A Freudenthal Supermagic Square

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1. Freudenthal Magic Square
2. Jordan superalgebras
3. Composition superalgebras
4. Supermagic Square
5. Some conclusions

1 Freudenthal Magic Square

2 Jordan superalgebras

3 Composition superalgebras

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5 Some conclusions

Exceptional Lie algebras

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

Exceptional Lie algebras

$$G_2, \quad F_4, \quad E_6, \quad E_7, \quad E_8$$

$G_2 = \mathfrak{der}\, \mathbb{O}$ (Cartan 1914)

$F_4 = \mathfrak{der}\, H_3(\mathbb{O})$

$E_6 = \mathfrak{str}_0\, H_3(\mathbb{O})$

(Chevalley-Schafer 1950)

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then

$$\mathcal{T}(C,J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

is a Lie algebra (char $\neq 2,3$) under a suitable Lie bracket:

$$[a\otimes x, b\otimes y] = \frac{1}{3}tr(xy)D_{a,b} + \Big([a,b]\otimes \big(xy - \frac{1}{3}tr(xy)1\big)\Big) + 2t(ab)d_{x,y}.$$

Freudenthal Magic Square

$\mathcal{T}(C,J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$
k	A_1	A_2	C_3	F_4
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7
$C(k)$	F_4	E_6	E_7	E_8

Tits construction rearranged

$$
\begin{aligned}
&J = H_3(C') \simeq k^3 \oplus \left(\oplus_{i=0}^2 \iota_i(C')\right),\\
&J_0 \simeq k^2 \oplus \left(\oplus_{i=0}^2 \iota_i(C')\right),\\
&\mathfrak{der}\, J \simeq \mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^2 \iota_i(C')\right),
\end{aligned}
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$$\mathfrak{der}\, J \simeq \mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right),$$

$$\begin{aligned}\mathcal{T}(C,J) &= \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq \mathfrak{der}\, C \oplus (C_0 \otimes k^2) \oplus \left(\oplus_{i=0}^{2} C_0 \otimes \iota_i(C')\right) \oplus \left(\mathfrak{tri}(C') \oplus \left(\oplus_{i=0}^{2} \iota_i(C')\right)\right) \\ &\simeq \left(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\right) \oplus \left(\oplus_{i=0}^{2} \iota_i(C \otimes C')\right)\end{aligned}$$

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$$\begin{aligned}\mathcal{T}(C,J) &= \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq \mathfrak{der}\, C \oplus (C_0 \otimes k^2) \oplus \big(\oplus_{i=0}^2 C_0 \otimes \iota_i(C')\big) \oplus \Big(\mathfrak{tri}(C') \oplus \big(\oplus_{i=0}^2 \iota_i(C')\big)\Big) \\ &\simeq \big(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\big) \oplus \big(\oplus_{i=0}^2 \iota_i(C \otimes C')\big)\end{aligned}$$

$$\mathfrak{tri}(C) = \{(d_0, d_1, d_2) \in \mathfrak{so}(C)^3 : \overline{d_0(\overline{xy})} = d_2(x)y + xd_1(y)\ \forall x, y \in C\}$$

is the triality Lie algebra of C.

Tits construction rearranged

The product in

$$\mathfrak{g}(C, C') = \big(\mathfrak{tri}(C) \oplus \mathfrak{tri}(C')\big) \oplus \big(\oplus_{i=0}^{2} \iota_i(C \otimes C')\big),$$

is given by:

Tits construction rearranged

The product in

$$\mathfrak{g}(C,C') = \big(\mathfrak{tri}(C)\oplus\mathfrak{tri}(C')\big)\oplus\big(\oplus_{i=0}^{2}\iota_i(C\otimes C')\big),$$

is given by:

