Some simple modular Lie superalgebras

Alberto Elduque

Universidad de Zaragoza

Freudenthal-Tits Magic Square

A supermagic rectangle

A supermagic square

Bouarroudj-Grozman-Leites Classification

Exceptional Lie algebras

Exceptional Lie algebras

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

Exceptional Lie algebras

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

$G_2 = \mathfrak{der}\,\mathbb{O}$ (Cartan 1914)

$F_4 = \mathfrak{der}\,H_3(\mathbb{O})$

$E_6 = \mathfrak{str}_0\,H_3(\mathbb{O})$ (Chevalley-Schafer 1950)

Tits construction (1966)

Tits construction (1966)

- C a Hurwitz algebra (unital composition algebra),
- J a central simple Jordan algebra of degree 3,

Tits construction (1966)

- C a Hurwitz algebra (unital composition algebra),
- J a central simple Jordan algebra of degree 3,

then

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

is a Lie algebra (char $\neq 2, 3$) under a suitable Lie bracket:

$$[a \otimes x, b \otimes y] = \frac{1}{3} tr(xy) D_{a,b} + \Big([a, b] \otimes \big(xy - \frac{1}{3} tr(xy)1\big)\Big) + 2t(ab) d_{x,y}.$$

Freudenthal-Tits Magic Square

$\mathcal{T}(C, J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$
k	A_1	A_2	C_3	F_4
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7
$C(k)$	F_4	E_6	E_7	E_8

Tits construction rearranged

Tits construction rearranged

$$J = H_3(C') \simeq k^3 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$J_0 \simeq k^2 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$\mathfrak{der}\, J \simeq \mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

Tits construction rearranged

$$
\begin{aligned}
J &= H_3(C') \simeq k^3 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right), \\
J_0 &\simeq k^2 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right), \\
\mathfrak{der}\, J &\simeq \mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}(C, J) &= \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\
&\simeq \mathfrak{der}\, C \oplus (C_0 \otimes k^2) \oplus \left(\oplus_{i=0}^{2} C_0 \otimes \iota_i(C')\right) \oplus \left(\mathfrak{tri}(C') \oplus (\oplus_{i=0}^{2} \iota_i(C'))\right) \\
&\simeq \left(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(C \otimes C')\right)
\end{aligned}
$$

Tits construction rearranged

$$J = H_3(C') \simeq k^3 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$
$$J_0 \simeq k^2 \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$
$$\mathfrak{der}\, J \simeq \mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$\begin{aligned} \mathcal{T}(C, J) &= \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq \mathfrak{der}\, C \oplus (C_0 \otimes k^2) \oplus \left(\oplus_{i=0}^{2} C_0 \otimes \iota_i(C')\right) \oplus \left(\mathfrak{tri}(C') \oplus (\oplus_{i=0}^{2} \iota_i(C'))\right) \\ &\simeq \left(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(C \otimes C')\right) \end{aligned}$$

$$\mathfrak{tri}(C) = \{(d_0, d_1, d_2) \in \mathfrak{so}(C)^3 : d_0(x \bullet y) = d_1(x) \bullet y + x \bullet d_2(y)\ \forall x, y\}$$

$(x \bullet y = \bar{x}\bar{y})$ is the triality Lie algebra of C.

Tits construction rearranged

Tits construction rearranged

The product in

$$\mathfrak{g}(C, C') = \big(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(C \otimes C')\big),$$

is given by:

Tits construction rearranged

The product in

$$\mathfrak{g}(C, C') = \big(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(C \otimes C')\big),$$

is given by:

- $\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')$ is a Lie subalgebra of $\mathfrak{g}(C, C')$,
- $[(d_0, d_1, d_2), \iota_i(x \otimes x')] = \iota_i\big(d_i(x) \otimes x'\big)$,
- $[(d_0', d_1', d_2'), \iota_i(x \otimes x')] = \iota_i\big(x \otimes d_i'(x')\big)$,
- $[\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y')] = \iota_{i+2}\big((\bar{x}\bar{y}) \otimes (\bar{x}'\bar{y}')\big)$ (indices modulo 3),
- $[\iota_i(x \otimes x'), \iota_i(y \otimes y')] = q'(x', y')\theta^i(t_{x,y}) + q(x, y)\theta'^i(t'_{x',y'})$,

  where
  $t_{x,y} = \big(q(x, .)y - q(y, .)x, \frac{1}{2}q(x, y)1 - R_{\bar{x}}R_y, \frac{1}{2}q(x, y)1 - L_{\bar{x}}L_y\big)$

