo

Theorem (Okubo-Osborn 1981, E.-Pérez-Izquierdo 1996)

Any symmetric composition algebra is either a form of a para-Hurwitz algebra or an Okubo algebra.

In other words, any symmetric composition algebra over an algebraically closed field $\mathbb{F}$ is, up to isomorphism, either the para-Hurwitz algebra associated to one of the four Hurwitz algebras: $\mathbb{F}$, $\mathbb{F} \times \mathbb{F}$, $\mathrm{Mat}_2(\mathbb{F})$ or $\mathcal{C}_s$, or the Okubo algebra $\mathcal{O}_{\mathbb{F}}$.

Symmetric composition algebras are (almost always) either para-Hurwitz or Okubo

Sketch of proof

- If $(\mathcal{C}, *.n)$ is a symmetric composition algebra over $\mathbb{F}$, there is a field extension $\mathbb{K}/\mathbb{F}$ of degree ≤ 3 such that $(\mathcal{C}_{\mathbb{K}}, *, n)$ contains a nonzero idempotent. Hence we may assume that there exists $0 \neq e \in \mathcal{C}$ with $e * e = e$. Then $n(e) = 1$.
- Consider the new multiplication

$$x \cdot y = (e * x) * (y * e).$$

Then $(\mathcal{C}, \cdot, n)$ is a Hurwitz algebra with unity $1 = e$, and the map $\tau : x \mapsto e * (e * x) = n(e, x)e - x * e$ is an automorphism of both $(\mathcal{C}, *, n)$ and of $(\mathcal{C}, \cdot, n)$, such that $\tau^3 = \mathrm{id}$.

- If $\tau = \mathrm{id}$, $(\mathcal{C}, *, n)$ is para-Hurwitz, otherwise it may be either para-Hurwitz or Okubo.

Forms of para-Hurwitz algebras (dim ≥ 4)

Theorem

For $m = 4$ or $m = 8$, any form of a para-Hurwitz algebra of dimension m is the para-Hurwitz algebra associated to a unique, up to isomorphism, Hurwitz algebra.

Sketch of proof

Let $(\mathcal{C}, \cdot, n)$ be a Hurwitz algebra of dimension m, and $(\mathcal{C}, \bullet, n)$ the corresponding Hurwitz algebra. Then

$$\{x \in \mathcal{C} : x \cdot y = y \cdot x \ \forall y \in \mathcal{C}\} = \{x \in \mathcal{C} : x \bullet y = y \bullet x \ \forall y \in \mathcal{C}\} = \mathbb{F}1.$$

Hence any automorphism of $(\mathcal{C}, \bullet, n)$ fixes 1 and $\mathbf{Aut}(\mathcal{C}, \cdot, n) = \mathbf{Aut}(\mathcal{C}, \bullet, n)$. □

Forms of para-Hurwitz algebras (dim = 2)

Restrict Okubo's construction to the diagonal part of $\mathfrak{sl}_3(\mathbb{F})$
$(\mathbb{F}^3)_0 := \{(a_1, a_2, a_3) \in \mathbb{F}^3 : a_1 + a_2 + a_3 = 0\}$

$$a \diamond b = ab - \frac{1}{3}t(ab)1 \quad t\big(a = (a_1, a_2, a_3)\big) = a_1 + a_2 + a_3$$

$$n(a) = \frac{1}{6}t(a^2) \quad \text{(valid in characteristic 2!)}$$

$\big((\mathbb{F}^3)_0, \diamond, n\big)$ is a two-dimensional symmetric composition algebra and, by restriction, we obtain a natural isomorphism

$$S_3 \simeq \mathbf{Aut}(\mathbb{F}^3) \to \mathbf{Aut}\big((\mathbb{F}^3)_0, \diamond, n\big)$$

of affine group schemes.