- $\mathfrak{tri}(C)\oplus\mathfrak{tri}(C')$ is a Lie subalgebra of $\mathfrak{g}(C,C')$,
- $[(d_0,d_1,d_2),\iota_i(x\otimes x')] = \iota_i\big(d_i(x)\otimes x'\big)$,
- $[(d_0',d_1',d_2'),\iota_i(x\otimes x')] = \iota_i\big(x\otimes d_i'(x')\big)$,
- $[\iota_i(x\otimes x'),\iota_{i+1}(y\otimes y')] = \iota_{i+2}\big((\bar{x}\bar{y})\otimes(\bar{x}'\bar{y}')\big)$ (indices modulo 3),
- $[\iota_i(x\otimes x'),\iota_i(y\otimes y')] = q'(x',y')\theta^i(t_{x,y}) + q(x,y)\theta'^i(t'_{x',y'})$,

  where $t_{x,y} = \big(q(x,.)y - q(y,.)x, \frac{1}{2}q(x,y)1 - R_{\bar{x}}R_y, \frac{1}{2}q(x,y)1 - L_{\bar{x}}L_y\big)$

(Vinberg, Allison-Faulkner, Barton-Sudbery, Landsberg-Manivel)

Freudenthal Magic Square (char 3)

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dim C'

	$\mathfrak{g}(C, C')$	1	2	4	8
	1	A_1	$\tilde{A}_2$	C_3	F_4
	2	$\tilde{A}_2$	$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$
dim C	4	C_3	$\tilde{A}_5$	D_6	E_7
	8	F_4	$\tilde{E}_6$	E_7	E_8

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2	$\tilde{A}_2$	$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$
4	C_3	$\tilde{A}_5$	D_6	E_7
8	F_4	$\tilde{E}_6$	E_7	E_8

- $\tilde{A}_2$ denotes a form of $\mathfrak{pgl}_3$, so $[\tilde{A}_2, \tilde{A}_2]$ is a form of $\mathfrak{psl}_3$.
- $\tilde{A}_5$ denotes a form of $\mathfrak{pgl}_6$, so $[\tilde{A}_5, \tilde{A}_5]$ is a form of $\mathfrak{psl}_6$.
- $\tilde{E}_6$ is not simple, but $[\tilde{E}_6, \tilde{E}_6]$ is a codimension 1 simple ideal.

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2 Jordan superalgebras

3 Composition superalgebras

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A look at the rows of Tits construction

$$\mathcal{T}(C, J) = \mathfrak{der}\, C \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

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Second row $\dim C = 2$, $\mathcal{T}(C,J) \simeq J_0 \oplus \mathfrak{der}\, J \simeq \mathfrak{str}_0(J)$.

Third row $\dim C = 4$, so $C = Q$ is a quaternion algebra and

$$\begin{aligned}\mathcal{T}(C,J) &= \mathrm{ad}_{Q_0} \oplus (Q_0 \otimes J_0) \oplus \mathfrak{der}\, J \\ &\simeq (Q_0 \otimes J) \oplus \mathfrak{der}\, J.\end{aligned}$$

(This is Tits version [Tits 1962] of the Tits-Kantor-Koecher construction $\mathcal{TKK}(J)$)

Fourth row

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Fourth row $\dim C = 8$. If the characteristic is $\neq 2, 3$, then $\mathfrak{der}\, C = \mathfrak{g}_2$ is simple of type G_2, C_0 is its smallest nontrivial irreducible module, and

$$\mathcal{T}(C, J) = \mathfrak{g}_2 \oplus (C_0 \otimes J_0) \oplus \mathfrak{der}\, J$$

is a G_2-*graded* Lie algebra. Essentially, all the G_2-graded Lie algebras appear in this way [Benkart-Zelmanov 1996].

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It makes sense to consider Jordan superalgebras, as long as its Grassmann envelope satisfies the Cayley-Hamilton equation of degree 3.

A Freudenthal Supermagic Rectangle (Benkart-E. 2003)

$\mathcal{T}(C,J)$	$H_3(k)$	$H_3(k\times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$	$J(V)$	D_t
k	A_1	A_2	C_3	F_4	A_1	$B(0,1)$
$k\times k$	A_2	$A_2\oplus A_2$	A_5	E_6	$B(0,1)$	$A(1,0)$
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7	$B(1,1)$	$D(2,1;t)$
$C(k)$	F_4	E_6	E_7	E_8	$G(3)$	$F(4)$ $(t=2)$

Kac superalgebra K_{10}

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Let $K_3 = ke \oplus V$ be the *tiny Kaplansky superalgebra* (V a two dim'l vector space with a nonzero alternating bilinear form $\langle .|. \rangle$), with multiplication:

$$e^2 = e, \quad ev = ve = \frac{1}{2}v, \quad uv = \langle u|v \rangle e.$$

Consider the supersymmetric bilinear form $(.|.)$ on K_3 with $(e|e) = \frac{1}{2}$, $(e|V) = 0$, and $(u|v) = \langle u|v \rangle \ \forall u, v \in V$.