(Vinberg, Allison-Faulkner, Barton-Sudbery, Landsberg-Manivel)

Freudenthal-Tits Magic Square (char 3)

Freudenthal-Tits Magic Square (char 3)

This more symmetric construction is valid too in characteristic 3:

Freudenthal-Tits Magic Square (char 3)

This more symmetric construction is valid too in characteristic 3:

		dim C'			
	$\mathfrak{g}(C, C')$	1	2	4	8
	1	A_1	$\tilde{A}_2$	C_3	F_4
	2	$\tilde{A}_2$	$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$
dim C	4	C_3	$\tilde{A}_5$	D_6	E_7
	8	F_4	$\tilde{E}_6$	E_7	E_8

Freudenthal-Tits Magic Square (char 3)

This more symmetric construction is valid too in characteristic 3:

dim C'

	$\mathfrak{g}(C, C')$	1	2	4	8
	1	A_1	$\tilde{A}_2$	C_3	F_4
	2	$\tilde{A}_2$	$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$
dim C	4	C_3	$\tilde{A}_5$	D_6	E_7
	8	F_4	$\tilde{E}_6$	E_7	E_8

- $\tilde{A}_2$ denotes a form of $\mathfrak{pgl}_3$, so $[\tilde{A}_2, \tilde{A}_2]$ is a form of $\mathfrak{psl}_3$.
- $\tilde{A}_5$ denotes a form of $\mathfrak{pgl}_6$, so $[\tilde{A}_5, \tilde{A}_5]$ is a form of $\mathfrak{psl}_6$.
- $\tilde{E}_6$ is not simple, but $[\tilde{E}_6, \tilde{E}_6]$ is a codimension 1 simple ideal.

Freudenthal-Tits Magic Square

A supermagic rectangle

A supermagic square

Bouarroudj-Grozman-Leites Classification

A look at the rows of Tits construction

A look at the rows of Tits construction

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

(C a composition algebra, J a suitable Jordan algebra.)

A look at the rows of Tits construction

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

(C a composition algebra, J a suitable Jordan algebra.)

First row $\quad \dim C = 1, \qquad \mathcal{T}(C, J) = \mathfrak{der}\, J.$

A look at the rows of Tits construction

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

(C a composition algebra, J a suitable Jordan algebra.)

First row $\quad \dim C = 1, \quad \mathcal{T}(C, J) = \mathfrak{der}\, J.$

Second row $\quad \dim C = 2, \quad \mathcal{T}(C, J) \simeq J_0 \oplus \mathfrak{der}\, J \simeq \mathfrak{str}_0(J).$

A look at the rows of Tits construction

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

(C a composition algebra, J a suitable Jordan algebra.)

First row dim $C = 1$, $\mathcal{T}(C, J) = \mathfrak{der}\, J$.

Second row dim $C = 2$, $\mathcal{T}(C, J) \simeq J_0 \oplus \mathfrak{der}\, J \simeq \mathfrak{str}_0(J)$.

Third row dim $C = 4$, so $C = Q$ is a quaternion algebra and

$$\begin{aligned} \mathcal{T}(C, J) &= \mathrm{ad}_{Q_0} \oplus (Q_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq (Q_0 \otimes J) \oplus \mathfrak{der}\, J. \end{aligned}$$

(This is Tits version (1962) of the Tits-Kantor-Koecher construction $\mathcal{TKK}(J)$)

A look at the rows of Tits construction

$$\mathcal{T}(C,J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

(C a composition algebra, J a suitable Jordan algebra.)

First row $\dim C = 1$, $\mathcal{T}(C,J) = \mathfrak{der}\, J$.

Second row $\dim C = 2$, $\mathcal{T}(C,J) \simeq J_0 \oplus \mathfrak{der}\, J \simeq \mathfrak{str}_0(J)$.