Remark

If $\operatorname{char}\mathbb{F} \neq 3$, $\big((\mathbb{F}^3)_0, \diamond, n\big)$ is the para-Hurwitz algebra corresponding to the Hurwitz algebra $\mathbb{F}[X]/(X^2 + X + 1)$.

Forms of para-Hurwitz algebras (dim $= 2$)

Theorem (E.-Myung 1991, 1993)

If $\operatorname{char}\mathbb{F} \neq 3$, *the map*

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{cubic étale algebras}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{two-dimensional symmetric}\\ \textit{composition algebras}\end{array}\right\}$$

$$[\mathbb{L}] \mapsto [(\mathbb{L}_0, \diamond, n)]$$

is bijective.
(Here $\mathbb{L}_0 = \{x \in \mathbb{L} : t(x) = 0\}$, t *is the generic trace,*
$a \diamond b = ab - \frac{1}{3}t(ab)1$, $n(a) = \frac{1}{6}t(a^2)$.*)*

Forms of para-Hurwitz algebras (dim = 2)

Theorem (E.-Myung 91, E. 1997)

Let $(\mathcal{C}, *, n)$ *be a two-dimensional symmetric composition algebra over a field* $\mathbb{F}$ *of characteristic* 3*.*

- $(\mathcal{C}, *, n)$ *contains an idempotent if and only if it is para-Hurwitz, and two such algebras are isomorphic if and only if so are the corresponding Hurwitz algebras.*
- $(\mathcal{C}, *, n)$ *does not contain idempotents if and only if* $\exists \lambda \in \mathbb{F} \setminus \mathbb{F}^3$ *and a basis* $\{u, v\}$ *such that*

$$u * u = v, \quad u * v = v * u = u, \quad v * v = \lambda u - v.$$

The algebras associated to the scalars λ *and* μ *are isomorphic if and only if* $\mathbb{F}^3\lambda + \mathbb{F}^3(\lambda^2 + 1) = \mathbb{F}^3\mu + \mathbb{F}^3(\mu^2 + 1)$*.*

Classification of Okubo algebras

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 3$, containing a primitive cubic root of 1. By restriction we obtain a natural isomorphism

$$\mathbf{PGL}_3 \simeq \mathbf{Aut}\big(\mathrm{Mat}_3(\mathbb{F})\big) \to \mathbf{Aut}\big((\mathfrak{sl}_3(\mathbb{F}), \diamond, q)\big)$$

of affine group schemes.

Theorem (E.-Myung 1991, 1993)

The map

$$\begin{Bmatrix} \textit{Isomorphism classes of} \\ \textit{central simple degree } 3 \\ \textit{associative algebras} \end{Bmatrix} \longrightarrow \begin{Bmatrix} \textit{Isomorphism classes} \\ \textit{of Okubo algebras} \end{Bmatrix}$$

$$[\mathcal{A}] \mapsto [(\mathcal{A}_0, \diamond, q)]$$

is bijective.

Classification of Okubo algebras

Let $\mathbb{F}$ be a field, $\operatorname{char}\mathbb{F} \neq 3$, not containing primitive cubic roots of 1. Let $\mathbb{K} = \mathbb{F}[X]/(X^2 + X + 1)$.

Theorem (E.-Myung 1991, 1993)

The map

$$\left\{\begin{array}{c}\text{Isomorphism classes of}\\ \text{pairs } (\mathcal{B},\sigma)\text{, where } \mathcal{B} \text{ is a simple}\\ \text{degree 3 associative algebra}\\ \text{over } \mathbb{K} \text{ and } \sigma \text{ a } \mathbb{K}/\mathbb{F}\text{-involution}\\ \text{of the second kind}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Isomorphism classes}\\ \text{of Okubo algebras}\end{array}\right\}$$

$$[(\mathcal{B},\tau)] \quad \mapsto \quad [(\mathrm{Sym}(\mathcal{B},\sigma)_0, \diamond, q)]$$

is bijective.

