Exceptional simple classical Lie superalgebras

1. Extended Freudenthal-Tits magic square.

2. Vector cross products and exceptional simple classical Lie superalgebras.

3. Forms of the exceptional simple classical Lie superalgebras.

1. Extended Freudenthal-Tits magic square.

(joint work with G. Benkart)

Throughout F will denote a field of characteristic $\neq 2, 3$.

Tits construction:

- **C a unital composition algebra over F:**

$$a^2 - \mathfrak{tr}(a)a - \mathfrak{n}(a)1 = 0,$$

$$\mathfrak{n}(ab) = \mathfrak{n}(a)\mathfrak{n}(b),$$

$$D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b] \in \mathrm{Der}(C).$$

- **J a unital Jordan algebra over F with a *normalized trace*:**

$$t(1) = 1, \qquad t\big((J, J, J)\big) = 0,$$

$$xy = t(xy)1 + x * y,$$

$$d_{x,y} = [l_x, l_y] \in \mathrm{Der}(J).$$

$$\boxed{\mathcal{T}(C,J) := D_{C,C} \oplus (C^0 \otimes J^0) \oplus d_{J,J}}$$

with the anticommutative product [,] specified by

- $D_{C,C}$ **and** $d_{J,J}$ **are Lie subalgebras,**
- $[D_{C,C}, d_{J,J}] = 0,$
- $[D, a \otimes x] = D(a) \otimes x, \qquad [d, a \otimes x] = a \otimes d(x),$
- $[a \otimes x, b \otimes y] = t(xy)D_{a,b} + [a,b] \otimes x * y + 2\mathrm{tr}(ab)d_{x,y}.$

1. Extended Freudenthal-Tits magic square.

$\mathcal{T}(C,J)$ is a Lie algebra if and only if

$$\text{(i)}\quad 0 = \sum_{\text{cyclic}} \operatorname{tr}\big([a_1,a_2]a_3\big)d_{(x_1*x_2),x_3},$$

$$\text{(ii)}\quad 0 = \sum_{\text{cyclic}} t\big((x_1*x_2)x_3\big)D_{[a_1,a_2],a_3}$$

$$\text{(iii)}\quad 0 = \sum_{\text{cyclic}} \Big(D_{a_1,a_2}(a_3)\otimes t\big(x_1x_2\big)x_3 + [[a_1,a_2],a_3]\otimes(x_1*x_2)*x_3 + 2\operatorname{tr}\big(a_1a_2\big)a_3\otimes d_{x_1,x_2}(x_3)\Big)$$

In particular, this happens if J satisfies the Cayley-Hamilton equation $ch_3(x)=0$, where

$$ch_3(x) = x^3 - 3t(x)x^2 + \big(\tfrac{9}{2}t(x)^2 - \tfrac{3}{2}t(x^2)\big)x - \Big(t(x^3) - \tfrac{9}{2}t(x^2)t(x) + \tfrac{9}{2}t(x)^3\Big)1$$

1. Extended Freudenthal-Tits magic square.

Replace Jordan algebra by Jordan superalgebra above.

Here, a *normalized trace* satisfies

$$t(1) = 1, \qquad t(J_{\bar{1}}) = 0, \qquad t\big((J, J, J)\big) = 0.$$

The only finite-dimensional simple unital Jordan superalgebras J with $J_{\bar{1}} \neq 0$, over a field of characteristic $\neq 2, 3$, whose Grassmann envelope $G(J)$ satisfies the trace identity $ch_3(x) = 0$, relative a normalized trace on J are:

i) **the Jordan superalgebra $J(V) = F1 \oplus V$ of a supersymmetric bilinear form such that $V = V_{\bar{1}}$ and $\dim V = 2$, and**

i) **$D_2 = (Fe \oplus Ff) \oplus (Fx \oplus Fy)$, with multiplication given by**

$$e^2 = e, \qquad f^2 = f, \qquad ef = 0$$
$$ex = \tfrac{1}{2}x = fx, \quad ey = \tfrac{1}{2}y = fy,$$
$$xy = e + 2f = -yx.$$

Therefore, $\mathcal{T}(C, J(V))$ and $\mathcal{T}(C, D_2)$ are Lie superalgebras.