Third row $\dim C = 4$, so $C = Q$ is a quaternion algebra and

$$\begin{aligned}\mathcal{T}(C,J) &= \mathrm{ad}_{Q_0} \oplus (Q_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq (Q_0 \otimes J) \oplus \mathfrak{der}\, J.\end{aligned}$$

(This is Tits version (1962) of the Tits-Kantor-Koecher construction $\mathcal{TKK}(J)$)

A look at the rows of Tits construction

$$\mathcal{T}(C,J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

(C a composition algebra, J a suitable Jordan algebra.)

First row $\dim C = 1$, $\mathcal{T}(C,J) = \mathfrak{der}\, J$.

Second row $\dim C = 2$, $\mathcal{T}(C,J) \simeq J_0 \oplus \mathfrak{der}\, J \simeq \mathfrak{str}_0(J)$.

Third row $\dim C = 4$, so $C = Q$ is a quaternion algebra and

$$\begin{aligned}\mathcal{T}(C,J) &= \mathrm{ad}_{Q_0} \oplus (Q_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq (Q_0 \otimes J) \oplus \mathfrak{der}\, J.\end{aligned}$$

(This is Tits version (1962) of the Tits-Kantor-Koecher construction $\mathcal{TKK}(J)$)

Up to now, everything works for arbitrary Jordan algebras in characteristic $\neq 2$, and even for Jordan superalgebras.

Fourth row

Fourth row

Fourth row

$\dim C = 8$. If the characteristic is $\neq 2, 3$, then $\mathfrak{Der}\, C = \mathfrak{g}_2$ is simple of type G_2, C_0 is its smallest nontrivial irreducible module, and

$$\mathcal{T}(C, J) = \mathfrak{g}_2 \oplus (C_0 \otimes J_0) \oplus \mathfrak{Der}\, J$$

is a G_2-*graded* Lie algebra. Essentially, all the G_2-graded Lie algebras appear in this way [Benkart-Zelmanov 1996].

Fourth row

Fourth row — $\dim C = 8$. If the characteristic is $\neq 2, 3$, then $\mathfrak{Der}\, C = \mathfrak{g}_2$ is simple of type G_2, C_0 is its smallest nontrivial irreducible module, and

$$\mathcal{T}(C, J) = \mathfrak{g}_2 \oplus (C_0 \otimes J_0) \oplus \mathfrak{Der}\, J$$

is a G_2-*graded* Lie algebra. Essentially, all the G_2-graded Lie algebras appear in this way [Benkart-Zelmanov 1996].

It makes sense to consider Jordan superalgebras, as long as their Grassmann envelopes satisfy the Cayley-Hamilton equation of degree 3.

Fourth row

Fourth row dim $C = 8$. If the characteristic is $\neq 2, 3$, then $\mathfrak{der}\, C = \mathfrak{g}_2$ is simple of type G_2, C_0 is its smallest nontrivial irreducible module, and

$$\mathcal{T}(C, J) = \mathfrak{g}_2 \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

is a G_2-*graded* Lie algebra. Essentially, all the G_2-graded Lie algebras appear in this way [Benkart-Zelmanov 1996].

It makes sense to consider Jordan superalgebras, as long as their Grassmann envelopes satisfy the Cayley-Hamilton equation of degree 3.

- $\mathcal{T}(C, J(V)) \simeq G(3) \quad (\dim V = 2,\ V = V_{\bar{1}})$.

Fourth row

Fourth row dim $C = 8$. If the characteristic is $\neq 2, 3$, then $\mathfrak{Der}\, C = \mathfrak{g}_2$ is simple of type G_2, C_0 is its smallest nontrivial irreducible module, and

$$\mathcal{T}(C, J) = \mathfrak{g}_2 \oplus (C_0 \otimes J_0) \oplus \mathfrak{Der}\, J$$

is a G_2-*graded* Lie algebra. Essentially, all the G_2-graded Lie algebras appear in this way [Benkart-Zelmanov 1996].

It makes sense to consider Jordan superalgebras, as long as their Grassmann envelopes satisfy the Cayley-Hamilton equation of degree 3.

- $\mathcal{T}(C, J(V)) \simeq G(3) \quad (\dim V = 2,\ V = V_{\bar{1}})$.
- $\mathcal{T}(C, D_2) \simeq F(4)$.