Classification of Okubo algebras (char $\mathbb{F} = 3$)

Theorem (Chernousov-E.-Knus-Tignol 2013)

Let $(\mathcal{O}, *, n)$ be the split Okubo algebra over a field $\mathbb{F}$ (char $\mathbb{F} = 3$).

- $\mathbf{Aut}(\mathcal{O}, *, n)$ *is not smooth:* $\dim \mathbf{Aut}(\mathcal{O}, *, n) = 8$ *while* $\mathfrak{Der}(\mathcal{O}, *, n)$ *is a simple (nonclassical) Lie algebra of dimension* 10*.*
- $\mathbf{Aut}(\mathcal{O}, *, n) = \mathbf{HD}$*, where* $\mathbf{H} = \mathbf{Aut}(\mathcal{O}, *, n)_{\text{red}}$ *and* $\mathbf{D} \simeq \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$*.*
- *The map*

$$H^1(\mathbb{F}, \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \to H^1(\mathbb{F}, \mathbf{Aut}(\mathcal{O}, *, n))$$

induced by the inclusion $\mathbf{D} \hookrightarrow \mathbf{Aut}(\mathcal{O}, *, n)$*, is surjective.*

Classification of Okubo algebras (char $\mathbb{F} = 3$)

Recall that $\mathcal{O}$ is spanned by elements $x_{i,j}$, $(i,j) \neq (0,0)$ (indices modulo 3). It is actually generated by $x_{1,0}$ and $x_{0,1}$. Given $0 \neq \alpha, \beta \in \mathbb{F}$, the elements

$$x_{1,0} \otimes \alpha^{\frac{1}{3}},\ x_{0,1} \otimes \beta^{\frac{1}{3}} \in \mathcal{O} \otimes_{\mathbb{F}} \overline{\mathbb{F}}$$

generates, by multiplication and linear combinations over $\mathbb{F}$, a twisted form of $(\mathcal{O}, *, n)$. Denote it by $\mathcal{O}_{\alpha,\beta}$.

Corollary

The following map is surjective:

$$\begin{array}{ccc} \mathbb{F}^\times/(\mathbb{F}^\times)^3 \times \mathbb{F}^\times/(\mathbb{F}^\times)^3 & \longrightarrow & \left\{ \begin{array}{c} \textit{Isomorphism classes} \\ \textit{of Okubo algebras} \end{array} \right\} \\ \big(\alpha(\mathbb{F}^\times)^3, \beta(\mathbb{F}^\times)^3\big) & \mapsto & [\mathcal{O}_{\alpha,\beta}] \end{array}$$

Classification of Okubo algebras (char $\mathbb{F} = 3$)

Theorem (E. 1997)

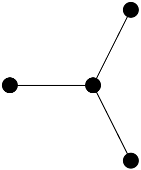
- *Any Okubo algebra over* $\mathbb{F}$ *(*char $\mathbb{F} = 3$*) is isomorphic to* $\mathcal{O}_{\alpha,\beta}$ *for some* $0 \neq \alpha, \beta \in \mathbb{F}$.

- *For* $0 \neq \alpha, \beta \in \mathbb{F}$*, let*

$$S_{\alpha,\beta} := span_{\mathbb{F}^3} \left\{ \alpha^{\pm 1}, \beta^{\pm 1}, \alpha^{\pm 1}\beta^{\pm 1} \right\}.$$

Then $\mathcal{O}_{\alpha,\beta}$ *is either isomorphic or antiisomorphic to* $\mathcal{O}_{\gamma,\delta}$ *if and only if* $S_{\alpha,\beta} = S_{\gamma,\delta}$.

Triality



The simple Lie algebra of type D_4 contains outer automorphisms of order 3.