Kac superalgebra K_{10}

Then the Kac superalgebra can be defined as:

$$K_{10} = k1 \oplus (K_3 \otimes K_3),$$

with multiplication determined by:

$$(a \otimes b)(c \otimes d) = (-1)^{bc}\Big(ac \otimes bd - \frac{3}{4}(a|c)(b|d)1\Big).$$

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In characteristic 3, K_{10} is no longer simple, as it contains the simple ideal $K_9 = K_3 \otimes K_3$.

Kac superalgebra K_{10}

The Grasmann envelope of K_{10} is a degree three algebra in characteristic 5 (McCrimmon 2005), so K_{10} can be plugged into the fourth row of Tits construction:

$\mathcal{T}(C(k), K_{10})$ in characteristic 5!!

Kac superalgebra K_{10}

The Grasmann envelope of K_{10} is a degree three algebra in characteristic 5 (McCrimmon 2005), so K_{10} can be plugged into the fourth row of Tits construction:

$\mathcal{T}(C(k), K_{10})$ in characteristic 5!!

This is a new simple modular Lie superalgebra, whose even part is $\mathfrak{so}_{11}$ and odd part its spin module (E. 2007).

A larger Freudenthal Supermagic Rectangle

$\mathcal{T}(C,J)$	$H_3(k)$	$H_3(k \times k)$	$H_3(\mathrm{Mat}_2(k))$	$H_3(C(k))$	$J(V)$	D_t	K_{10}
k	A_1	A_2	C_3	F_4	A_1	$B(0,1)$	$B(0,1) \oplus B(0,1)$
$k \times k$	A_2	$A_2 \oplus A_2$	A_5	E_6	$B(0,1)$	$A(1,0)$	$C(3)$
$\mathrm{Mat}_2(k)$	C_3	A_5	D_6	E_7	$B(1,1)$	$D(2,1;t)$	$F(4)$
$C(k)$	F_4	E_6	E_7	E_8	$G(3)$	$F(4)$ $(t=2)$	$\mathcal{T}(C(k), K_{10})$ (char 5)

Fourth row, characteristic 3

If the characteristic is 3 and $\dim C = 8$, then $\mathfrak{Der}\, C$ is no longer simple, but contains the simple ideal $\mathrm{ad}\, C_0$ (a form of $\mathfrak{psl}_3$). It makes sense to consider:

$$\begin{aligned}\tilde{\mathcal{T}}(C, J) &= \mathrm{ad}\, C_0 \oplus (C_0 \otimes J_0) \oplus \mathfrak{Der}\, J \\ &\simeq (C_0 \otimes J) \oplus \mathfrak{Der}\, J.\end{aligned}$$

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$\tilde{\mathcal{T}}(C, J)$ becomes a Lie algebra if and only if J is a commutative and alternative algebra (these conditions imply the Jordan identity).

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The simple commutative alternative algebras are just the fields, so nothing interesting appears here.

Fourth "superrow", characteristic 3

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(i) fields,

(ii) $J(V)$, the Jordan superalgebra of a superform on a two dimensional odd space V,

(iii) $B = B(\Gamma, D, 0) = \Gamma \oplus \Gamma u$, where

- Γ is a commutative associative algebra,
- $D \in \mathfrak{Der}\,\Gamma$ such that Γ is D-simple,
- $a(bu) = (ab)u = (au)b, \quad (au)(bu) = aD(b) - D(a)b, \, \forall a, b \in \Gamma.$

Fourth "superrow", characteristic 3

Example (Divided powers)