Fourth row

Fourth row dim $C = 8$. If the characteristic is $\neq 2, 3$, then $\mathfrak{der}\, C = \mathfrak{g}_2$ is simple of type G_2, C_0 is its smallest nontrivial irreducible module, and

$$\mathcal{T}(C, J) = \mathfrak{g}_2 \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

is a G_2-*graded* Lie algebra. Essentially, all the G_2-graded Lie algebras appear in this way [Benkart-Zelmanov 1996].

It makes sense to consider Jordan superalgebras, as long as their Grassmann envelopes satisfy the Cayley-Hamilton equation of degree 3.

- $\mathcal{T}(C, J(V)) \simeq G(3) \quad (\dim V = 2,\ V = V_{\bar{1}})$.
- $\mathcal{T}(C, D_2) \simeq F(4)$.
- $\mathcal{T}(C, K_{10})$ in characteristic 5!!
  This is a new simple modular Lie superalgebra, whose even part is $\mathfrak{so}_{11}$ and odd part its spin module.

A supermagic rectangle

$\mathcal{T}(C, J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$	$J(V)$	D_t	K_{10}
k	A_1	A_2	C_3	F_4	A_1	$B(0,1)$	$B(0,1) \oplus B(0,1)$
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6	$B(0,1)$	$A(1,0)$	$C(3)$
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7	$B(1,1)$	$D(2,1;t)$	$F(4)$
$C(k)$	F_4	E_6	E_7	E_8	$G(3)$	$F(4)$ $(t = 2)$	$\mathcal{T}(C(k), K_{10})$ (char 5)

A supermagic rectangle: the new columns

$\mathcal{T}(C,J)$	$J(V)$	D_t	K_{10}
k	A_1	$B(0,1)$	$B(0,1) \oplus B(0,1)$
$k \times k$	$B(0,1)$	$A(1,0)$	$C(3)$
$\mathrm{Mat}_2(k)$	$B(1,1)$	$D(2,1;t)$	$F(4)$
$C(k)$	$G(3)$	$F(4)$ $(t=2)$	$\mathcal{T}(C(k),K_{10})$ (char 5)

Fourth row, characteristic 3

Fourth row, characteristic 3

If the characteristic is 3 and $\dim C = 8$, then $\mathfrak{der}\, C$ is no longer simple, but contains the simple ideal $\operatorname{ad} C_0$ (a form of $\mathfrak{psl}_3$). It makes sense to consider:

$$\begin{aligned} \tilde{\mathcal{T}}(C, J) &= \operatorname{ad} C_0 \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq (C_0 \otimes J) \oplus \mathfrak{der}\, J. \end{aligned}$$

Fourth row, characteristic 3

If the characteristic is 3 and $\dim C = 8$, then $\mathfrak{der}\, C$ is no longer simple, but contains the simple ideal $\operatorname{ad} C_0$ (a form of $\mathfrak{psl}_3$). It makes sense to consider:

$$\begin{aligned}\tilde{\mathcal{T}}(C, J) &= \operatorname{ad} C_0 \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq (C_0 \otimes J) \oplus \mathfrak{der}\, J.\end{aligned}$$

$\tilde{\mathcal{T}}(C, J)$ becomes a Lie algebra if and only if J is a commutative and alternative algebra (these conditions imply the Jordan identity).

Fourth row, characteristic 3

If the characteristic is 3 and $\dim C = 8$, then $\mathfrak{Der}\, C$ is no longer simple, but contains the simple ideal $\mathrm{ad}\, C_0$ (a form of $\mathfrak{psl}_3$). It makes sense to consider:

$$\begin{aligned}\tilde{\mathcal{T}}(C, J) &= \mathrm{ad}\, C_0 \oplus (C_0 \otimes J_0) \oplus \mathfrak{Der}\, J \\ &\simeq (C_0 \otimes J) \oplus \mathfrak{Der}\, J.\end{aligned}$$

$\tilde{\mathcal{T}}(C, J)$ becomes a Lie algebra if and only if J is a commutative and alternative algebra (these conditions imply the Jordan identity).

The simple commutative alternative algebras are just the fields, so nothing interesting appears here.