Symmetric composition algebras and triality

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra. Write

$$L_x(y) = x * y = R_y(x).$$

$$L_x R_x = n(x)\mathrm{id} = R_x L_x \quad \Longrightarrow \quad \begin{pmatrix} 0 & L_x \\ R_x & 0 \end{pmatrix}^2 = n(x)\mathrm{id}$$

Therefore, the map $x \mapsto \begin{pmatrix} 0 & L_x \\ R_x & 0 \end{pmatrix}$ extends to an isomorphism of algebras with involution

$$\Phi : \big(\mathfrak{Cl}(\mathcal{C}, n), \tau\big) \longrightarrow \big(\mathrm{End}(\mathcal{C} \oplus \mathcal{C}), \sigma_{n \perp n}\big)$$

Spin group

Consider the *spin group*:

$$\mathrm{Spin}(\mathcal{C}, n) = \{u \in \mathfrak{Cl}(\mathcal{C}, n)^{\times}_{\bar{0}} : u \cdot x \cdot u^{-1} \in \mathcal{C},\ u \cdot \tau(u) = 1,\ \forall x \in \mathcal{C}\}.$$

For any $u \in \mathrm{Spin}(\mathcal{C}, n)$,

$$\Phi(u) = \begin{pmatrix} \rho_u^- & 0 \\ 0 & \rho_u^+ \end{pmatrix}$$

for some $\rho_u^{\pm} \in \mathrm{O}(\mathcal{C}, n)$ such that

$$\chi_u(x * y) = \rho_u^+(x) * \rho_u^-(y)$$

for any $x, y \in \mathcal{C}$, where $\chi_u(x) = u \cdot x \cdot u^{-1}$.

The natural and the two half-spin representations are linked!

Spin group

This last condition is equivalent to:

$$\langle \chi_u(x), \rho_u^+(y), \rho_u^-(z) \rangle = \langle x, y, z \rangle$$

for any $x, y, z \in \mathcal{C}$, where

$$\langle x, y, z \rangle = n(x, y * z),$$

and this has cyclic symmetry!!

$$\langle x, y, z \rangle = \langle y, z, x \rangle.$$

Spin group

Theorem

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra. Then:

$$\mathrm{Spin}(\mathcal{C}, n) \simeq \{(f_0, f_1, f_2) \in \mathrm{O}^+(\mathcal{C}, n)^3 : \\ f_0(x * y) = f_1(x) * f_2(y) \ \forall x, y \in \mathcal{C}\}.$$

Moreover, the set of related triples *(the set on the right hand side) has cyclic symmetry.*

The cyclic symmetry on the right hand side induces an outer automorphism of order 3 (*trialitarian automorphism*) of $\mathrm{Spin}(\mathcal{C}, n)$.

The Principle of Triality

Theorem

Let $(\mathcal{C}, *, n)$ be an eight-dimensional symmetric composition algebra. Then, for any $f_0 \in \mathrm{O}^+(\mathcal{C}, n)$, there are elements $f_1, f_2 \in \mathrm{O}^+(\mathcal{C}, n)$, unique up to scalar multiplication of both by -1, such that (f_0, f_1, f_2) is a related triple.

Remark

All this is functorial, and we get three exact sequences

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \mathbf{Spin}(\mathcal{C}, n) \longrightarrow \mathbf{O}^+(\mathcal{C}, n) \longrightarrow 1.$$

Trialitarian automorphisms

Theorem (Chernousov,Knus,Tignol,E. 2012-2015)

- *A simply connected simple group of type* $^1D_4$ *admits trialitarian automorphisms if and only if it is isomorphic to* $\mathbf{Spin}(n)$ *for a 3-fold Pfister form; i.e., the norm of an eight-dimensional composition algebra.*
- *The set of conjugacy classes of these automorphisms is in one-to-one correspondence with the set of isomorphism classes of symmetric composition algebras with norm* n*.*
- *The groups of type* $^2D_4$ *and* $^6D_4$ *do not admit trialitarian automorphisms.*
- *The trialitarian automorphisms of the groups of type* $^3D_4$ *are related too to symmetric composition algebras.*

Thank you!