$\Gamma = \mathcal{O}(1;n) = \text{span}\left\{t^{(r)} : 0 \leq r \leq 3^n - 1\right\}$,

$$t^{(r)}t^{(s)} = \binom{r+s}{r}t^{(r+s)},$$

$$D(t^{(r)}) = t^{(r-1)}.$$

Bouarroudj-Leites superalgebras

Over an algebraically closed field of characteristic 3:

- $\tilde{\mathcal{T}}\big(C(k), J(V)\big)$ is a simple Lie superalgebra specific of characteristic 3 of (super)dimension $10|14$,
- $\tilde{\mathcal{T}}\big(C(k), \mathcal{O}(1,n) \oplus \mathcal{O}(1,n)u\big) = \mathrm{Bj}(1;n|7)$ is a simple Lie superalgebra of (super)dimension $2^3 \times 3^n|2^3 \times 3^n$.

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Both simple Lie superalgebras have been considered in a completely different way by Bouarroudj and Leites (2006).

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Composition superalgebras

Definition

A superalgebra $C = C_{\bar{0}} \oplus C_{\bar{1}}$, endowed with a regular quadratic superform $q = (q_{\bar{0}}, b)$, called the *norm*, is said to be a *composition superalgebra* in case

$$q_{\bar{0}}(x_{\bar{0}}y_{\bar{0}}) = q_{\bar{0}}(x_{\bar{0}})q_{\bar{0}}(y_{\bar{0}}),$$

$$b(x_{\bar{0}}y, x_{\bar{0}}z) = q_{\bar{0}}(x_{\bar{0}})b(y, z) = b(yx_{\bar{0}}, zx_{\bar{0}}),$$

$$b(xy, zt) + (-1)^{|x||y|+|x||z|+|y||z|}b(zy, xt) = (-1)^{|y||z|}b(x, z)b(y, t),$$

The unital composition superalgebras are termed *Hurwitz superalgebras*.

Composition superalgebras: examples (Shestakov)

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$$B(1,2) = k1 \oplus V,$$

char $k = 3$, V a two dim'l vector space with a nonzero alternating bilinear form $\langle .|.\rangle$, with

$$1x = x1 = x, \quad uv = \langle u|v\rangle 1, \qquad q_{\bar{0}}(1) = 1, \quad b(u,v) = \langle u|v\rangle,$$

is a Hurwitz superalgebra. (As a superalgebra, this is just our previous $J(V)$)

Composition superalgebras: examples (Shestakov)

$$B(4,2) = \mathrm{End}_k(V) \oplus V,$$

k and V as before, $\mathrm{End}_k(V)$ is equipped with the symplectic involution $f \mapsto \bar{f}$, $(\langle f(u)|v\rangle = \langle u|\bar{f}(v)\rangle)$, the multiplication is given by:

- the usual multiplication (composition of maps) in $\mathrm{End}_k(V)$,
- $v \cdot f = f(v) = \bar{f} \cdot v$ for any $f \in \mathrm{End}_k(V)$ and $v \in V$,
- $u \cdot v = \langle .|u\rangle v \ \big(w \mapsto \langle w|u\rangle v\big) \in \mathrm{End}_k(V)$ for any $u, v \in V$,

and with quadratic superform

$$q_{\bar{0}}(f) = \det f, \quad b(u,v) = \langle u|v\rangle,$$

is a Hurwitz superalgebra.

Composition superalgebras: classification

Theorem (E.-Okubo 2002)

Any unital composition superalgebra is either:

- *a Hurwitz algebra,*
- *a $\mathbb{Z}_2$-graded Hurwitz algebra in characteristic* 2,
- *isomorphic to either $B(1,2)$ or $B(4,2)$ in characteristic* 3.

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Hurwitz superalgebras can be plugged into the symmetric construction of Freudenthal Magic Square $\mathfrak{g}(C, C')$.