Fourth “superrow”, characteristic 3

Fourth "superrow", characteristic 3

But there are simple commutative alternative superalgebras [Shestakov 1997] (characteristic 3):

Fourth "superrow", characteristic 3

But there are simple commutative alternative superalgebras [Shestakov 1997] (characteristic 3):

(i) fields,

Fourth "superrow", characteristic 3

But there are simple commutative alternative superalgebras [Shestakov 1997] (characteristic 3):

(i) fields,

(ii) $J(V)$, the Jordan superalgebra of a superform on a two dimensional odd space V,

Fourth "superrow", characteristic 3

But there are simple commutative alternative superalgebras [Shestakov 1997] (characteristic 3):

(i) fields,

(ii) $J(V)$, the Jordan superalgebra of a superform on a two dimensional odd space V,

(iii) $B = B(\Gamma, D, 0) = \Gamma \oplus \Gamma u$, where

- Γ is a commutative associative algebra,
- $D \in \mathfrak{Der}\,\Gamma$ such that Γ is D-simple,
- $a(bu) = (ab)u = (au)b, \quad (au)(bu) = aD(b) - D(a)b,$ $\forall a, b \in \Gamma.$

Fourth "superrow", characteristic 3

Example (Divided powers)

$\Gamma = \mathcal{O}(1; n) = \operatorname{span}\left\{ t^{(r)} : 0 \leq r \leq 3^n - 1 \right\}$,

$$t^{(r)} t^{(s)} = \binom{r+s}{r} t^{(r+s)},$$

$$D(t^{(r)}) = t^{(r-1)}.$$

Bouarroudj-Leites superalgebras

Bouarroudj-Leites superalgebras

Over an algebraically closed field of characteristic 3:

Bouarroudj-Leites superalgebras

Over an algebraically closed field of characteristic 3:

- $\tilde{\mathcal{T}}\big(C(k), J(V)\big)$ is a simple Lie superalgebra specific of characteristic 3 of (super)dimension $10|14$,

Bouarroudj-Leites superalgebras

Over an algebraically closed field of characteristic 3:

- $\tilde{\mathcal{T}}\big(C(k), J(V)\big)$ is a simple Lie superalgebra specific of characteristic 3 of (super)dimension $10|14$,
- $\tilde{\mathcal{T}}\big(C(k), \mathcal{O}(1,n) \oplus \mathcal{O}(1,n)u\big) = \mathrm{Bj}(1;n|7)$ is a simple Lie superalgebra of (super)dimension $2^3 \times 3^n|2^3 \times 3^n$.

Bouarroudj-Leites superalgebras

Over an algebraically closed field of characteristic 3:

- $\tilde{\mathcal{T}}\big(C(k), J(V)\big)$ is a simple Lie superalgebra specific of characteristic 3 of (super)dimension $10|14$,
- $\tilde{\mathcal{T}}\big(C(k), \mathcal{O}(1,n) \oplus \mathcal{O}(1,n)u\big) = \mathrm{Bj}(1; n|7)$ is a simple Lie superalgebra of (super)dimension $2^3 \times 3^n | 2^3 \times 3^n$.

Bouarroudj-Leites superalgebras

Over an algebraically closed field of characteristic 3:

- $\tilde{\mathcal{T}}\big(C(k), J(V)\big)$ is a simple Lie superalgebra specific of characteristic 3 of (super)dimension $10|14$,
- $\tilde{\mathcal{T}}\big(C(k), \mathcal{O}(1,n) \oplus \mathcal{O}(1,n)u\big) = \mathrm{Bj}(1; n|7)$ is a simple Lie superalgebra of (super)dimension $2^3 \times 3^n | 2^3 \times 3^n$.

Both simple Lie superalgebras have been considered in a completely different way by Bouarroudj and Leites (2006).

Freudenthal-Tits Magic Square

A supermagic rectangle

A supermagic square

Bouarroudj-Grozman-Leites Classification

Composition superalgebras

Composition superalgebras

Definition

A superalgebra $C = C_{\bar{0}} \oplus C_{\bar{1}}$, endowed with a regular quadratic superform $q = (q_{\bar{0}}, b)$, called the *norm*, is said to be a *composition superalgebra* in case

$$q_{\bar{0}}(x_{\bar{0}}y_{\bar{0}}) = q_{\bar{0}}(x_{\bar{0}})q_{\bar{0}}(y_{\bar{0}}),$$

$$b(x_{\bar{0}}y, x_{\bar{0}}z) = q_{\bar{0}}(x_{\bar{0}})b(y,z) = b(yx_{\bar{0}}, zx_{\bar{0}}),$$

$$b(xy, zt) + (-1)^{|x||y|+|x||z|+|y||z|}b(zy, xt) = (-1)^{|y||z|}b(x,z)b(y,t),$$

The unital composition superalgebras are termed *Hurwitz superalgebras*.

