Exceptional numbers

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$$SO(2) \simeq \{z \in \mathbb{C} : |z| = 1\} \simeq S^1$$

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Problem: ij?, ji?

After years of trying hard, he found a solution on October 16, 1843.

A spark flashed forth

Letter of Sir W. R. Hamilton to his son Rev. Archibald H. Hamilton, dated August 5, 1865:

MY DEAR ARCHIBALD -
(1) I had been wishing for an occasion of corresponding a little with you on QUATERNIONS: and such now presents itself, by your mentioning in your note of yesterday, received this morning, that you "have been reflecting on several points connected with them" (the quaternions), "particularly on the Multiplication of Vectors."

(2) No more important, or indeed fundamental question, in the whole Theory of Quaternions, can be proposed than that which thus inquires What is such MULTIPLICATION? What are its Rules, its Objects, its Results? What Analogies exist between it and other Operations, which have received the same general Name? And finally, what is (if any) its Utility?

A spark flashed forth

(3) If I may be allowed to speak of myself in connexion with the subject, I might do so in a way which would bring you in, by referring to an ante-quaternionic time, when you were a mere child, but had caught from me the conception of a Vector, as represented by a Triplet: and indeed I happen to be able to put the finger of memory upon the year and month - October, 1843 - when having recently returned from visits to Cork and Parsonstown, connected with a meeting of the British Association, the desire to discover the laws of the multiplication referred to regained with me a certain strength and earnestness, which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above-cited month, on my coming down to breakfast, your (then) little brother William Edwin, and yourself, used to ask me, "Well, Papa, can you multiply triplets"? Whereto I was always obliged to reply, with a sad shake of the head: "No, I can only add and subtract them."

A spark flashed forth

(4) But on the 16th day of the same month - which happened to be a Monday, and a Council day of the Royal Irish Academy - I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery.

A spark flashed forth

Nor could I resist the impulse -unphilosophical as it may have been- to cut with a knife on a stone of Brougham Bridge[1], as we passed it, the fundamental formula with the symbols, i, j, k; namely,

$$i^2 = j^2 = k^2 = ijk = -1$$

which contains the Solution of the Problem, but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (October 16th, 1843), which records the fact, that I then asked for and obtained leave to read a Paper on Quaternions, at the First General Meeting of the session: which reading took place accordingly, on Monday the 13th of the November following.
With this quaternion of paragraphs I close this letter I.; but I hope to follow it up very shortly with another.
Your affectionate father, WILLIAM ROWAN HAMILTON.

[1] The actual name of the bridge is Broome

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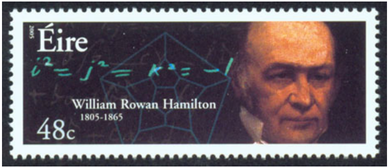

Hamilton and his quaternions

Some properties of $\mathbb{H}$

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- $|q_1 q_2| = |q_1||q_2| \ \forall q_1, q_2 \in \mathbb{H}$
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- $\forall q = a1 + u \in \mathbb{H}$, $q^2 = (a^2 - u \cdot u) + 2au$, so

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  $$\boxed{q^2 - (2a)q + |q|^2 = 0} \qquad (\mathbb{H} \text{ is a quadratic algebra})$$
- The map $q = a + u \mapsto \bar{q} = a - u$ is an involution, with $q + \bar{q} = 2a$ and $q\bar{q} = \bar{q}q = |q|^2$.

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Consider the linear map:

$$\varphi_q : \mathbb{H}_0 \longrightarrow \mathbb{H}_0,$$
$$x \mapsto qxq^{-1} = qx\bar{q}.$$

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&= \big((\cos\alpha)v + (\sin\alpha)u \times v\big)\big((\cos\alpha)1 - (\sin\alpha)u\big) \\
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$$\varphi_q(u \times v) = \ldots = -(\sin 2\alpha)v + (\cos 2\alpha)u \times v.$$

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Therefore the coordinate matrix of φ_q relative to the basis $\{u, v, u \times v\}$ is

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φ_q is the rotation of angle 2α relative to the axis $\mathbb{R}^+ u$.

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(S^3 is the universal cover of $SO(3)$)

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Now it is straightforward to deduce the formulae by Olinde Rodrigues (1840) for the composition of rotations.

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(The isomorphism $SO(3) \simeq PSU(2)$ is deduced too.)

Rotations in $\mathbb{R}^4$

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The multiplications by norm 1 quaternions are rotations of $\mathbb{H} \simeq \mathbb{R}^4$.

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- If ψ is a rotation of $\mathbb{R}^4 \simeq \mathbb{H}$, $a = \psi(1)$ is a norm 1 element in $\mathbb{H}$, so

$$L_{\bar{a}} \circ \psi(1) = \bar{a}a = |a|^2 = 1,$$

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- Therefore, there is a norm 1 element $q \in \mathbb{H}$, such that

$$\bar{a}\psi(x) = qxq^{-1}$$

  for any $x \in \mathbb{H}$. that is,:

$$\psi(x) = (aq)xq^{-1} \quad \forall x \in \mathbb{H}.$$

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$$\Psi : S^3 \times S^3 \longrightarrow SO(4),$$
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It is enough to multiply pairs of norm 1 quaternions!
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Exercise

What kind of rotation is $\psi_{p,q}$ if $p + \bar{p} = 2\cos\alpha$ and $q + \bar{q} = 2\cos\beta$?

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Solution: It is a "double rotation" of angles $\alpha + \beta$ and $\alpha - \beta$.

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If another doubling is performed, there appear the octonions (Graves – Cayley):

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}l.$$

Octonions (1843-1845)

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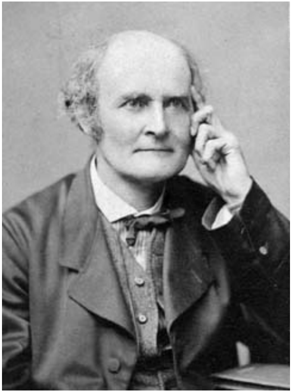

Arthur Cayley

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These are the same formulae that allow to go from $\mathbb{C}$ to $\mathbb{H}$!

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- $\mathbb{O}$ is *quadratic*: $\forall x = a1 + u \in \mathbb{O}$, $x^2 - 2ax + |x|^2 = 0$.

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- *Non-Desarguesian projective plane* $\mathbb{OP}^2$.
- The only spheres which appear as homogeneous spaces of non classical groups are $S^6 = \mathrm{Aut}\,\mathbb{O}/SU(3)$, $S^7 = Spin_7/\,\mathrm{Aut}\,\mathbb{O}$ and $S^{15} = Spin_9/Spin_7$.

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Each such algebra has a unique real *compact form*.

G_2 and F_4

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Cartan (1914)

$$\mathfrak{der}(\mathbb{O}) = \{d \in \mathrm{End}(\mathbb{O}) : d(xy) = d(x)y + xd(y),\ \forall x, y \in \mathbb{O}\}$$

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Chevalley-Schafer (1950)

$\mathfrak{der}\big(\mathfrak{h}_3(\mathbb{O})\big)$ is the compact simple Lie algebra of type F_4.

Freudenthal's Magic Square

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	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	A_1	A_2	C_3	F_4
$\mathbb{C}$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathbb{H}$	C_3	A_5	D_6	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

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H. Freudenthal

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Thanks