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Supermagic Square (char 3, Cunha-E. 2007)

$\mathfrak{g}(C, C')$	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$	$B(1,2)$	$B(4,2)$
k	A_1	$\tilde{A}_2$	C_3	F_4	6|8	21|14
$k \times k$		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	11|14	35|20
$\mathrm{Mat}_2(k)$			D_6	E_7	24|26	66|32
$C(k)$				E_8	55|50	133|56
$B(1,2)$					21|16	36|40
$B(4,2)$						78|64

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$k \times k$		$\tilde{A}_2 \oplus \tilde{A}_2$	$\tilde{A}_5$	$\tilde{E}_6$	$11\|14$	$35\|20$
$\mathrm{Mat}_2(k)$			D_6	E_7	$24\|26$	$66\|32$
$C(k)$				E_8	$55\|50$	$133\|56$
$B(1,2)$					$21\|16$	$36\|40$
$B(4,2)$						$78\|64$

Notation: $\mathfrak{g}(n, m)$ will denote the superalgebra $\mathfrak{g}(C, C')$, with $\dim C = n$, $\dim C' = m$.

Lie superalgebras in the Supermagic Square

	$B(1,2)$	$B(4,2)$
k	$\mathfrak{psl}_{2,2}$	$\mathfrak{sp}_6 \oplus (14)$
$k \times k$	$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus ((2) \otimes \mathfrak{psl}_3)$	$\mathfrak{pgl}_6 \oplus (20)$
$\mathrm{Mat}_2(k)$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$\mathfrak{so}_{12} \oplus spin_{12}$
$C(k)$	$(\mathfrak{sl}_2 \oplus \mathfrak{f}_4) \oplus ((2) \otimes (25))$	$\mathfrak{e}_7 \oplus (56)$
$B(1,2)$	$\mathfrak{so}_7 \oplus 2spin_7$	$\mathfrak{sp}_8 \oplus (40)$
$B(4,2)$	$\mathfrak{sp}_8 \oplus (40)$	$\mathfrak{so}_{13} \oplus spin_{13}$

Lie superalgebras in the Supermagic Square

All these Lie superalgebras are simple, with the exception of $\mathfrak{g}(2,3)$ and $\mathfrak{g}(2,6)$, both of which contain a codimension one simple ideal.

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Only $\mathfrak{g}(1,3) \simeq \mathfrak{psl}_{2,2}$ has a counterpart in Kac's classification in characteristic 0. The other Lie superalgebras in the Supermagic Square, or their derived algebras, are new simple Lie superalgebras, specific of characteristic 3.

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The simple Lie superalgebra $\mathfrak{g}(2,3)' = [\mathfrak{g}(2,3), \mathfrak{g}(2,3)]$ is isomorphic to our previous $\tilde{\mathcal{T}}(C(k), J(V))$.

The Supermagic Square and Jordan algebras

	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$
$B(1,2)$	$\mathfrak{psl}_{2,2}$	$(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3) \oplus ((2) \otimes \mathfrak{psl}_3)$	$(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6) \oplus ((2) \otimes (13))$	$(\mathfrak{sl}_2 \oplus \mathfrak{f}_4) \oplus ((2) \otimes (25))$
$B(4,2)$	$\mathfrak{sp}_6 \oplus (14)$	$\mathfrak{pgl}_6 \oplus (20)$	$\mathfrak{so}_{12} \oplus spin_{12}$	$\mathfrak{e}_7 \oplus (56)$

The Supermagic Square and Jordan algebras

	k	$k \times k$	$\mathrm{Mat}_2(k)$	$C(k)$
$B(1,2)$	$\mathfrak{psl}_{2,2}$	$\big(\mathfrak{sl}_2 \oplus \mathfrak{pgl}_3\big) \oplus \big((2) \otimes \mathfrak{psl}_3\big)$	$\big(\mathfrak{sl}_2 \oplus \mathfrak{sp}_6\big) \oplus \big((2) \otimes (13)\big)$	$\big(\mathfrak{sl}_2 \oplus \mathfrak{f}_4\big) \oplus \big((2) \otimes (25)\big)$
$B(4,2)$	$\mathfrak{sp}_6 \oplus (14)$	$\mathfrak{pgl}_6 \oplus (20)$	$\mathfrak{so}_{12} \oplus \mathit{spin}_{12}$	$\mathfrak{e}_7 \oplus (56)$

$$\mathfrak{g}(3,r) = \big(\mathfrak{sl}_2 \oplus \mathfrak{der}\, J\big) \oplus \big((2) \otimes \hat{J}\big),$$
$$r = 1,2,4,8, \qquad \hat{J} = J_0/k1, \quad J = H_3(C).$$