Composition superalgebras: examples (Shestakov)

Composition superalgebras: examples (Shestakov)

$$B(1,2) = k1 \oplus V,$$

char $k = 3$, V a two dim'l vector space with a nonzero alternating bilinear form $\langle .|. \rangle$, with

$$1x = x1 = x, \quad uv = \langle u|v \rangle 1, \qquad q_{\bar{0}}(1) = 1, \quad b(u,v) = \langle u|v \rangle,$$

is a Hurwitz superalgebra. (As a superalgebra, this is just our previous $J(V)$.)

Composition superalgebras: examples (Shestakov)

Composition superalgebras: examples (Shestakov)

$$B(4,2) = \mathrm{End}_k(V) \oplus V,$$

k and V as before, $\mathrm{End}_k(V)$ is equipped with the symplectic involution $f \mapsto \bar{f}$, $(\langle f(u)|v\rangle = \langle u|\bar{f}(v)\rangle)$, the multiplication is given by:

- the usual multiplication (composition of maps) in $\mathrm{End}_k(V)$,
- $v \cdot f = f(v) = \bar{f} \cdot v$ for any $f \in \mathrm{End}_k(V)$ and $v \in V$,
- $u \cdot v = \langle .|u\rangle v \ \big(w \mapsto \langle w|u\rangle v\big) \in \mathrm{End}_k(V)$ for any $u, v \in V$,

and with quadratic superform

$$q_{\bar{0}}(f) = \det f, \quad b(u,v) = \langle u|v\rangle,$$

is a Hurwitz superalgebra.

Composition superalgebras: classification

Composition superalgebras: classification

Theorem (E.-Okubo 2002)

Any unital composition superalgebra is either:

- *a Hurwitz algebra,*
- *a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,*
- *isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.*

Composition superalgebras: classification

Theorem (E.-Okubo 2002)

Any unital composition superalgebra is either:

- *a Hurwitz algebra,*
- *a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic 2,*
- *isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic 3.*

Hurwitz superalgebras can be plugged into the symmetric construction of Freudenthal-Tits Magic Square $\mathfrak{g}(C, C')$.

Supermagic Square (char 3, Cunha-E. 2007)

$\mathfrak{g}(C, C')$	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$	$B(1,2)$	$B(4,2)$
k	A_1	$\tilde{A}_2$	C_3	F_4	$6\|8$	$21\|14$
$k \times k$		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	$11\|14$	$35\|20$
$\mathrm{Mat}_2(k)$			D_6	E_7	$24\|26$	$66\|32$
$C(k)$				E_8	$55\|50$	$133\|56$
$B(1,2)$					$21\|16$	$36\|40$
$B(4,2)$						$78\|64$

Supermagic Square (char 3, Cunha-E. 2007)

$\mathfrak{g}(C, C')$	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$	$B(1,2)$	$B(4,2)$
k	A_1	$\tilde{A}_2$	C_3	F_4	$6\|8$	$21\|14$
$k \times k$		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	$11\|14$	$35\|20$
$\mathrm{Mat}_2(k)$			D_6	E_7	$24\|26$	$66\|32$
$C(k)$				E_8	$55\|50$	$133\|56$
$B(1,2)$					$21\|16$	$36\|40$
$B(4,2)$						$78\|64$

Notation: $\mathfrak{g}(n, m)$ will denote the superalgebra $\mathfrak{g}(C, C')$, with $\dim C = n$, $\dim C' = m$.