1. Extended Freudenthal-Tits magic square.

However, consider $\mu \neq 0$ and D_μ the simple Jordan superalgebra $D_\mu = (Fe \oplus Ff) \oplus (Fx \oplus Fy)$, with multiplication given by

$$e^2 = e, \qquad f^2 = f, \qquad ef = 0$$

$$ex = \tfrac{1}{2}x = fx, \qquad ey = \tfrac{1}{2}y = fy$$

$$xy = e + \mu f = -yx.$$

Then

$$C \text{ associative} \implies \mathcal{T}(C, D_\mu) \text{ is a Lie superalgebra}$$

$$\forall \mu \neq 0, -1.$$

1. Extended Freudenthal-Tits magic square.

Freudenthal-Tits Magic Square

$C \setminus J$	F	$H_3(F)$	$H_3(K)$	$H_3(Q)$	$H_3(C)$
F	0	A_1	A_2	C_3	F_4
K	0	A_2	$A_2 \oplus A_2$	A_5	E_6
Q	A_1	C_3	A_5	D_6	E_7
C	G_2	F_4	E_6	E_7	E_8

$C \setminus J$	$J(V)$	D_μ $(\mu \neq 0, -1)$
F	A_1	$B(0,1)$
K	$B(0,1)$	$A(1,0)$
Q	$B(1,1)$	$D(2,1;\mu)$
C	$G(3)$	$F(4)$ $(\mu = 2, 1/2)$

1. Extended Freudenthal-Tits magic square.

2. Vector cross products and exceptional simple classical Lie superalgebras.

(based on joint work with N. Kamiya and S. Okubo)

Vector cross product: $\big(V, \langle\ |\ \rangle\big)$

$$X : V \times \overset{r}{\cdots} \times V \longrightarrow V$$

$$(v_1, \ldots, v_r) \mapsto X(v_1, \ldots, v_r)$$

such that

- $\langle X(v_1, \ldots, v_r) \mid v_{r+1} \rangle$ **is skew-symmetric,**
- $\langle X(v_1, \ldots, v_r) \mid X(v_1, \ldots, v_r) \rangle = \det\big(\langle v_i \mid v_j \rangle\big)$.

Possibilities:

$$\begin{cases} n \text{ even}, \quad r = 1, \\ n \text{ arbitrary}, \quad r = n-1, \\ n = 3, 7, \quad r = 2, \\ n = 4, 8, \quad r = 3. \end{cases}$$

$n = 4,\ r = 3$:

$\Phi : V \times V \times V \times V \longrightarrow F$, nonzero, skew-symmetric multilinear map.

$$\boxed{\langle X(v_1, v_2, v_3) \mid v_4\rangle = \Phi(v_1, v_2, v_3, v_4)}$$

then

$$\langle X(v_1, v_2, v_3) \mid X(w_1, w_2, w_3)\rangle = \mu \det\big(\langle v_i \mid w_j\rangle\big)$$

for some $0 \neq \mu \in F$.

Consider the operators

$$d_{u,v} = X(u, v, -) + \sigma_{u,v}$$

where $\sigma_{u,v}(w) = \langle u \mid w\rangle\, v - \langle v \mid w\rangle\, u$.

$d_{V,V}$ is a Lie algebra

(isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ in the 'split' case)

$$n = 8,\ r = 3:$$

X a 3-fold vector cross product on $\big(V, \langle \mid \rangle\big)$. Then:

$$\langle X(a_1, a_2, a_3) \mid X(b_1, b_2, b_3)\rangle$$

$$= \det\big(\langle a_i \mid b_j \rangle\big)$$

$$+ \epsilon \sum_{\substack{\sigma \\ even}} \sum_{\substack{\tau \\ even}} \Big\langle a_{\sigma(1)} \mid b_{\tau(1)} \Big\rangle \Big\langle a_{\sigma(2)} \mid X\big(a_{\sigma(3)}, b_{\tau(2)}, b_{\tau(3)}\big) \Big\rangle$$

where $\epsilon = \pm 1$.

Consider the operators

$$d_{u,v} = \frac{\epsilon}{3} X(u, v, -) + \sigma_{u,v}$$

$d_{V,V}$ is a Lie algebra

(isomorphic to $\mathfrak{o}_7$ in the 'split' case)

2. Cross products and exceptional simple classical Lie superalgebras.

$n = 7,\ r = 2$:

$u \times v$ a (2-fold) vector cross product on $\big(V, \langle\ |\ \rangle\big)$. Then:

$$(u \times v) \times v = \sigma_{u,v}(v)$$

Consider the operators

$$d_{u,v}(w) = \frac{1}{2}\Big(-(u \times v) \times w + 3\sigma_{u,v}(w)\Big)$$

$d_{V,V}$ is a Lie algebra of type G_2

2. Cross products and exceptional simple classical Lie superalgebras.

Let (U, φ) be a two dimensional vector space U endowed with a nonzero skew-symmetric bilinear form φ. For any $a, b \in U$, let $\varphi_{a,b} = \varphi(a,-)b + \varphi(b,-)a$.