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$$\mathfrak{g}(6,r) = (\mathfrak{der}\, T) \oplus T,$$

$$r = 1,2,4,8, \qquad T = \begin{pmatrix} k & J \\ J & k \end{pmatrix}, \quad J = H_3(C).$$

The Supermagic Square and Jordan superalgebras

	$B(1,2)$	$B(4,2)$
k	$\mathfrak{der}\big(H_3(B(1,2))\big)$	$\mathfrak{der}\big(H_3(B(4,2))\big)$
$k \times k$	$\mathfrak{pstr}\big(H_3(B(1,2))\big)$	$\mathfrak{pstr}\big(H_3(B(4,2))\big)$
$\mathrm{Mat}_2(k)$	$\mathcal{TKK}\big(H_3(B(1,2))\big)$	$\mathcal{TKK}\big(H_3(B(4,2))\big)$
$C(k)$		
$B(1,2)$	$\mathcal{TKK}(K_9)$	
$B(4,2)$		

Simple modular Lie superalgebras with a Cartan matrix

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For characteristic $p \geq 3$, apart from the Lie superalgebras obtained as the analogues of the Lie superalgebras in the classification in characteristic 0, by reducing the Cartan matrices modulo p, there are only the following exceptions:

Simple modular Lie superalgebras with a Cartan matrix

1. Two exceptions in characteristic 5: $\mathfrak{br}(2;5)$ and $\mathfrak{el}(5;5)$. (Dimensions $10|12$ and $55|32$.)

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The superalgebra $\mathfrak{el}(5;5)$ is the Lie superalgebra $\mathcal{T}(C(k), K_{10})$ considered previously.

$\mathfrak{el}(5;3)$

The superalgebra $\mathfrak{el}(5;3)$ lives (as a natural maximal subalgebra) in the Lie superalgebra $\mathfrak{g}(3,8)$ of the Supermagic Square as follows:

- $\mathfrak{el}(5;3)_{\bar{0}} = \mathfrak{sl}_2 \oplus \mathfrak{so}_9 \leq \mathfrak{sl}_2 \oplus \mathfrak{f}_4 = \mathfrak{g}(3,8)_{\bar{0}}$,
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- $\mathfrak{el}(5;3)_{\bar{1}} = (2) \otimes \big(C(k) \oplus C(k)\big) \leq (2) \otimes \hat{J} = \mathfrak{g}(3,8)_{\bar{1}}$,
  $\Big(\hat{J} = J_0/k1$ contains three copies of $C(k)$ in the off-diagonal entries.$\Big)$

1 Freudenthal Magic Square

2 Jordan superalgebras

3 Composition superalgebras

4 Supermagic Square

5 Some conclusions

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char. 5: In characteristic 5 one can add the new simple Lie superalgebra (without counterpart in Kac's classification) $\mathfrak{el}(5; 5) = \mathcal{T}(C(k), K_{10})$.

Some conclusions

char. 3:

• By using a symmetric construction in terms of two Hurwitz algebras, and extending it (only in characteristic 3) with the use of composition superalgebras.
Ten new simple Lie superalgebras are obtained:
$\mathfrak{g}(r,3)'$ ($r = 2,4,8$), $\mathfrak{g}(r,6)'$ ($r = 1,2,4,8$), $\mathfrak{g}(3,3)$, $\mathfrak{g}(3,6)$ and $\mathfrak{g}(6,6)$.

• The new simple Lie superalgebra $\mathfrak{el}(5;3)$ appears as a maximal subalgebra of $\mathfrak{g}(8,3)$.

• The new simple Lie superalgebra

$$\tilde{\mathcal{T}}(C(k), \mathcal{O}(1,n) \oplus \mathcal{O}(1,n)u) = \mathrm{Bj}(1; n|7)$$

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That's all. Thanks