Lie superalgebras in the Supermagic Square

	$B(1,2)$	$B(4,2)$
k	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
$k \times k$	$\left(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3\right) \oplus \left((2) \otimes \mathfrak{psl}_3\right)$	$\mathfrak{pgl}_6 \oplus (20)$
$\mathrm{Mat}_2(k)$	$\left(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6\right) \oplus \left((2) \otimes (13)\right)$	$\mathfrak{so}_{12} \oplus spin_{12}$
$C(k)$	$\left(\mathfrak{sl}_2 \oplus \mathfrak{f}_4\right) \oplus \left((2) \otimes (25)\right)$	$\mathfrak{e}_7 \oplus (56)$
$B(1,2)$	$\mathfrak{so}_7 \oplus 2spin_7$	$\mathfrak{sp}_8 \oplus (40)$
$B(4,2)$	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus spin_{13}$

Lie superalgebras in the Supermagic Square

Lie superalgebras in the Supermagic Square

All these Lie superalgebras are simple, with the exception of $\mathfrak{g}(2,3)$ and $\mathfrak{g}(2,6)$, both of which contain a codimension one simple ideal.

Lie superalgebras in the Supermagic Square

All these Lie superalgebras are simple, with the exception of $\mathfrak{g}(2,3)$ and $\mathfrak{g}(2,6)$, both of which contain a codimension one simple ideal.

Only $\mathfrak{g}(1,3) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac's classification in characteristic 0. The other Lie superalgebras in the Supermagic Square, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

Lie superalgebras in the Supermagic Square

All these Lie superalgebras are simple, with the exception of $\mathfrak{g}(2,3)$ and $\mathfrak{g}(2,6)$, both of which contain a codimension one simple ideal.

Only $\mathfrak{g}(1,3) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac's classification in characteristic 0. The other Lie superalgebras in the Supermagic Square, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

The simple Lie superalgebra $\mathfrak{g}(2,3)' = [\mathfrak{g}(2,3), \mathfrak{g}(2,3)]$ is isomorphic to our previous $\tilde{\mathcal{T}}(C(k), J(V))$.

The Supermagic Square and Jordan algebras

The Supermagic Square and Jordan algebras

	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$
$B(1,2)$	$\mathfrak{psl}_{2,2}$	$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus ((2) \otimes \mathfrak{psl}_3)$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$(\mathfrak{sl}_2 \oplus \mathfrak{f}_4) \oplus ((2) \otimes (25))$
$B(4,2)$	$\mathfrak{sp}_6 \oplus (14)$	$\mathfrak{pgl}_6 \oplus (20)$	$\mathfrak{so}_{12} \oplus spin_{12}$	$\mathfrak{e}_7 \oplus (56)$

The Supermagic Square and Jordan algebras

	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$
$B(1,2)$	$\mathfrak{psl}_{2,2}$	$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus ((2) \otimes \mathfrak{psl}_3)$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$(\mathfrak{sl}_2 \oplus \mathfrak{f}_4) \oplus ((2) \otimes (25))$
$B(4,2)$	$\mathfrak{sp}_6 \oplus (14)$	$\mathfrak{pgl}_6 \oplus (20)$	$\mathfrak{so}_{12} \oplus spin_{12}$	$\mathfrak{e}_7 \oplus (56)$

$$\mathfrak{g}(3,r) = \big(\mathfrak{sl}_2 \oplus \mathfrak{der}\, J\big) \oplus \big((2) \otimes \hat{J}\big),$$

$$r = 1,2,4,8, \qquad \hat{J} = J_0/k1, \quad J = H_3(C).$$

The Supermagic Square and Jordan algebras

	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$
$B(1,2)$	$\mathfrak{psl}_{2,2}$	$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus ((2) \otimes \mathfrak{psl}_3)$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$(\mathfrak{sl}_2 \oplus \mathfrak{f}_4) \oplus ((2) \otimes (25))$
$B(4,2)$	$\mathfrak{sp}_6 \oplus (14)$	$\mathfrak{pgl}_6 \oplus (20)$	$\mathfrak{so}_{12} \oplus spin_{12}$	$\mathfrak{e}_7 \oplus (56)$

$$\mathfrak{g}(3,r) = \big(\mathfrak{sl}_2 \oplus \mathfrak{der}\, J\big) \oplus \big((2) \otimes \hat{J}\big),$$
$$r = 1,2,4,8, \qquad \hat{J} = J_0/k1, \quad J = H_3(C).$$

$$\mathfrak{g}(6,r) = (\mathfrak{der}\, T) \oplus T,$$
$$r = 1,2,4,8, \qquad T = \begin{pmatrix} k & J \\ J & k \end{pmatrix}, \quad J = H_3(C).$$

Freudenthal-Tits Magic Square

A supermagic rectangle

A supermagic square

Bouarroudj-Grozman-Leites Classification

Simple modular Lie superalgebras with a Cartan matrix

Simple modular Lie superalgebras with a Cartan matrix

The finite dimensional modular Lie superalgebras with indecomposable symmetrizable Cartan matrices (or *contragredient Lie superalgebras*) over algebraically closed fields have been classified by Bouarroudj, Grozman and Leites (2009), under some extra technical hypotheses.