For any of the three classes of vector cross products above consider the superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$, where

- **$\mathfrak{g}_{\bar 0} = \mathfrak{sp}(U, \varphi) \oplus d_{V,V}$,**
- **$\mathfrak{g}_{\bar 1} = U \otimes V$,**

with multiplication given by

* **the usual Lie bracket on $\mathfrak{g}_{\bar 0}$,**
* **the natural action of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$,**
* **$[a \otimes x, b \otimes y] = \langle u \mid v \rangle \varphi_{a,b} + \varphi(a,b) d_{u,v}$.**

$\mathfrak{g}$ is then a Lie superalgebra and

- **$\mathfrak{g}(V_4, \Phi)$ is a form of $D(2,1;\mu)$,**
- **$\mathfrak{g}(V_8, X)$ is a form of $F(4)$,**
- **$\mathfrak{g}(V_7, \times)$ is a form of $G(3)$.**

3. Forms of the exceptional simple classical Lie superalgebras.

$G(3)$: Both the $\mathcal{T}\big(C, J(V)\big)$'s and the $\mathfrak{g}(V_7, \times)$'s exhaust the forms of $G(3)$.

$F(4)$: Both the $\mathcal{T}\big(C, D_2\big)$'s and the $\mathfrak{g}(V_8, X)$'s exhaust the forms of $F(4)$ whose even part contains an ideal isomorphic to $\mathfrak{sl}_2$.

There is another family of forms of $F(4)$ with

$$\mathfrak{g}_{\bar{0}} = [Q, Q] \oplus \mathfrak{o}(W, q), \quad \dim W = 7 \text{ and}$$

$$\text{Clifford invariant of } (W, q) = [Q],$$

$\mathfrak{g}_{\bar{1}}$ is the irreducible module for the

Clifford algebra of (W, q).

The forms $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ of the Lie superalgebras $D(2,1;\mu)$ satisfy:

- **$\mathfrak{g}_{\bar{0}} = Q^0$, where Q is a quaternion algebra over a cubic étale extension L/F, with trivial corestriction $N_{L/F}(Q)$ (isomorphic to $\mathrm{Mat}_8(F)$).**

- **$\mathfrak{g}_{\bar{1}}$ is the irreducible module for $N_{L/F}(Q)$.**

The $\mathcal{T}(Q, D_\mu)$'s correspond to the case where $L = F \times F \times F$ (so $Q = Q_1 \times Q_2 \times Q_3$) and (say) $Q_1 \cong \mathrm{Mat}_2(F)$.

The $\mathfrak{g}(V_4, \Phi)$'s correspond to the case where $L = F \times K$, K/F a quadratic étale extension (so $Q = Q_1 \times Q_2$) and $Q_1 \cong \mathrm{Mat}_2(F)$).

For real forms:

a) $\boldsymbol{G(3)}$ **has, up to isomorphism, two real forms.**

b) $\boldsymbol{F(4)}$ **has, up to isomorphism, four real forms.**

c) **If** $\boldsymbol{\alpha \in \mathbb{C} \setminus \Big(\mathbb{R} \cup \{z \in \mathbb{C} : |z| = 1\} \cup \{z \in \mathbb{C} : |z+1| = 1\} \cup \{z \in \mathbb{C} : z + \bar{z} = -1\}\Big)}$**, then** $\boldsymbol{D(2,1;\alpha)}$ **has no real form.**

d) **If** $\boldsymbol{\alpha \in \mathbb{R} \setminus \{0, -1, 1, -2, -1/2\}}$**, then** $\boldsymbol{D(2,1;\alpha)}$ **has four nonisomorphic real forms.**

e) **If** $\boldsymbol{\alpha = 1, -2}$ **or** $\boldsymbol{-1/2}$**, then** $\boldsymbol{D(2,1;\alpha) = osp(4,2)}$ **has four nonisomorphic real forms.**

f) **If** $\boldsymbol{\alpha \in \{z \in \mathbb{C} : |z| = 1\} \cup \{z \in \mathbb{C} : |z+1| = 1\} \cup \{z \in \mathbb{C} : z + \bar{z} = -1\}}$**, then** $\boldsymbol{D(2,1;\alpha)}$ **has exactly, up to isomorphism, one real form.**

Comments and summary:

1) The Freudenthal-Tits square can be extended to include (as the 'Jordan ingredient') the Jordan superalgebras D_μ and $J(V)$. The split exceptional classical simple Lie superalgebras appear then in the 'rectangle'.

2) Any Lie superalgebra $\mathfrak{g}$ with $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}_2 \oplus \mathfrak{a}$ and $\mathfrak{g}_{\bar{1}} = U \otimes V$ is determined by $\bigl(V, \langle\ |\ \rangle\bigr)$ and $d : V \times V \to \mathrm{End}(V)$ $(u, v) \mapsto d_{u,v}$, satisfying certain conditions (which, with a minor modification, define the structure of $(-1,-1)$–balanced Freudenthal-Kantor triple system). These are satisfied in particular for the vector cross products, which induce exceptional simple classical Lie superalgebras.

3) Many, but not all, of the forms of the exceptional simple classical Lie superalgebras are obtained by the previous constructions.