Simple modular Lie superalgebras with a Cartan matrix

The finite dimensional modular Lie superalgebras with indecomposable symmetrizable Cartan matrices (or *contragredient Lie superalgebras*) over algebraically closed fields have been classified by Bouarroudj, Grozman and Leites (2009), under some extra technical hypotheses.

For characteristic $p \geq 3$, apart from the Lie superalgebras obtained as the analogues of the Lie superalgebras in the classification in characteristic 0, by reducing the Cartan matrices modulo p, there are only the following exceptions:

Simple modular Lie superalgebras with a Cartan matrix

Simple modular Lie superalgebras with a Cartan matrix

1. Two exceptions in characteristic 5: $\mathfrak{br}(2;5)$ and $\mathfrak{el}(5;5)$.
   (Dimensions $10|12$ and $55|32$.)

Simple modular Lie superalgebras with a Cartan matrix

1. Two exceptions in characteristic 5: $\mathfrak{br}(2;5)$ and $\mathfrak{el}(5;5)$. (Dimensions $10|12$ and $55|32$.)
2. The family of exceptions given by the Lie superalgebras in the Supermagic Square in characteristic 3.

Simple modular Lie superalgebras with a Cartan matrix

1. Two exceptions in characteristic 5: $\mathfrak{br}(2;5)$ and $\mathfrak{el}(5;5)$. (Dimensions $10|12$ and $55|32$.)

2. The family of exceptions given by the Lie superalgebras in the Supermagic Square in characteristic 3.

3. Another two exceptions in characteristic 3, similar to the ones in characteristic 5: $\mathfrak{br}(2;3)$ and $\mathfrak{el}(5;3)$. (Dimensions $10|8$ and $39|32$.)

Moreover,

- The superalgebra $\mathfrak{br}(2;3)$ appeared first related to an eight dimensional *symplectic triple system* (E. 2006).

Moreover,

- The superalgebra $\mathfrak{br}(2;3)$ appeared first related to an eight dimensional *symplectic triple system* (E. 2006).
- The superalgebra $\mathfrak{el}(5;5)$ is the Lie superalgebra $\mathcal{T}(C(k), K_{10})$ considered previously.

Moreover,

- The superalgebra $\mathfrak{br}(2;3)$ appeared first related to an eight dimensional *symplectic triple system* (E. 2006).
- The superalgebra $\mathfrak{el}(5;5)$ is the Lie superalgebra $\mathcal{T}(C(k), K_{10})$ considered previously.
- The superalgebra $\mathfrak{el}(5;3)$ lives (as a natural maximal subalgebra) in the Lie superalgebra $\mathfrak{g}(3,8)$ of the Supermagic Square.

Moreover,

- The superalgebra $\mathfrak{br}(2;3)$ appeared first related to an eight dimensional *symplectic triple system* (E. 2006).
- The superalgebra $\mathfrak{el}(5;5)$ is the Lie superalgebra $\mathcal{T}(C(k), K_{10})$ considered previously.
- The superalgebra $\mathfrak{el}(5;3)$ lives (as a natural maximal subalgebra) in the Lie superalgebra $\mathfrak{g}(3,8)$ of the Supermagic Square.

Moreover,

- The superalgebra $\mathfrak{br}(2;3)$ appeared first related to an eight dimensional *symplectic triple system* (E. 2006).
- The superalgebra $\mathfrak{el}(5;5)$ is the Lie superalgebra $\mathcal{T}(C(k), K_{10})$ considered previously.
- The superalgebra $\mathfrak{el}(5;3)$ lives (as a natural maximal subalgebra) in the Lie superalgebra $\mathfrak{g}(3,8)$ of the Supermagic Square.

That's all. Thanks